Planar Embeddings of Graphs with Specified Edge Lengths

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Erik D. Demaine	MIT
Günter Rote	FU Berlin

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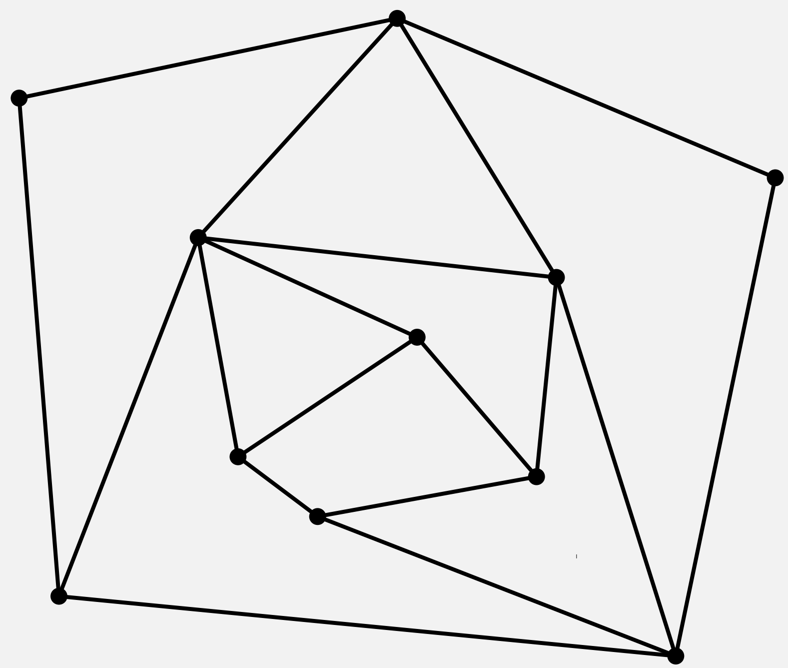
Universiteit Utrecht

Problem and motivation

Given a graph G



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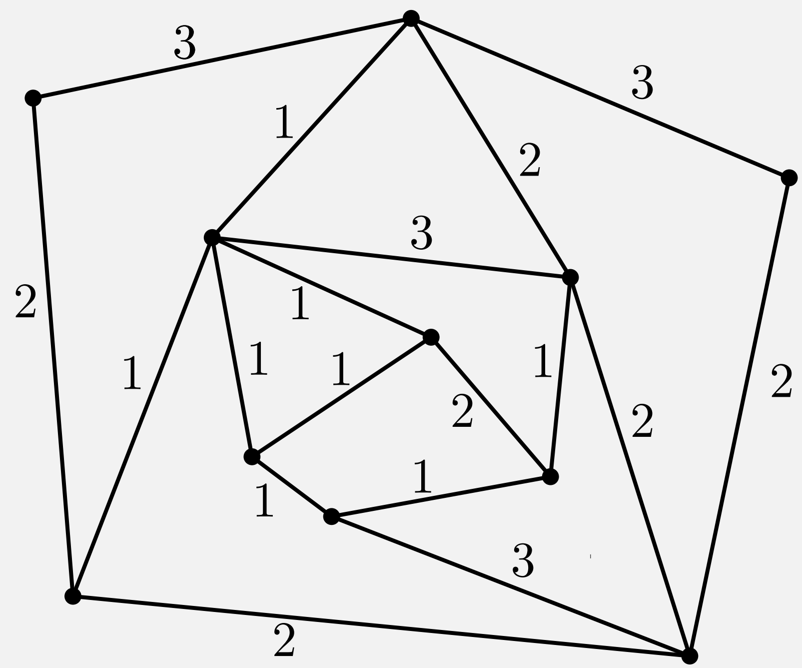


and specified edge lengths

Question: Can we draw G with these edge lengths?

Problem and motivation

Applications:

- Sensor networks
- Structural analysis of molecules
- Linear cartograms

Our results

Restriction to planar embeddings (so planar graphs):

- Triangulated graphs $\Rightarrow$ linear time decision
- $\left\{ \begin{array}{l} \text{3-connected } (\rightarrow \text{ fixed topology}) \\ \text{unit edge lengths,} \\ \text{bounded face degree, and} \\ \text{generically rigid} \end{array} \right\} \Rightarrow$ NP-hard.

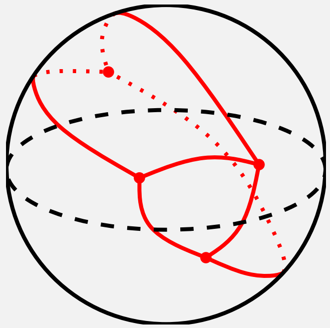
Improves Eades and Wormald '90 $\left\{ \begin{array}{l} \text{2-connected + unit length, or} \\ \text{3-connected} \end{array} \right\}$

also in simplicity

Triangulated graphs

Planar triangulation ⇒ Planar 3-connected

Planar 3-connected ⇒ One topological embedding in $\mathbb{S}^2$ [Whitney]



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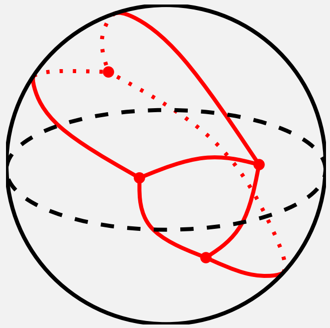

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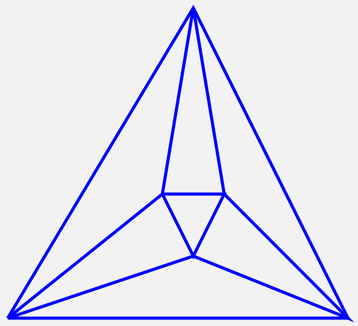

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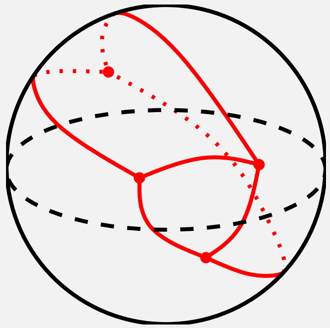

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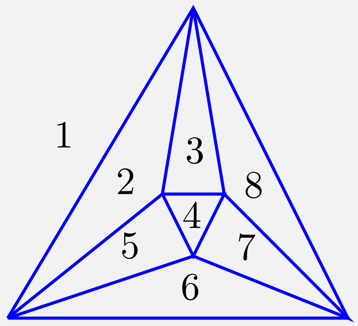

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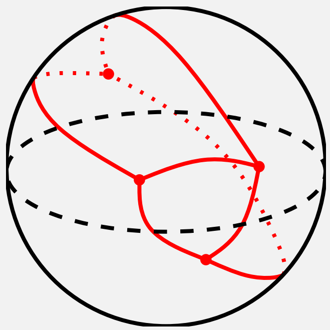

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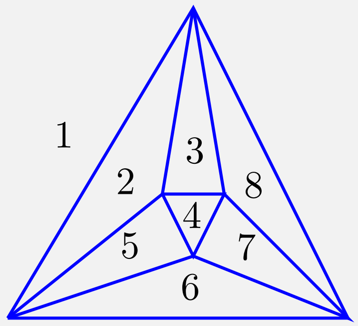

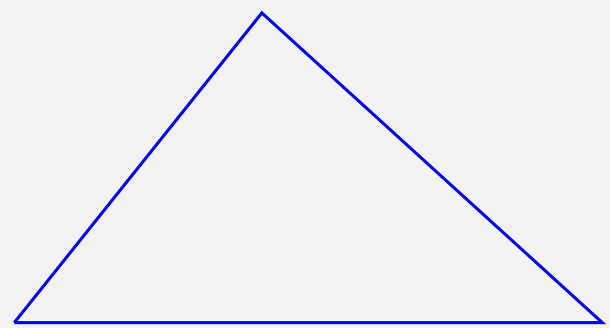

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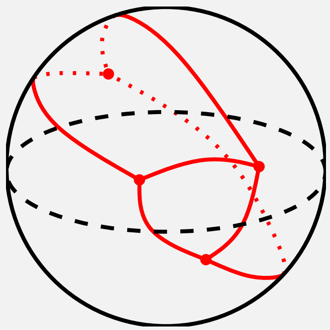

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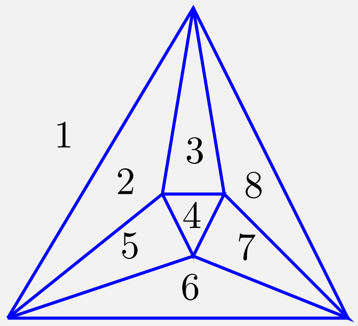

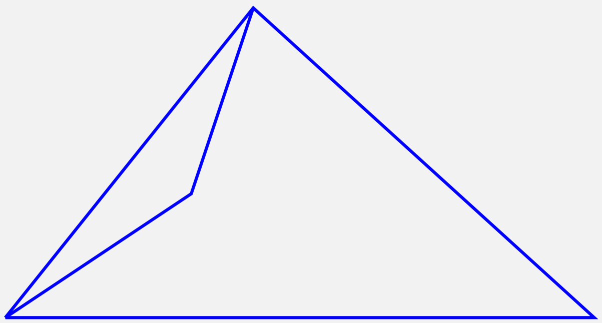

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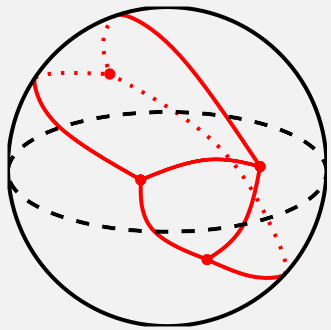

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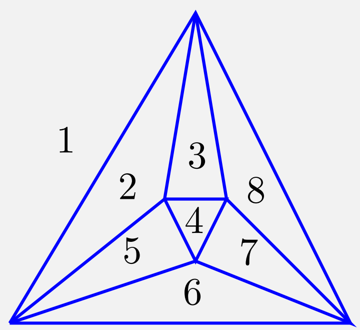

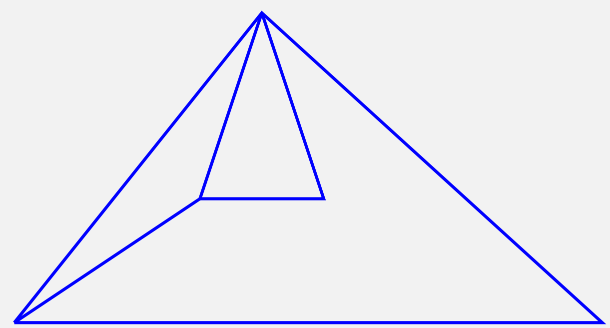

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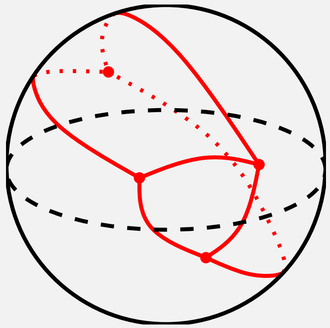

