Packing d-dimensional balls into a $d+1$-dimensional container

Sergio Cabello
University of Ljubljana and IMFM, Slovenia

Joint work with
Helmut Alt (FU Berlin)
Otfried Cheong (SCALGO)
Ji-won Park (Université de Lorraine, CNRS, Inria, LORIA)
Nadja Seiferth (FU Berlin)

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The problem

- We are in $\mathbb{R}^{d+1}$
- A (unit) hyperdisk is a (unit) d-dim ball in $\mathbb{R}^{d+1}$
- Today, all hyperdisks are unit

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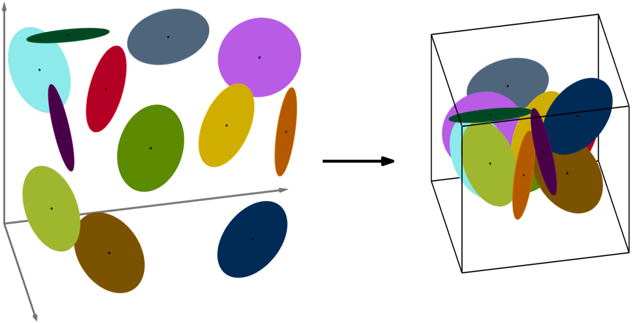
- Task: pack the hyperdisks using translations into a $d+1$-dim container of min volume



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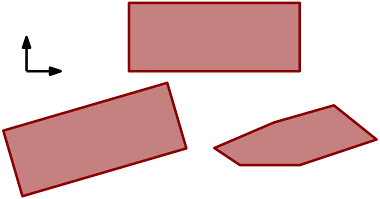
Pack: pairwise disjoint relative interiors

Container:

- axis-parallel box
- arbitrarily oriented box
- convex body



Allowing rotations — not interesting.

The problems

- Algorithmic problem:
  Given n unit hyperdisks in $\mathbb{R}^{d+1}$, find the min-volume
  - axis-parallel box
  - arbitrarily oriented box
  - convex body

  where they can be packed under translations.
- Mathematical problem:
  Find a tight bound $f(n, d)$ such that each set of n unit hyperdisks in $\mathbb{R}^{d+1}$ can be packed with translations into
  - an axis-parallel box
  - an arbitrarily oriented box
  - a convex body

  of volume $f(n, d)$.

What do we show?

- Algorithmic problem:
  Given n unit hyperdisks in $\mathbb{R}^{d+1}$, find the min-volume container where they can be packed under translations.
  - $d^{O(d)}$-approximation algorithms
  - $O(1)$-approximation algorithms for each d

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  Find a tight bound $f(n,d)$ such that each set of n unit hyperdisks in $\mathbb{R}^{d+1}$ can be packed with translations into a container of volume $f(n,d)$.
  - $f(n,d) = \Theta(n^{\frac{d-1}{d}})$
  - for $d = 1$, $f(n,1) = \Theta(1)$, independent of n
  - for $d = 2$, $f(n,2) = \Theta(n^{1/2})$, unbounded for $n \to \infty$

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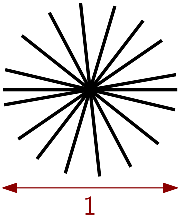
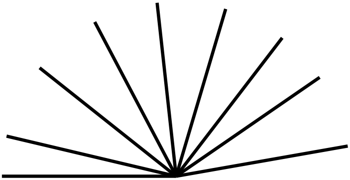
- Quite some geometry, simple algorithms

Container for all unit segments





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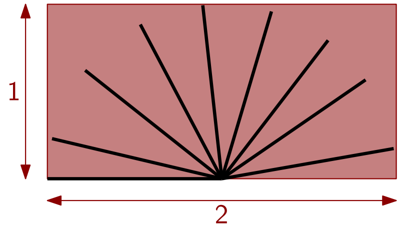

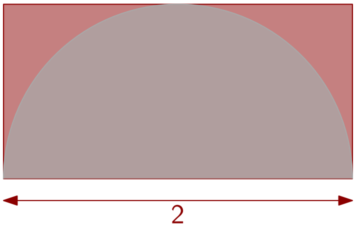

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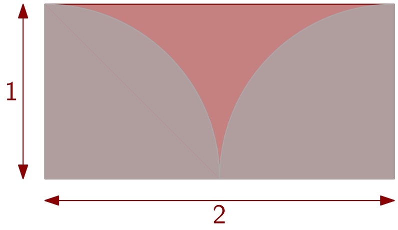