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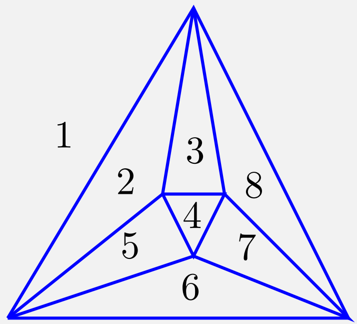

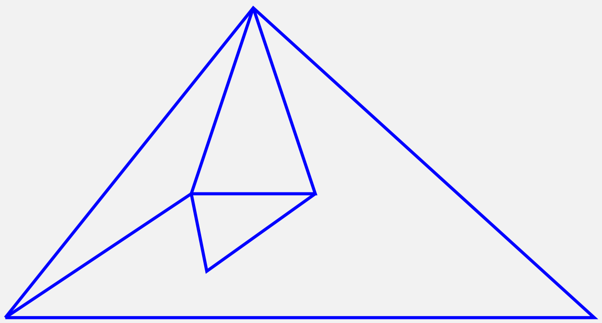

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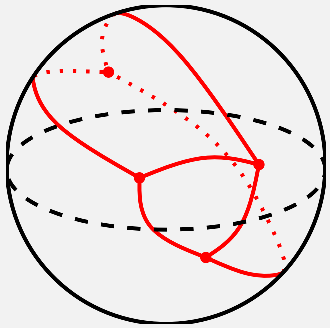

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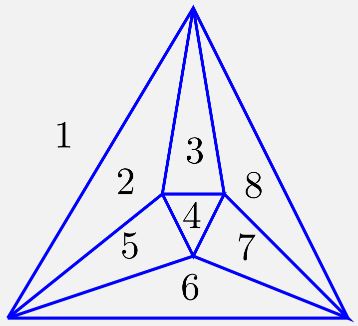

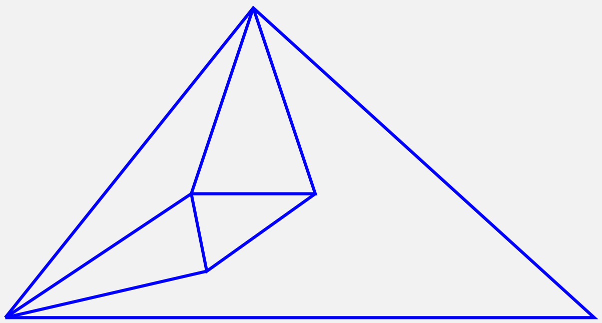

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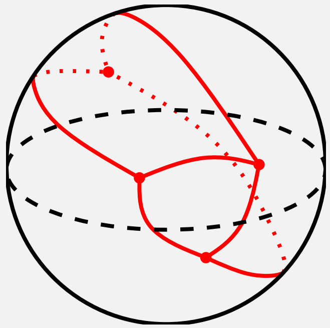

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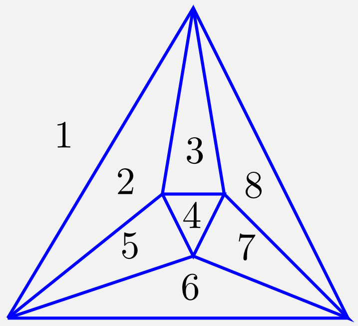

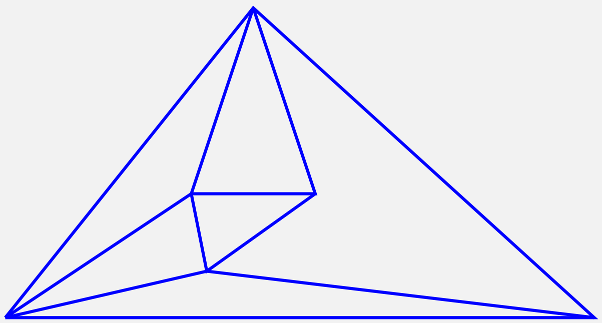

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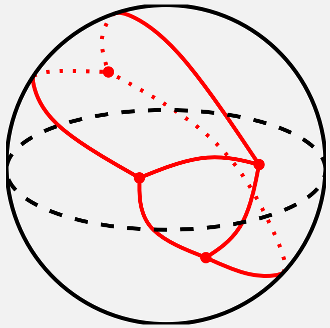

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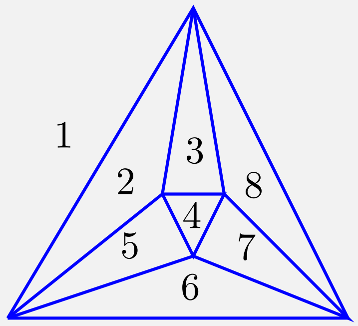

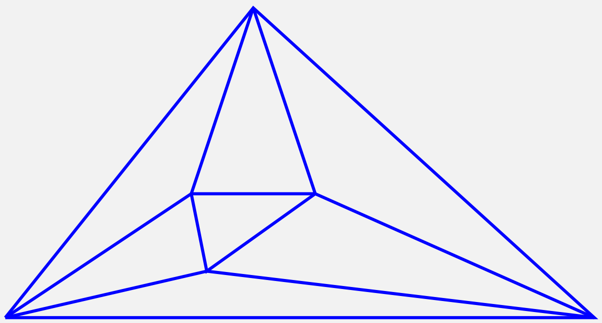

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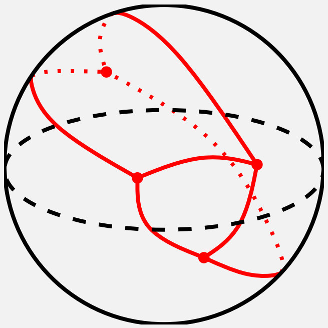

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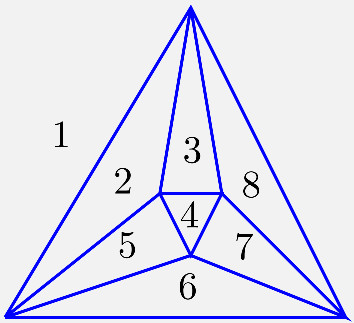

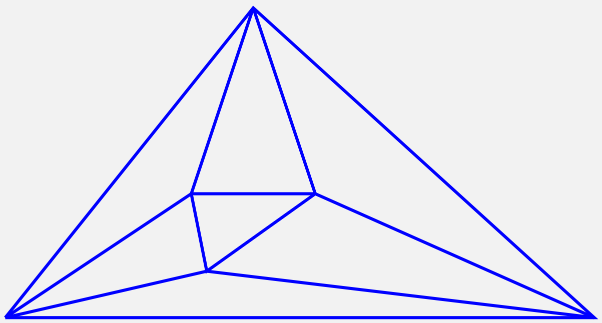

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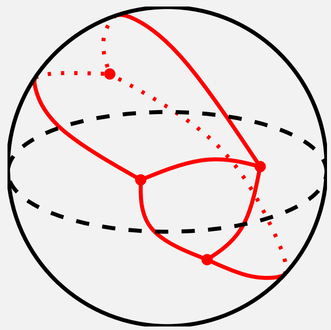

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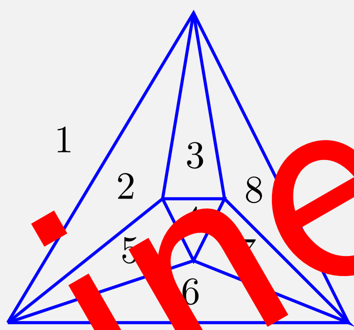

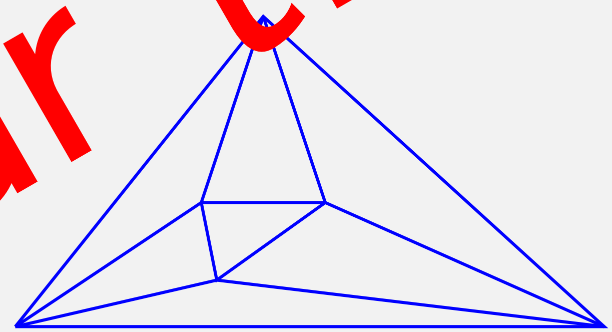

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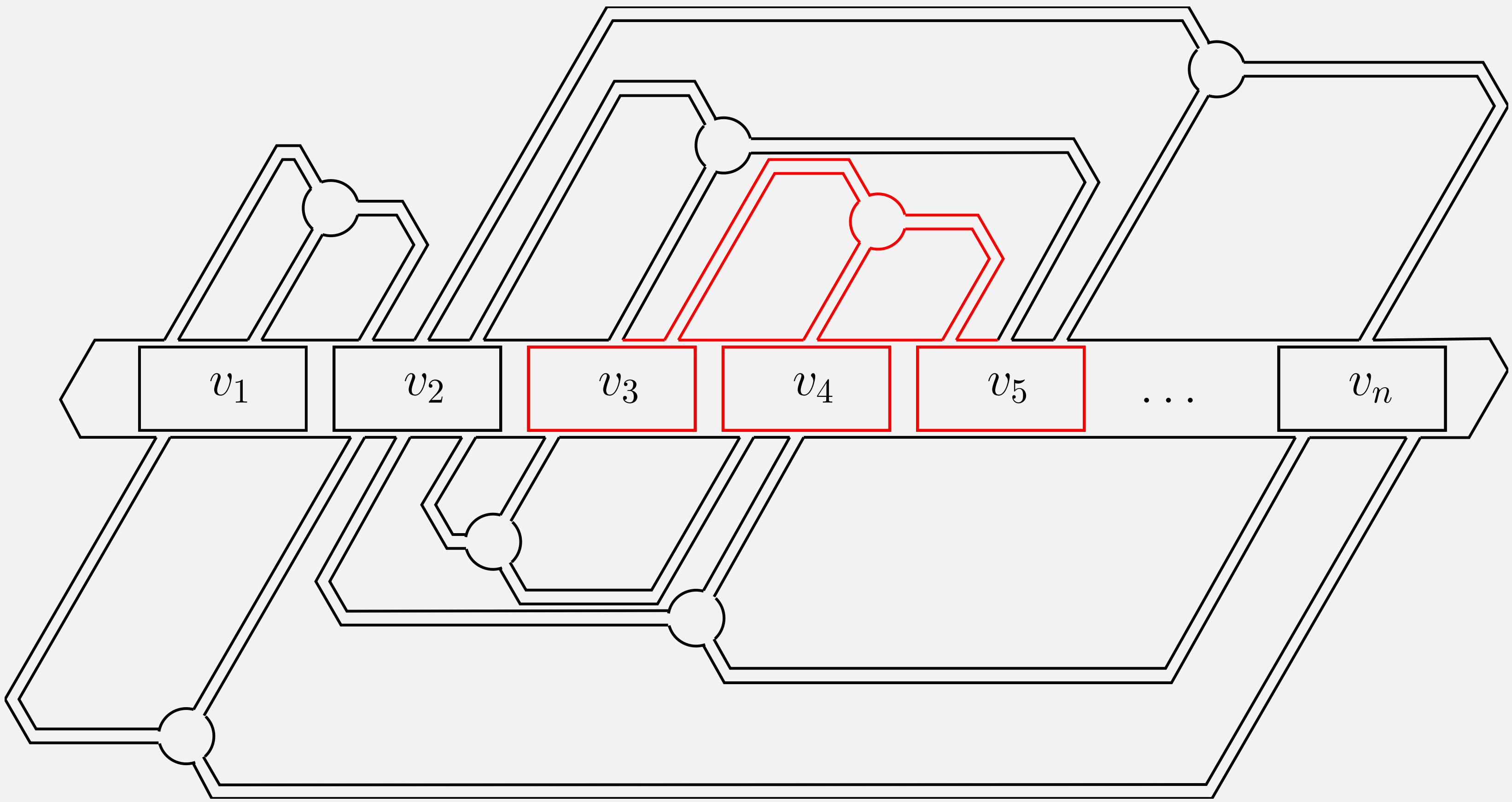
Linear time.

NP-hardness

Th: It is NP-hard for planar 3-connected graphs.

Reduction from planar 3-SAT



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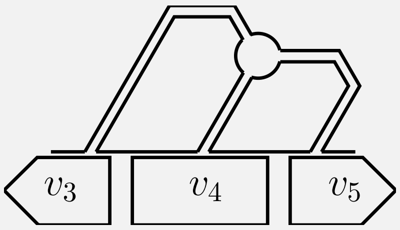

The holder gadget.

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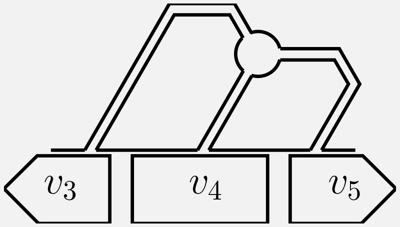

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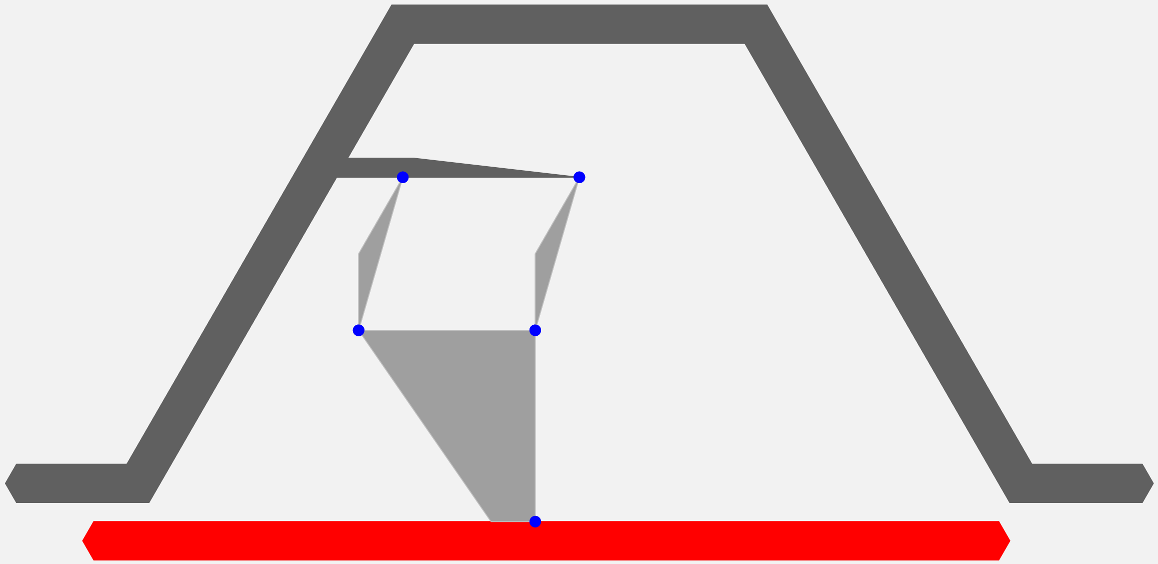

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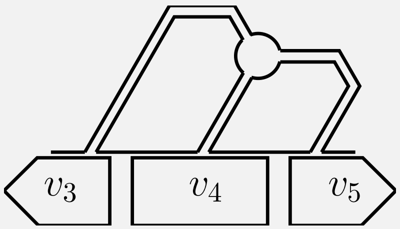

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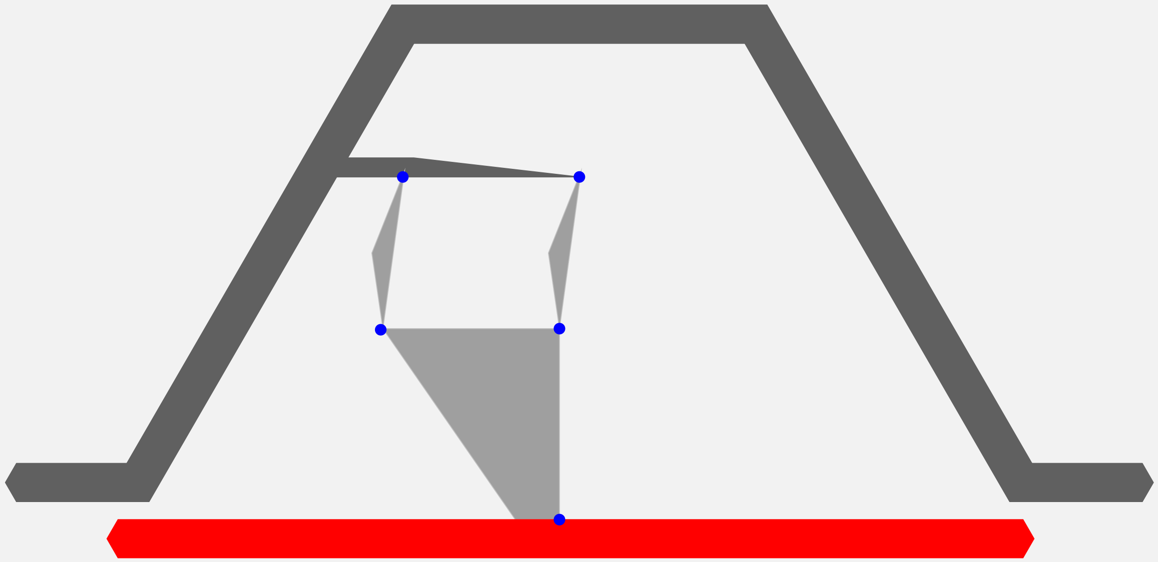

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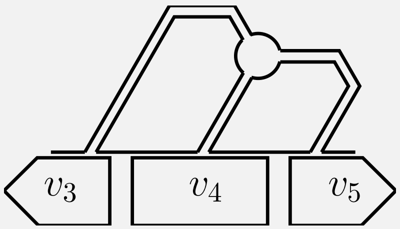

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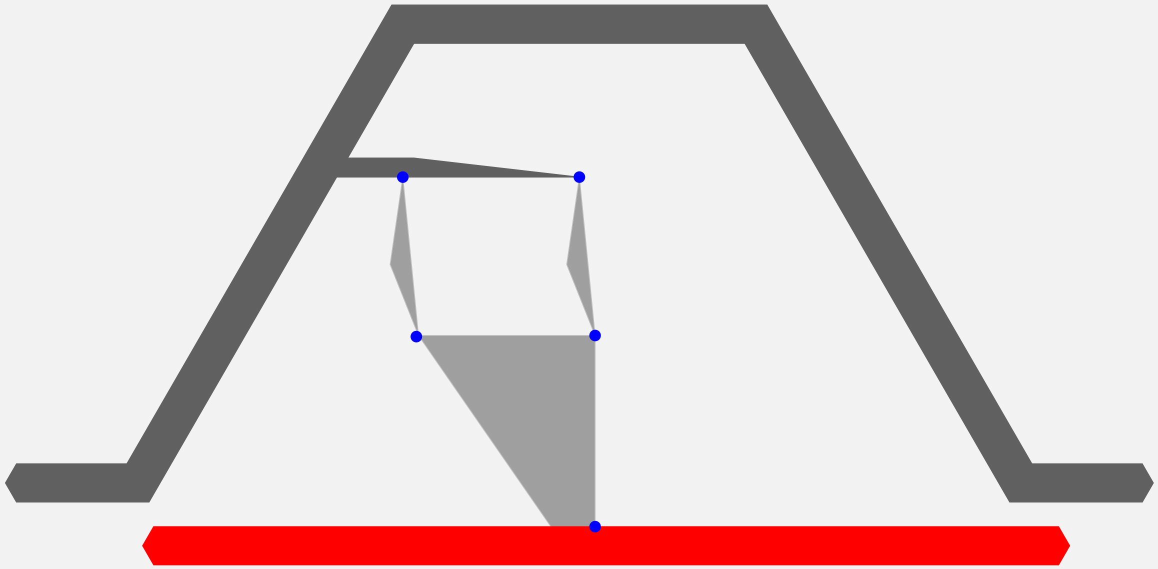

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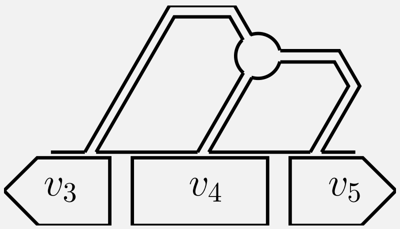

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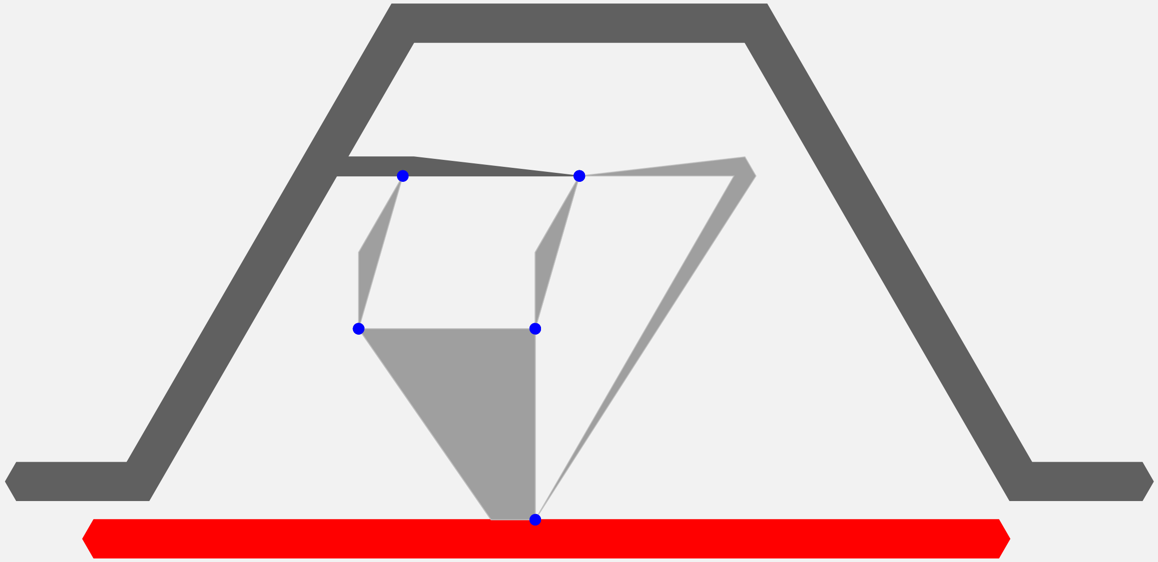