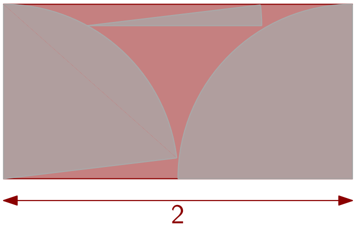

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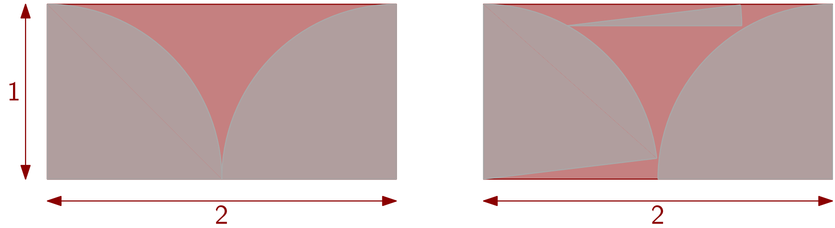


Such a result is not possible for 2-dim unit disks in $\mathbb{R}^3$.
For some n unit disks in $\mathbb{R}^3$, any container needs volume $\Theta(n^{1/2})$.

Potato Sack Theorem

- $(K_i)_{i=1}^{\infty}$ a sequence of convex bodies in $\mathbb{R}^d$
- $\mathrm{diam}(K_i) \leq a$ for all i
- $\sum_{i=1}^{\infty} \mathrm{vol}\, K_i \leq b$
- There exists a cube of volume $c = f(a, b, d)$ where we can pack $(K_i)_{i=1}^{\infty}$ using rigid motions

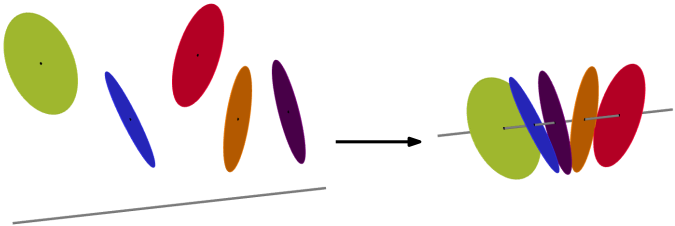
- Problem 10.1 by Auerbach, Banach, Mazur, and Ulam in the Scottish Book
- Solution published by Kosiński (1957)
- Survey by Fejes Tóth, 2023 – Packing and covering properties of sequences of convex bodies

The ideas

- Stabbing problem
- Connection between packing and stabbing for similar hyperdisks
- Clustering of hyperdisks with similar normals
- Quite some geometry, easy algorithms

Stabbing problem

- ℓ a line in $\mathbb{R}^{d+1}$
- n unit hyperdisks in $\mathbb{R}^{d+1}$
- Task: pack the hyperdisks using translations such that each center lies on ℓ
  - minimize the length of the projection on ℓ
  - minimize the furthest center-to-center distance



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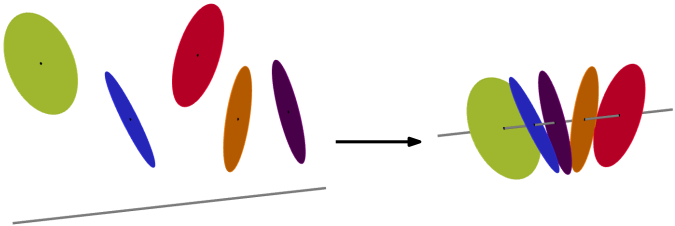
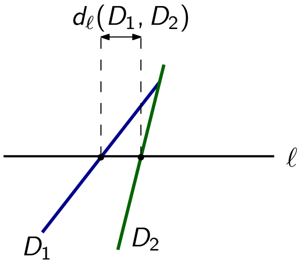


- Discretization: The order of the hyperdisks decides the packing
- Key property: we only need to care about consecutive hyperdisks

A metric on unit hyperdisks

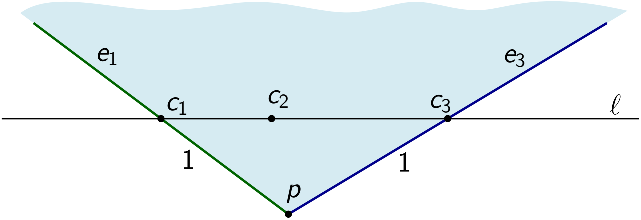
- ℓ a line in $\mathbb{R}^{d+1}$
- D_1 and D_2 two unit hyperdisks in $\mathbb{R}^{d+1}$
- $d_\ell(D_1, D_2) =$ distance between the centers of D_1 and D_2 when they are touching and pierced by ℓ on their centers



- Key property: this is a metric on unit hyperdisks

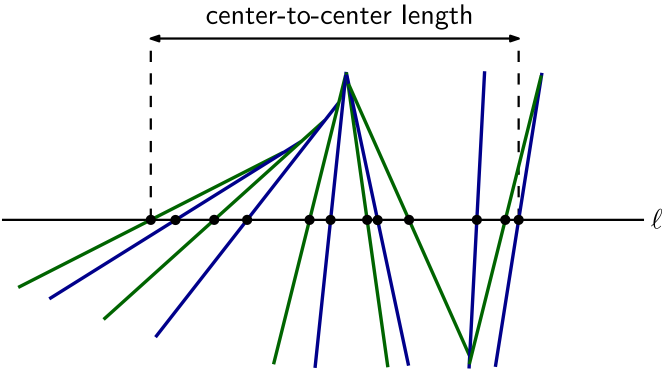
A metric on unit hyperdisks – proof overview

- challenging part: triangular inequality
- assume $d_\ell(D_1, D_2) + d_\ell(D_2, D_3) < d_\ell(D_1, D_3)$
- place D_1 and D_3 touching at a point p
- plane π containing ℓ and the touching point p
- $e_i = D_i \cap \pi$ a segment of unit length $(i = 1, 2, 3)$
- between e_1 and e_3 there is not enough space for e_2



Stabbing problem – $O(1)$-approximation

- Hamiltonian path with weights $d_\ell(\cdot,\cdot)$.
- $\frac{3}{2}$-approximation for minimizing the center-to-center length



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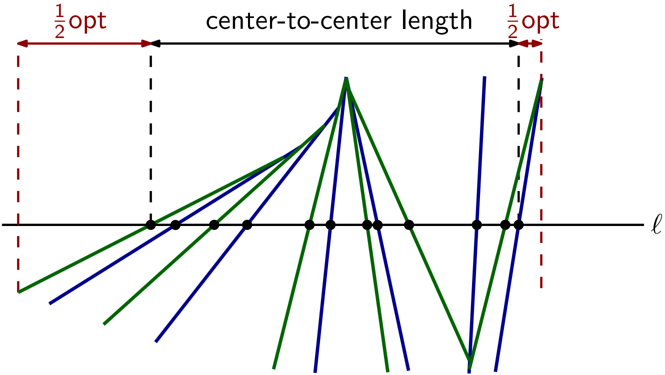
- $\frac{5}{2}$-approximation for minimizing the length of projection



A metric on unit hyperdisks – bounds

- ℓ a line in $\mathbb{R}^{d+1}$
- D_1 and D_2 two unit hyperdisks in $\mathbb{R}^{d+1}$ with normals n_1 and n_2
- $\xi = \measuredangle(n_1, n_2)$
- ϕ such that $\measuredangle(n_1, \ell) \leq \phi$ and $\measuredangle(n_2, \ell) \leq \phi$

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- ϕ such that $\measuredangle(n_1, \ell) \leq \phi$ and $\measuredangle(n_2, \ell) \leq \phi$
- Then

$$\sin \xi \ \leq \ d_\ell(D_1, D_2) \ \leq \ \frac{\sin \xi}{\cos \phi}$$

- $\sin \xi$ is a $O(1)$-approximation to $d_\ell(D_1, D_2)$,
  when ϕ not very large.

A metric on unit hyperdisks – bounds

- consider D_1, D_2 defining $d_\ell(D_1, D_2)$, p touching point
- h_i hyperplane containing D_i
- $g = h_1 \cap h_2$ a $(n-2)$-dim flat, $p \in g$

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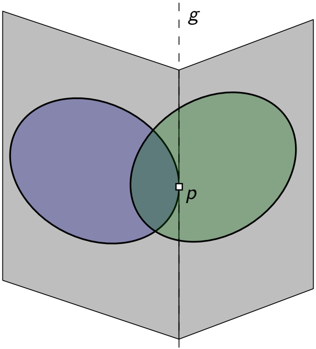
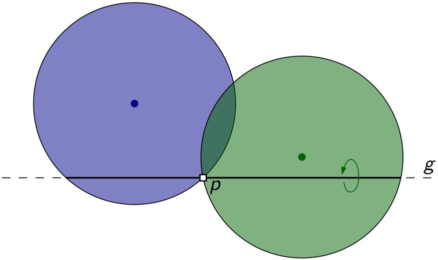
  - one $D_i \cap g$ contains only p OR
  - D_1 and D_2 both contain a piece of g in the interior