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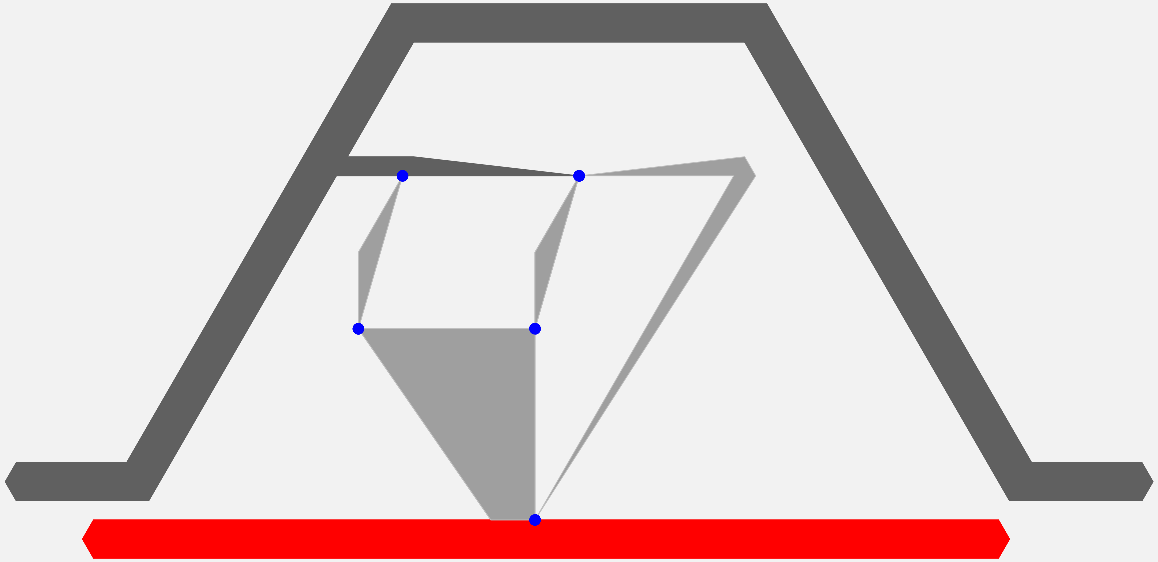

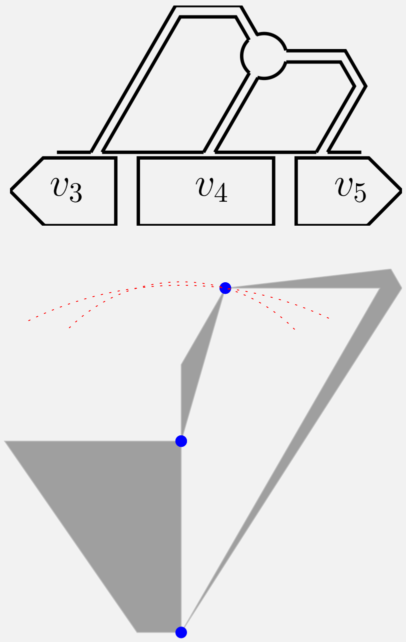

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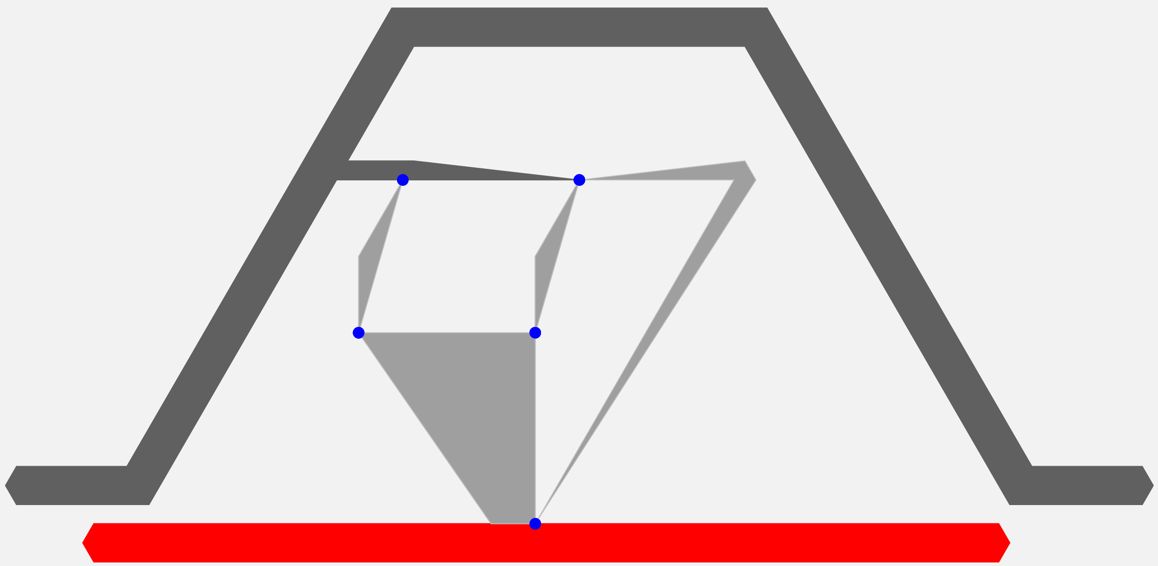

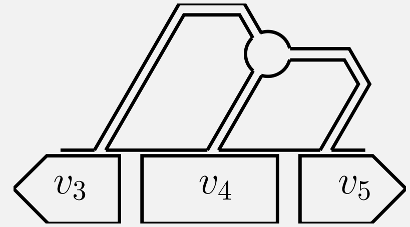

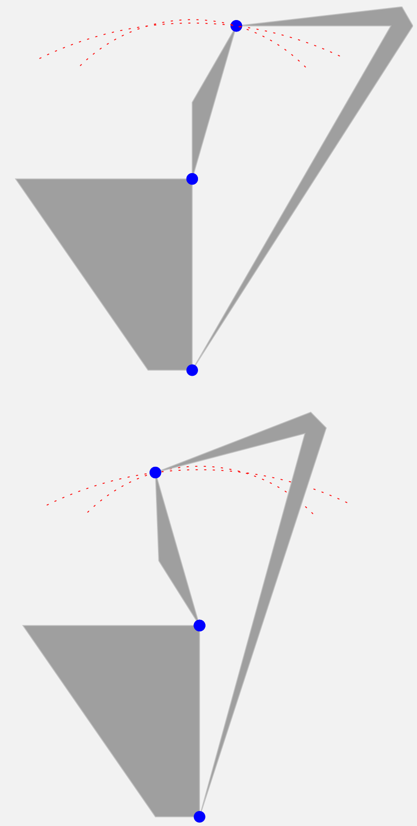

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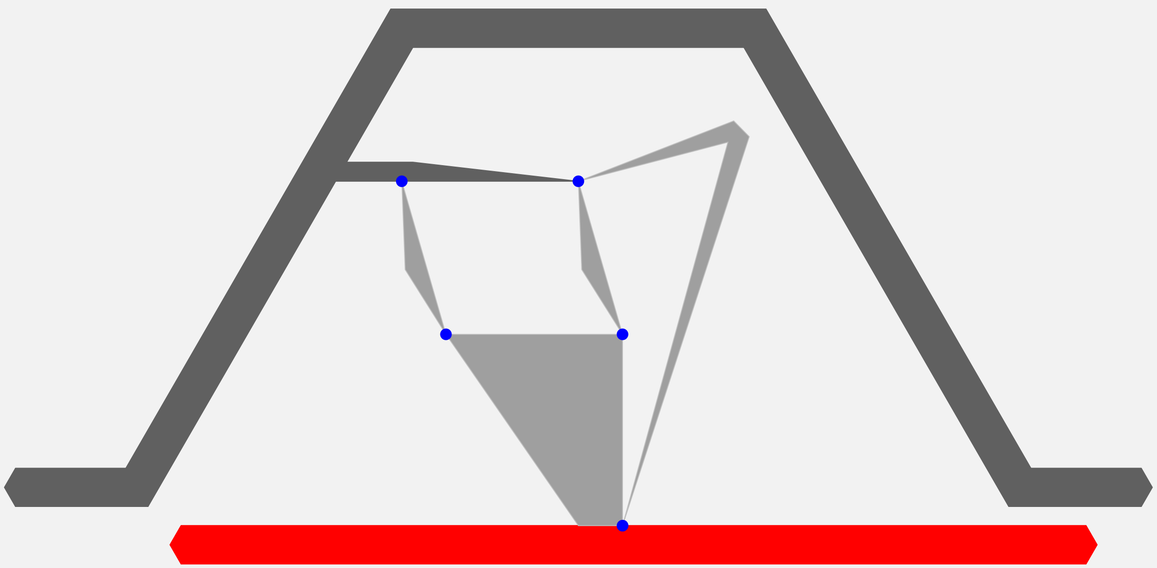

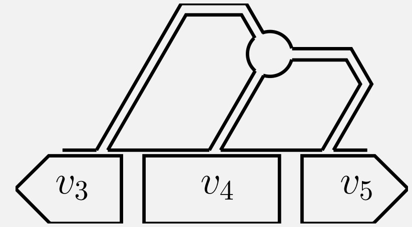

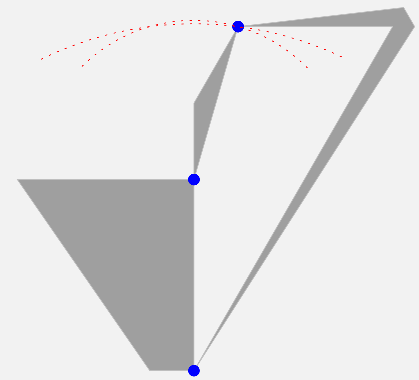

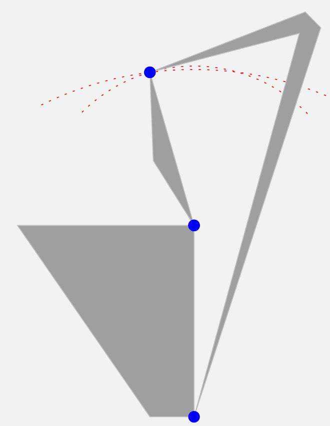

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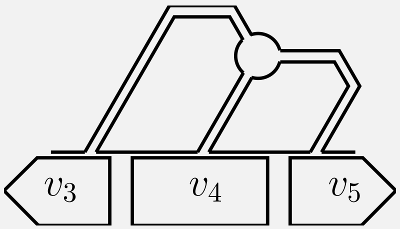

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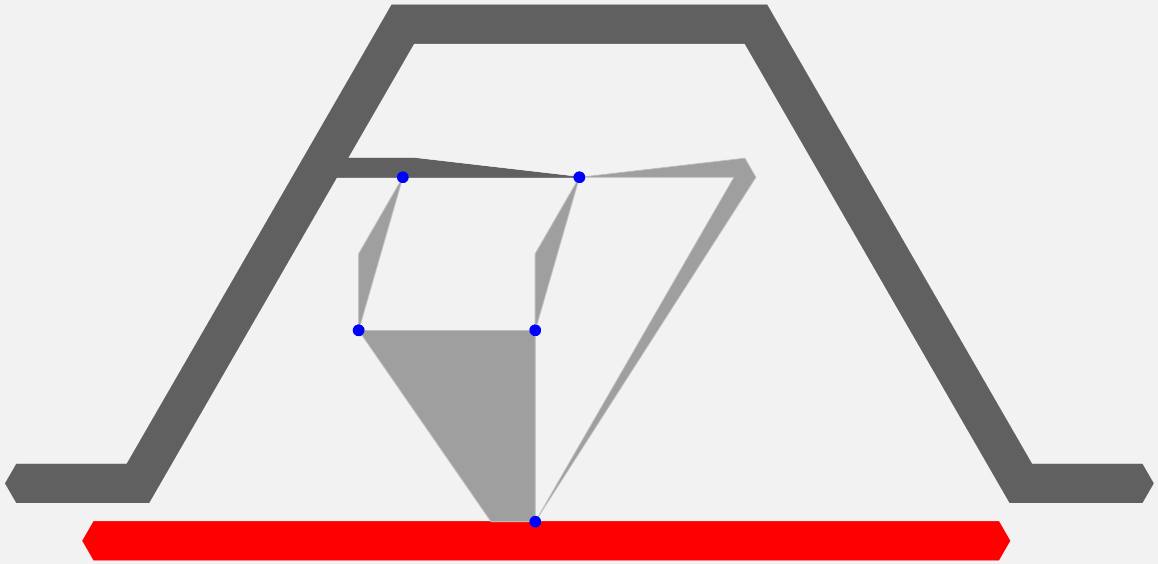

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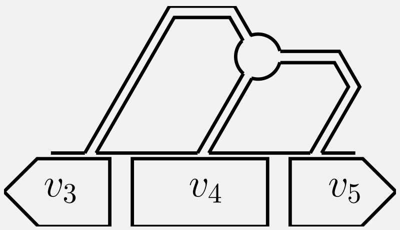

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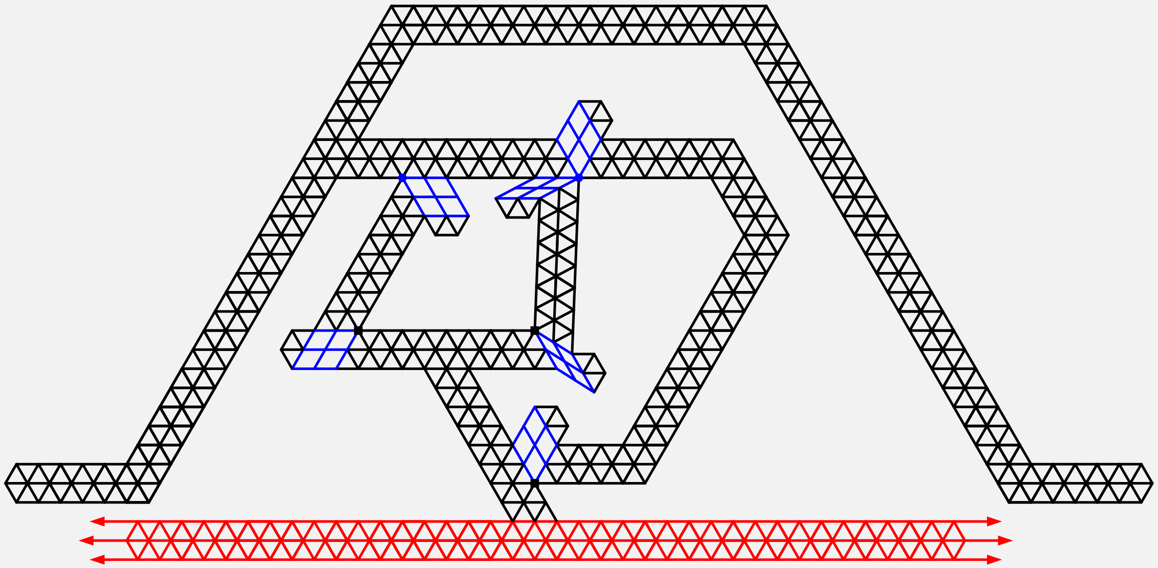

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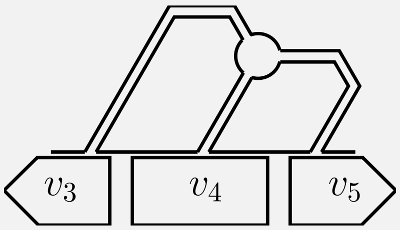

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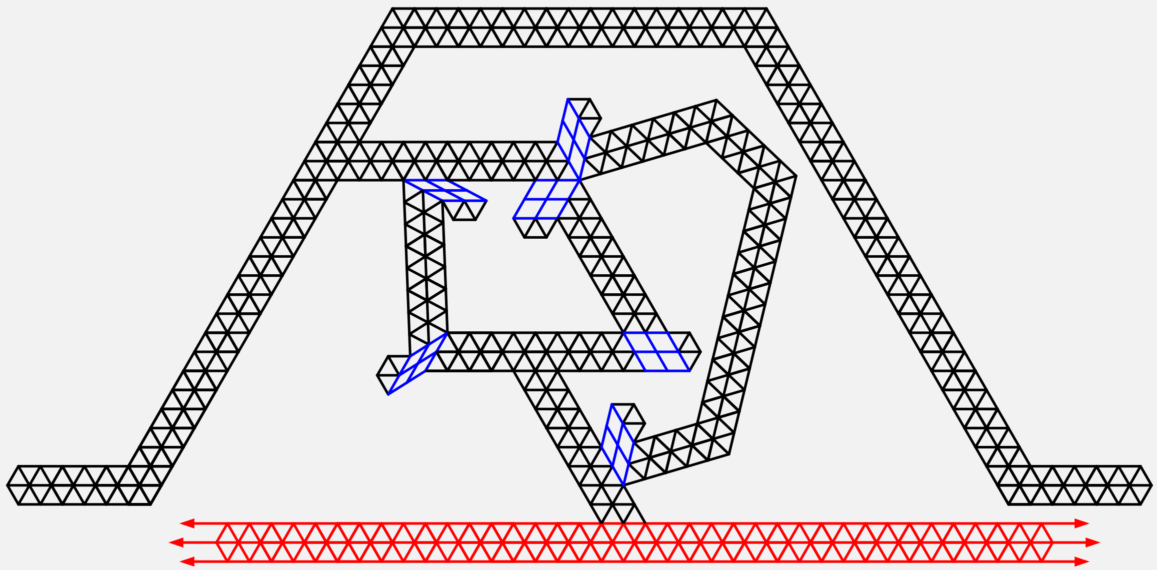

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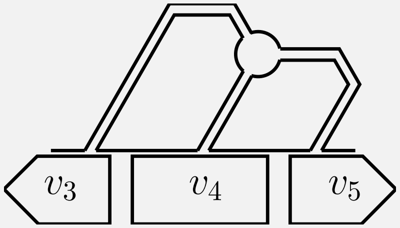

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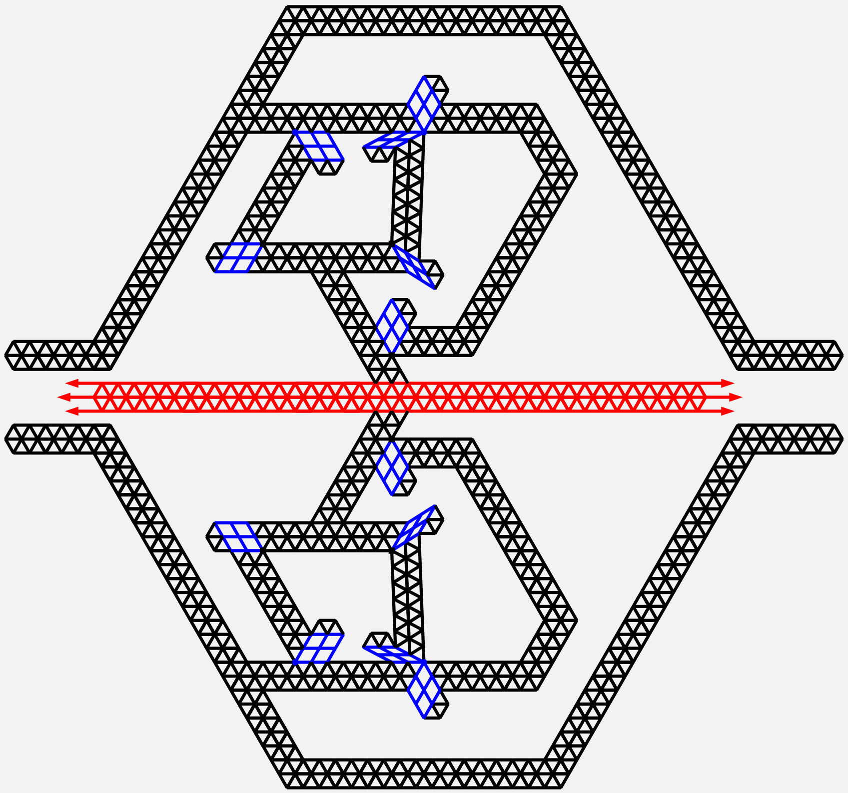

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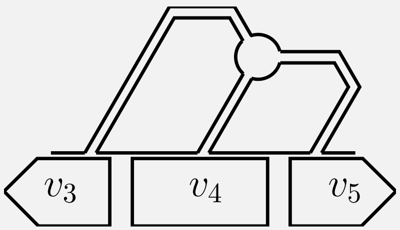