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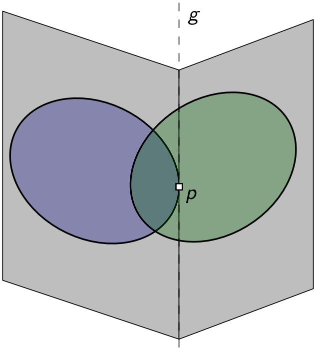

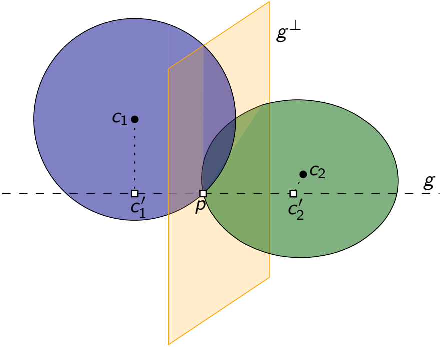

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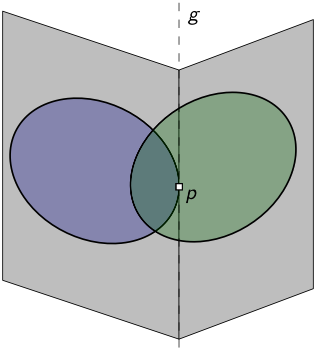

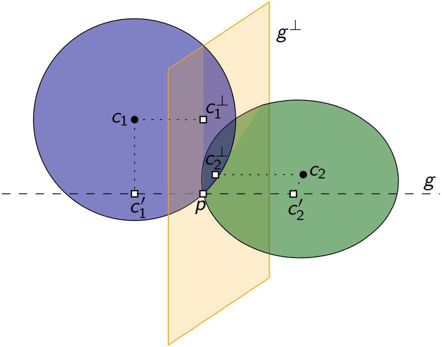

Packing similar unit hyperdisks in AA Box

- $\phi_0 = \arccos(1/\sqrt{d+1})$
- assume each hyperdisk D_i has normal n_i with $\measuredangle(n_i, x_{d+1}) \leq \phi_0$
- opt = volume optimal AA box

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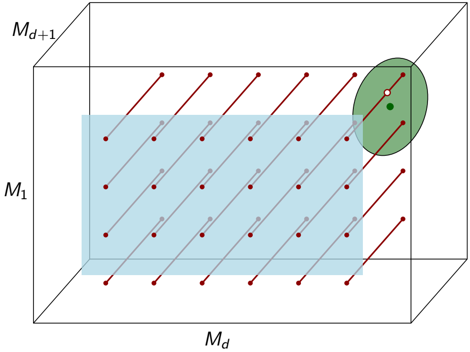
- the volume of that stabbing is $O(1) \cdot O(\text{length}) = O(\text{opt})$



$O(M_1 \times M_2 \times \cdots \times M_d)$ segments of length M_{d+1}

Each hyperdisk pierced at distance $\leq \frac{1}{2}$ from its center

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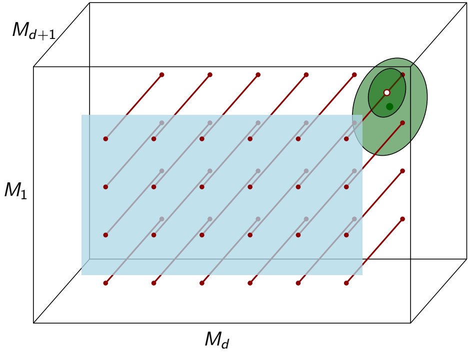

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A collection of hyperdisks of radius $\frac{1}{2}$ pierced through the centers

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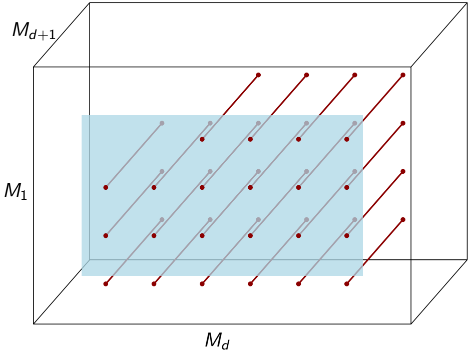
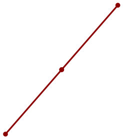