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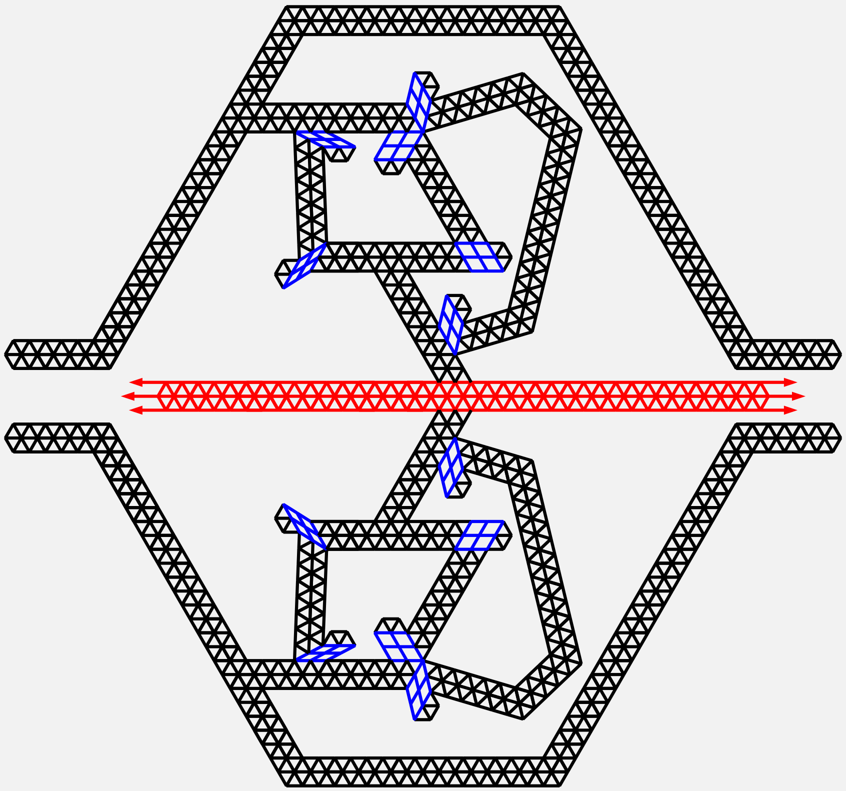

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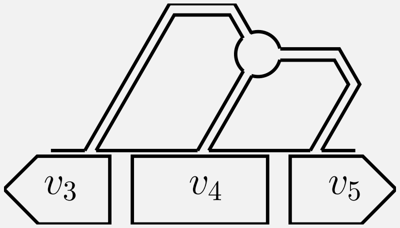
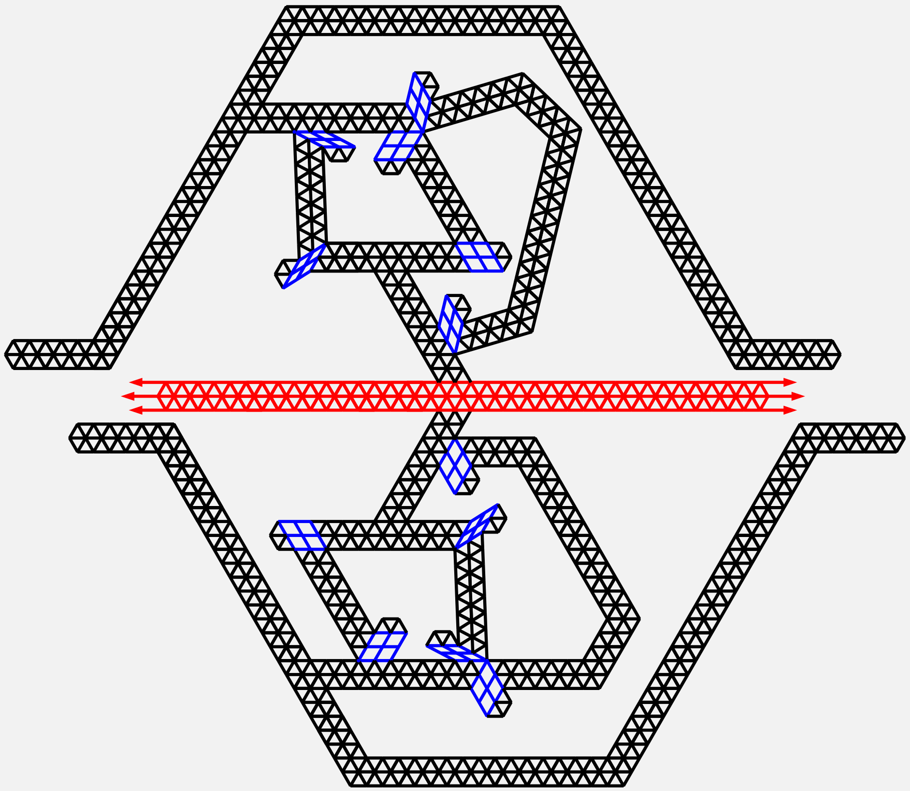
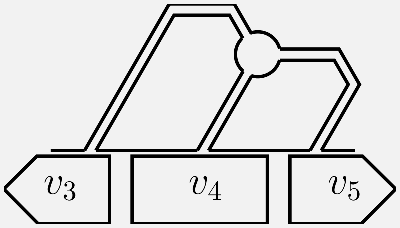


The holder gadget.



NP-hardness



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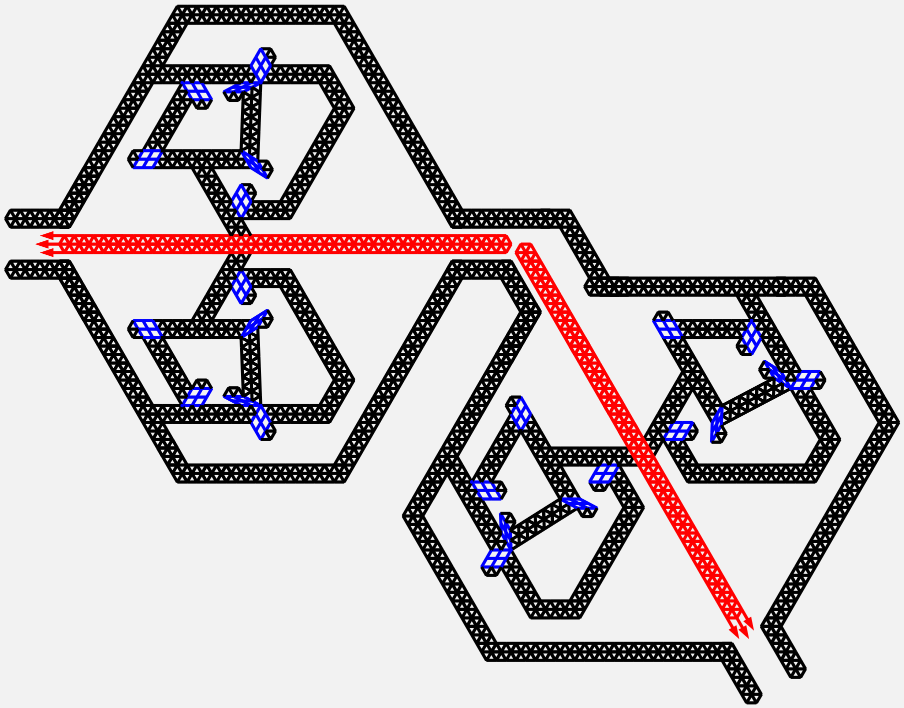
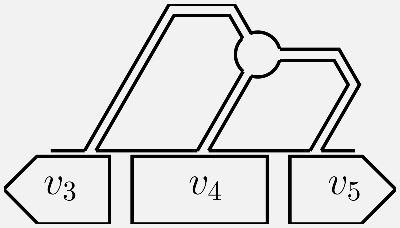
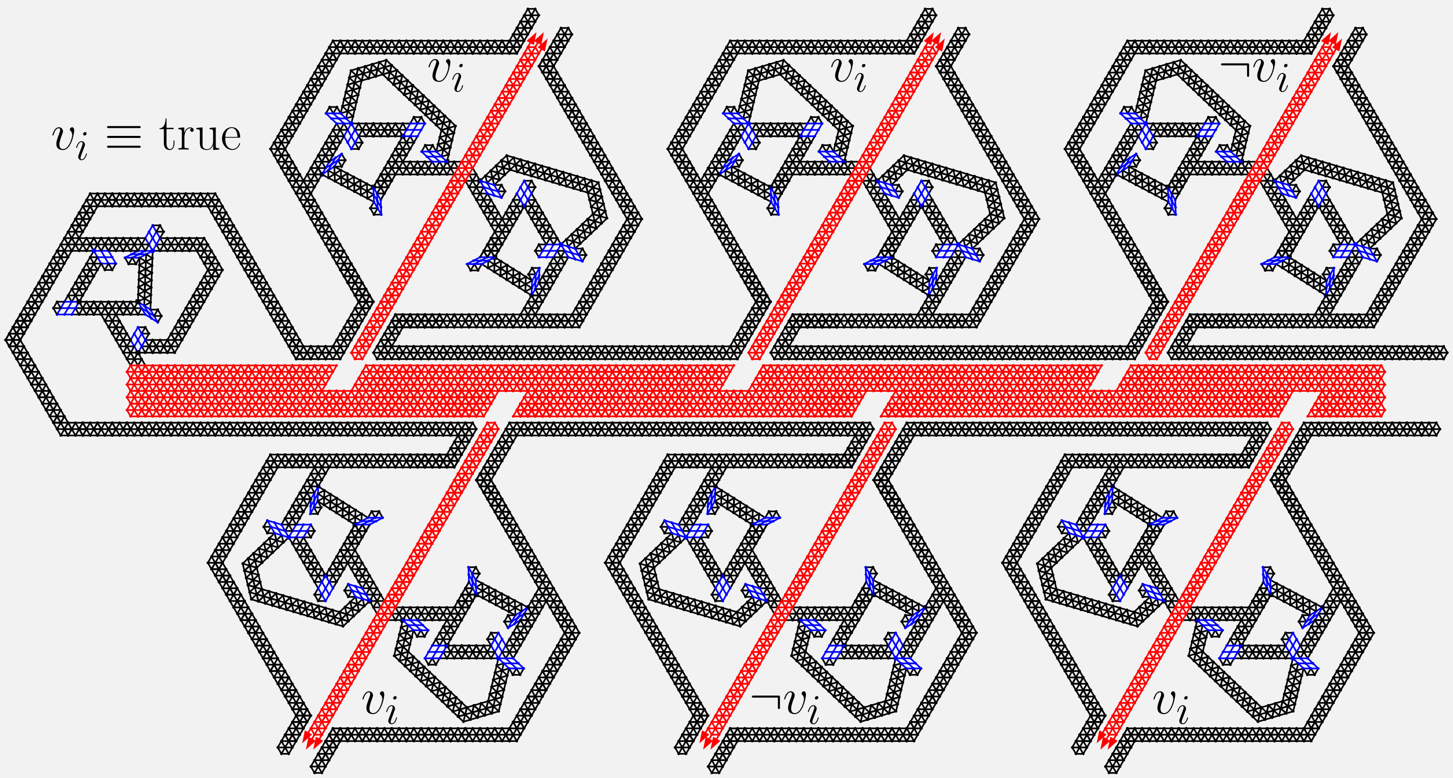


NP-hardness

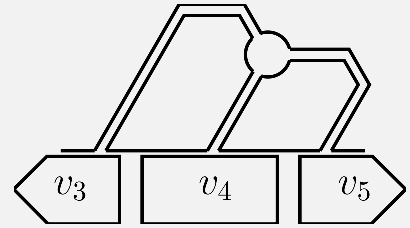
The variable gadget.