A stabbing of hyperdisks of radius $\frac{1}{2}$



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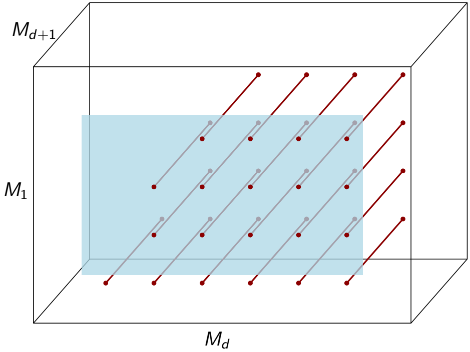

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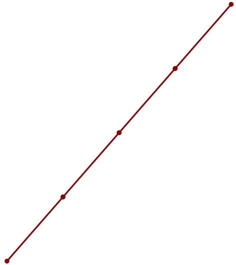

Packing different unit hyperdisks in AA Box

- $\phi_0 = \arccos(1/\sqrt{d+1})$
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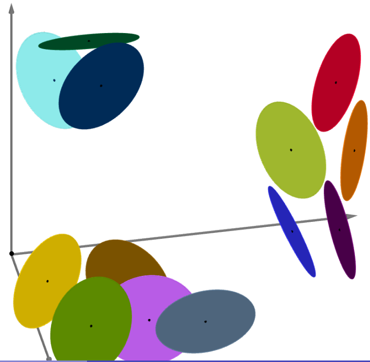

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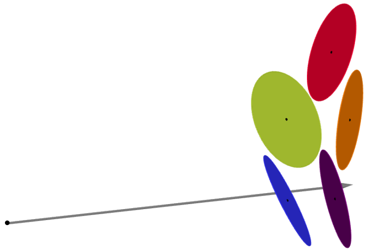

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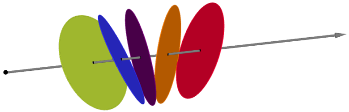

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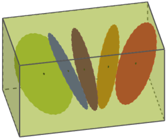

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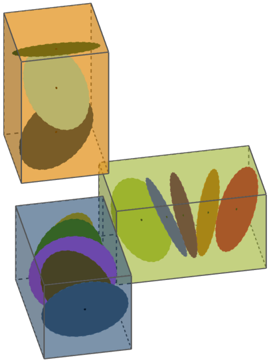

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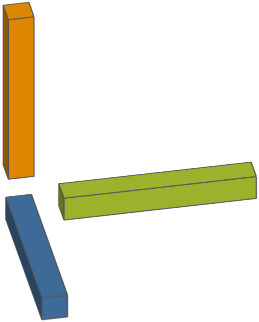

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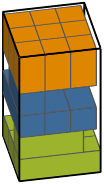

Packing unit hyperdisks in arbitrary box

- choose arbitrary D and make its normal the x_{d+1}-axis
- if all hyperdisks have similar normals, stabbing is good
- stabbing with ℓ or ℓ', if ℓ and ℓ' similar, is $O(1)$-approximation because

$$d_\ell(D_1, D_2) \;=\; \Theta\left(d_{\ell'}(D_1, D_2)\right)$$

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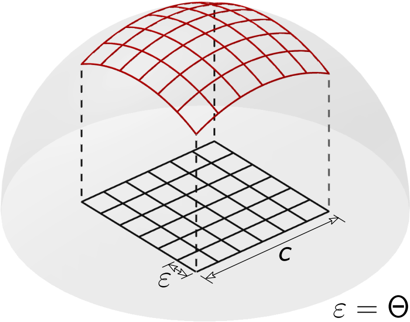
- convex container
  - each convex body K in $\mathbb{R}^d$ contained in a box of volume $d^{3d/2}\,\mathrm{vol}(K))$
  - compute the smallest-volume box and return it

Worst case bounds

- a hyperdisk is represented by a point in $\mathbb{S}^d$
- geodesic distance inside $\mathbb{S}^d \sim d_\ell(\cdot,\cdot)$, for points near ℓ
- an n-point instance such that any MST/TSP has $\Omega(n^{\frac{d-1}{d}})$
- each n points on $\mathbb{S}^d$ have a MST/TSP of length $O(n^{\frac{d-1}{d}})$



$$\varepsilon = \Theta\left(\frac{1}{n^{1/d}}\right)$$

Conclusions

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THANKS for your time!!