(true literal $\equiv$ pushing towards variable)





NP-hardness



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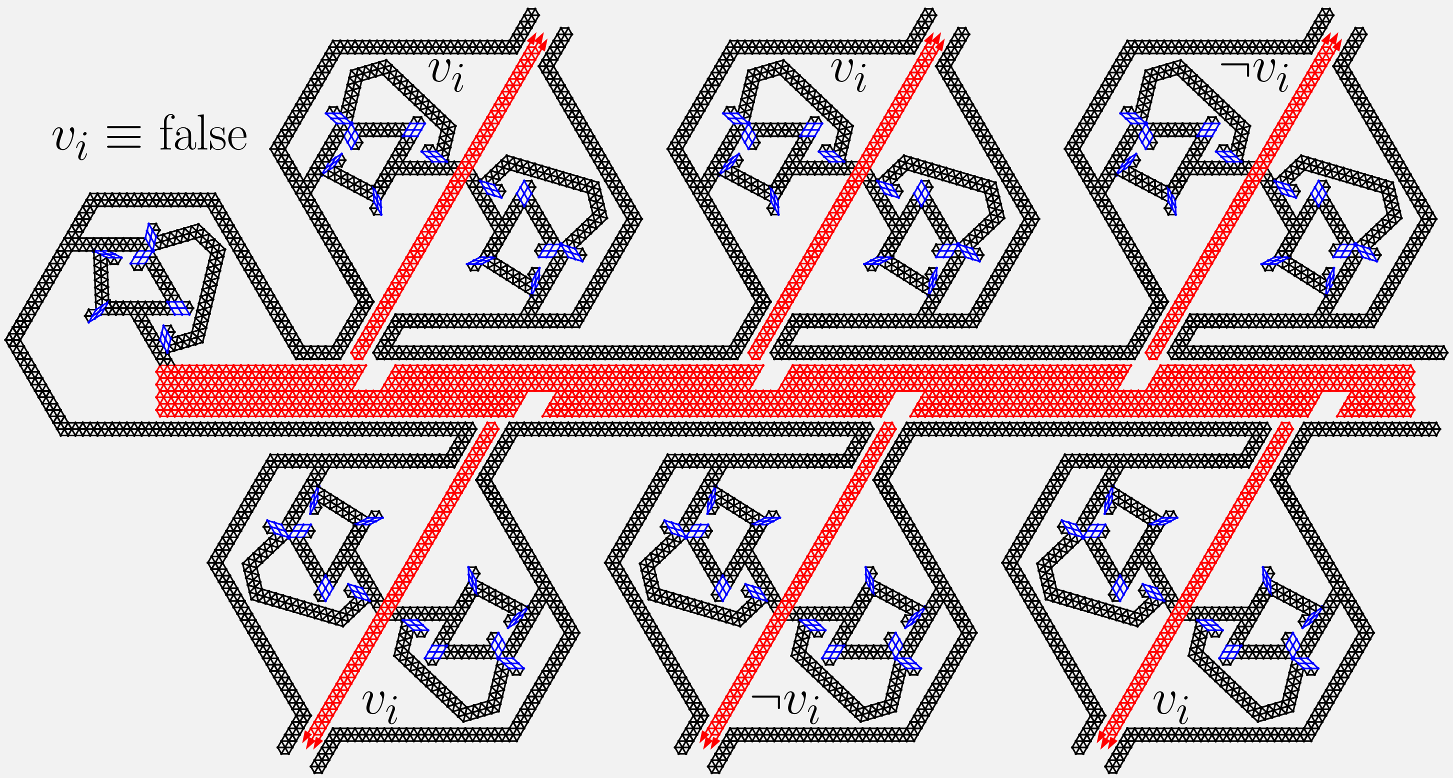


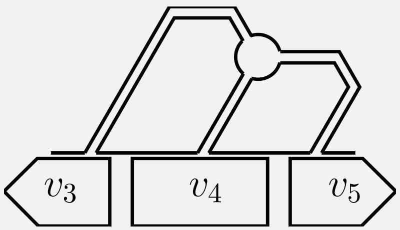
NP-hardness

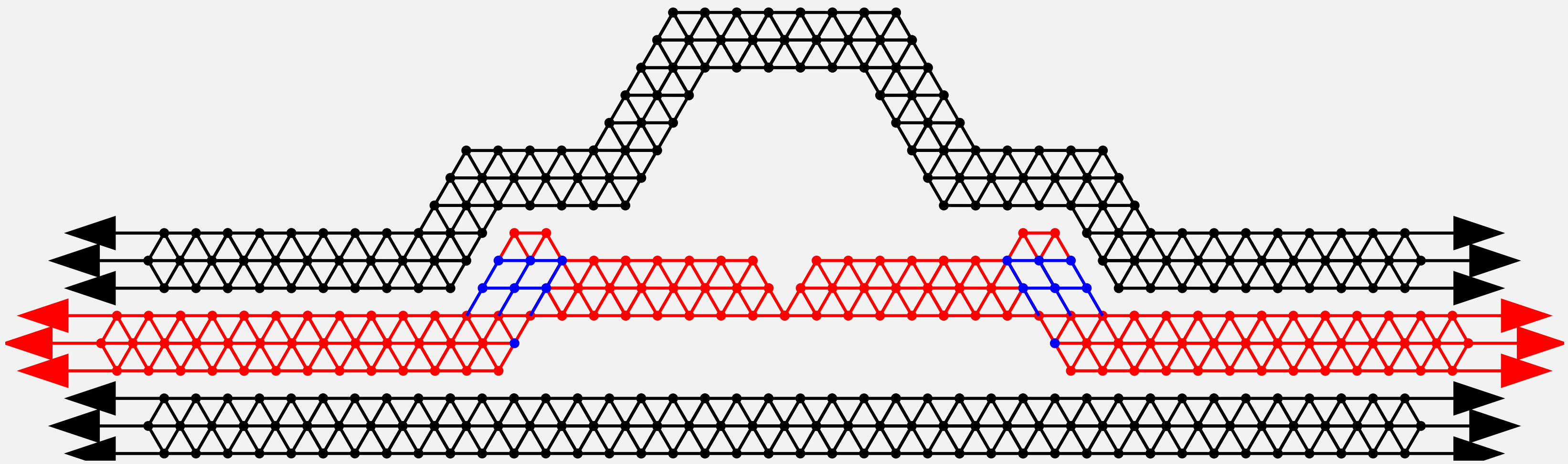
The inverter gadget.

pushing towards variable

⇕

pushing towards clause





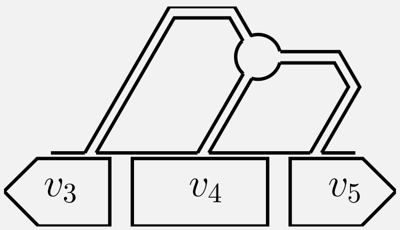
NP-hardness

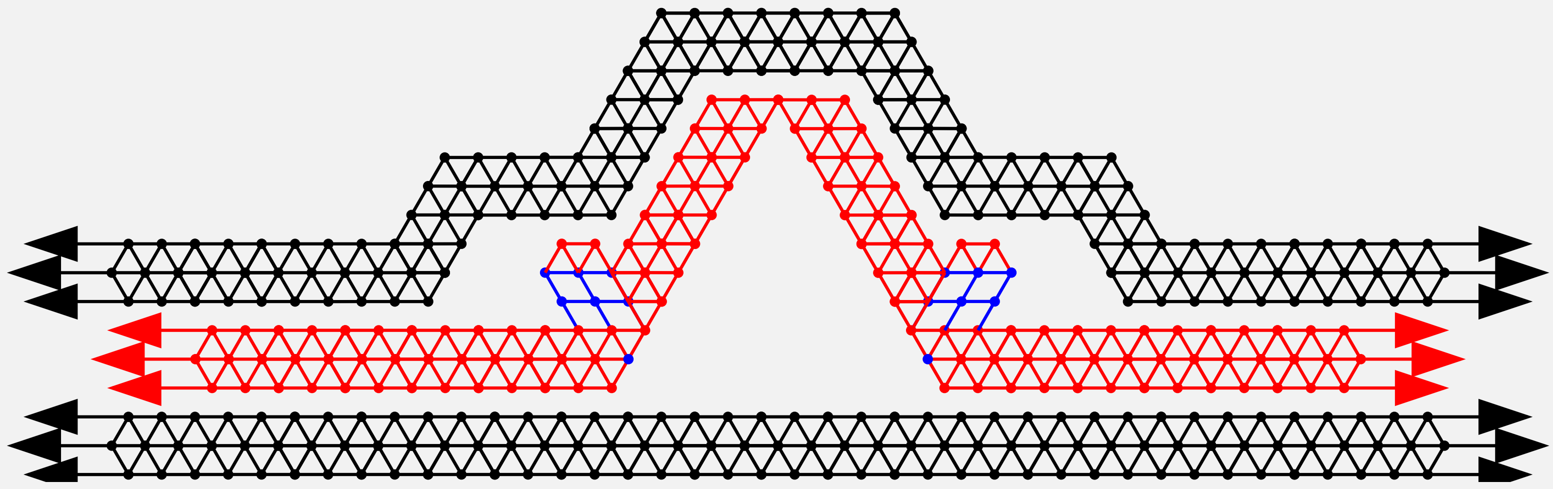
The inverter gadget.

pushing towards variable

⇕

pushing towards clause





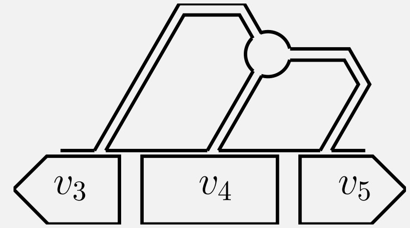
NP-hardness

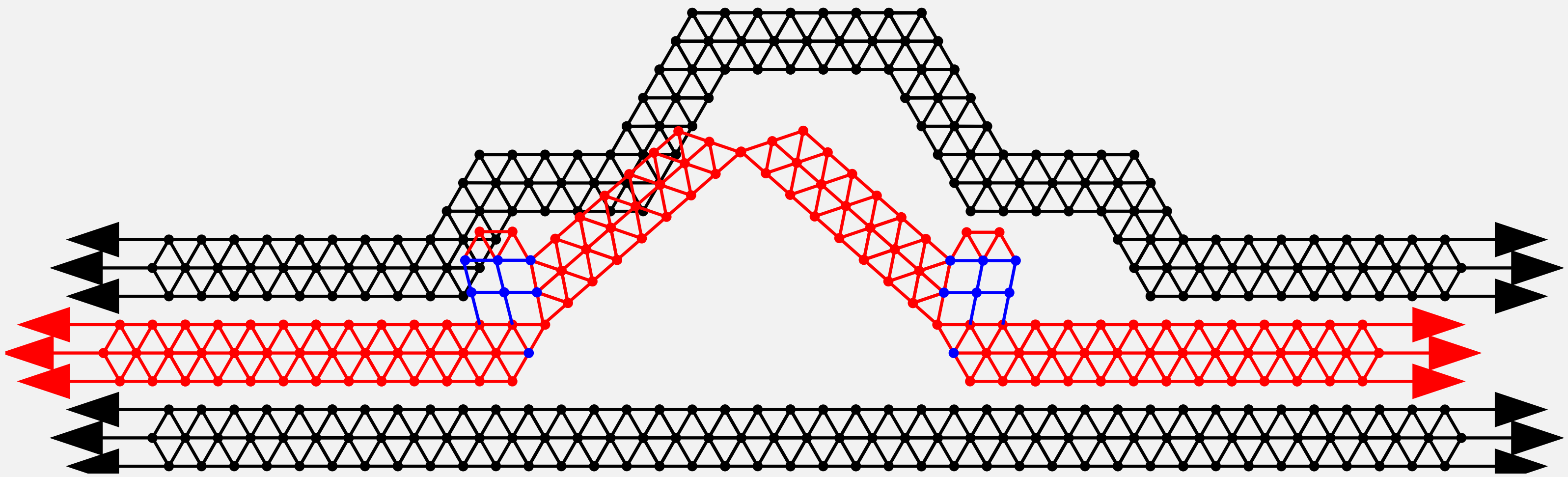
The inverter gadget.

pushing towards variable

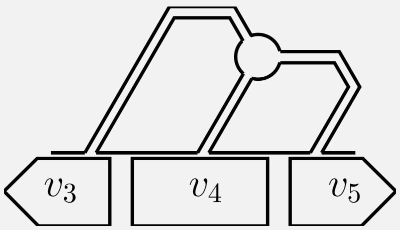
$\Updownarrow$

pushing towards clause

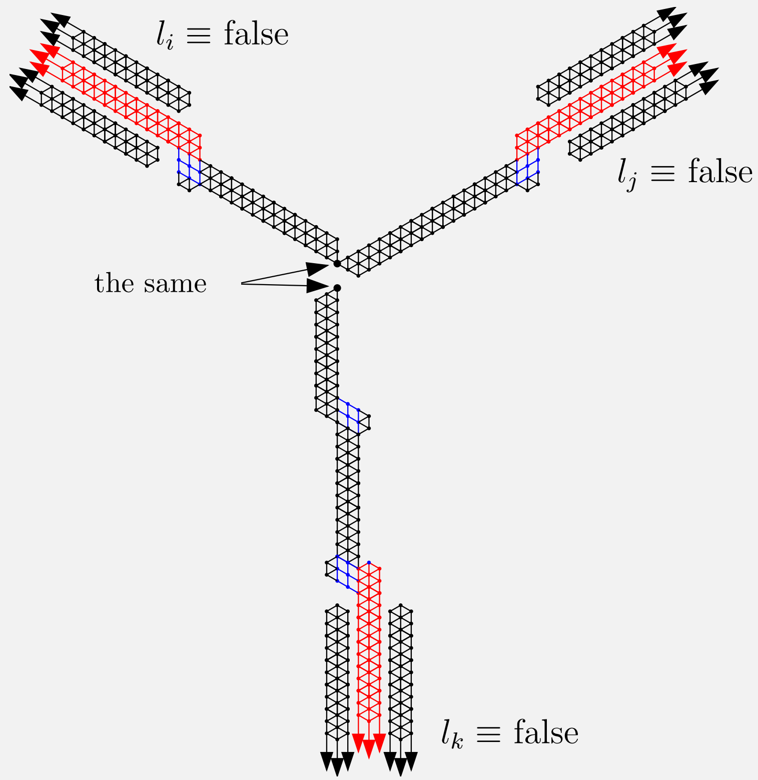




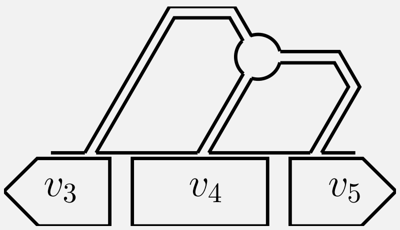
NP-hardness



The clause gadget.
(realizable $\Leftrightarrow$ $\exists$ literal pushing towards clause)



NP-hardness



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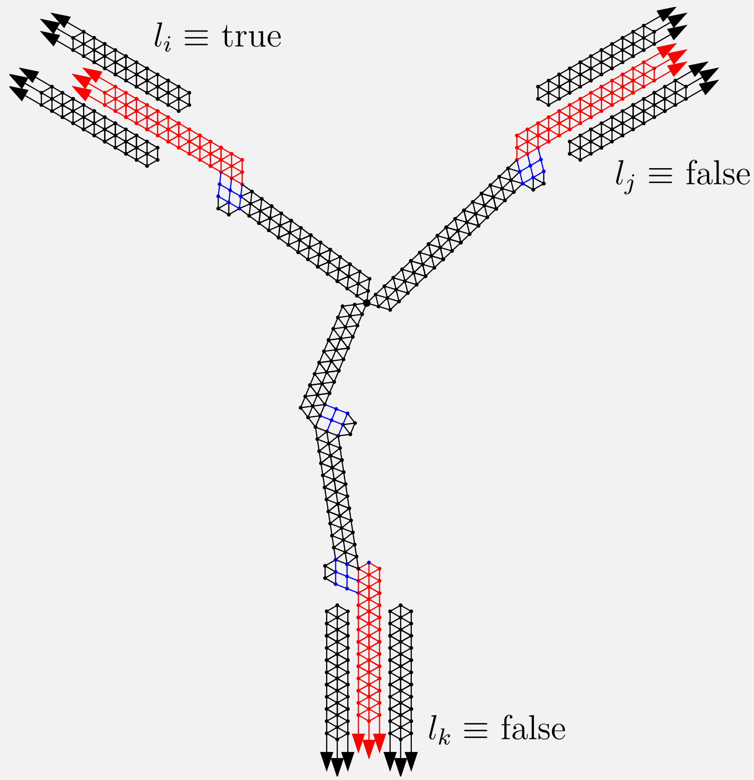
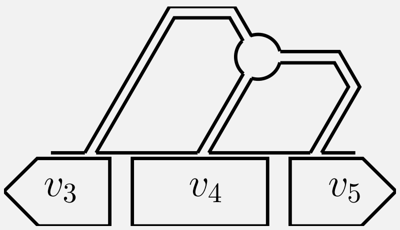


NP-hardness



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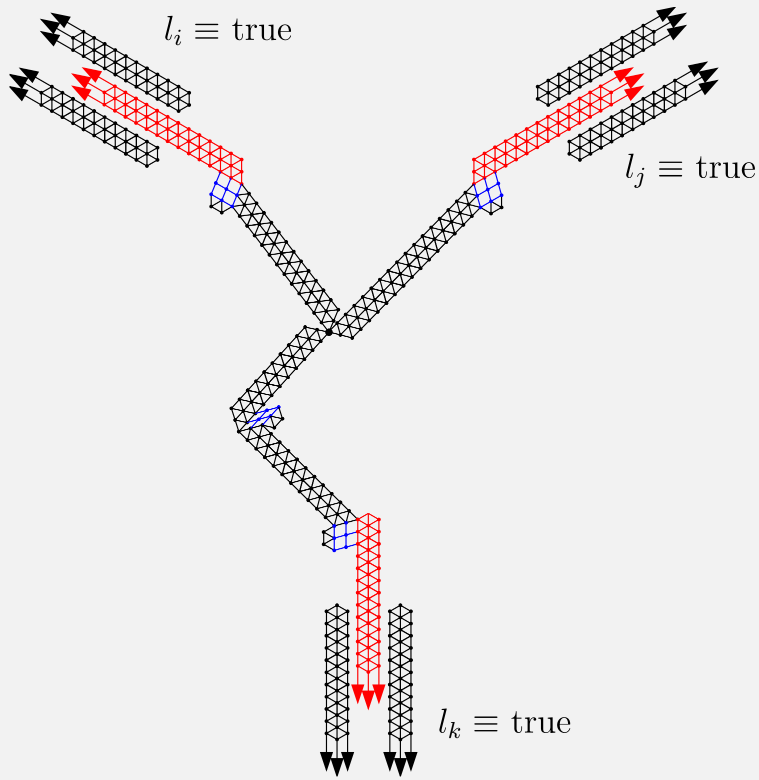
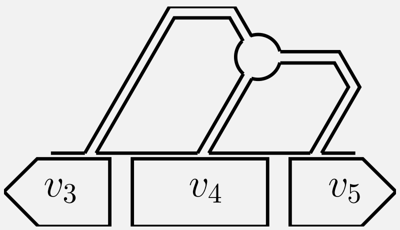
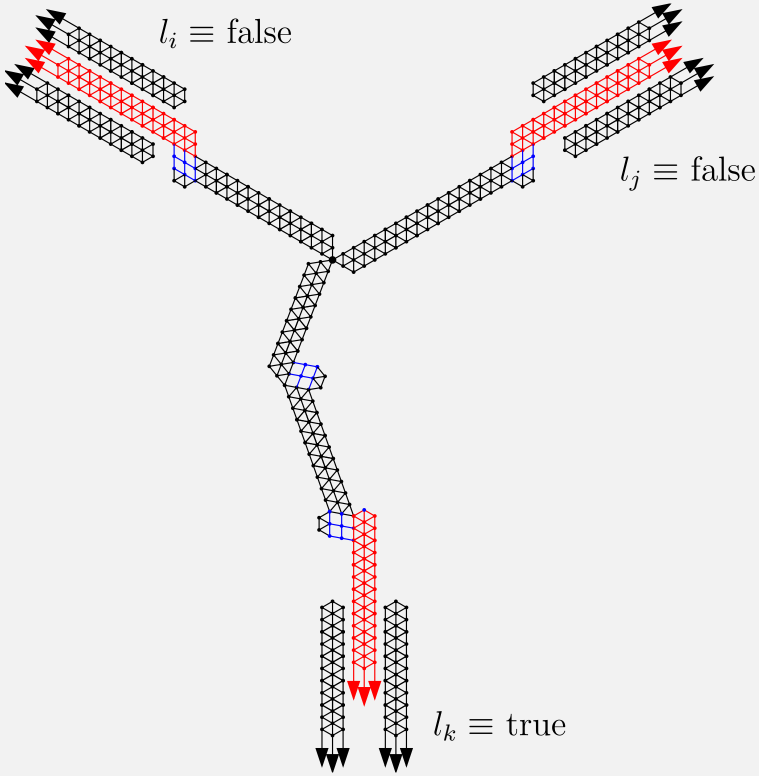
(realizable $\Leftrightarrow$ $\exists$ literal pushing towards clause)



NP-hardness



The clause gadget.
(realizable ⇔ ∃ literal pushing towards clause)



What did I explain?

Planar drawing of graphs with specified edge lengths.

- Triangulated graphs $\Rightarrow$ linear time decision
- $\left\{\begin{array}{l}\text{3-connected (} \longrightarrow \text{ fixed topology)} \\ \text{unit edge lengths,} \\ \text{bounded face degree, and} \\ \text{generically rigid}\end{array}\right\} \Rightarrow$ NP-hard.

Contents

- Problem and motivation
- Our results
- Triangulated graphs
- NP-hardness
- What did I explain?