Finding Shortest Non-Contractible and Non-Separating Cycles for Topologically Embedded Graphs

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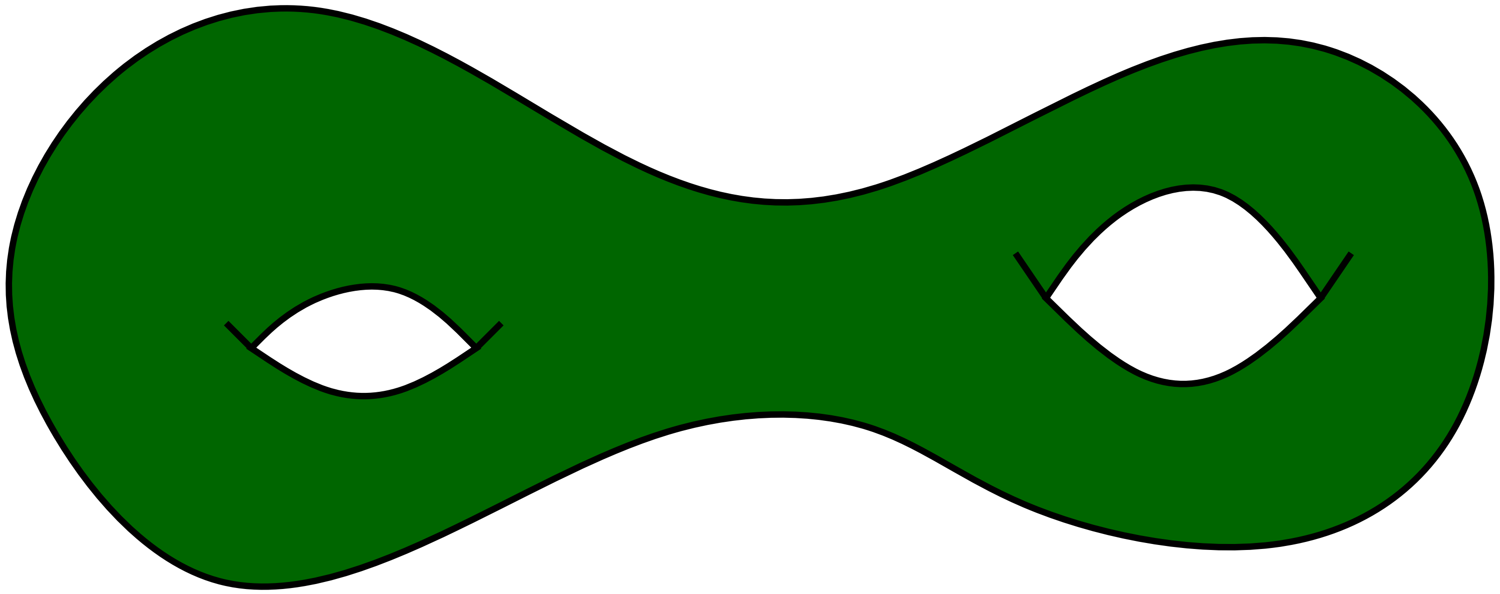
Overview

- surfaces and graphs
- old and new results
- other similar work
- key points for the non-separating case
- key point for the non-contractible case

Surfaces and graphs

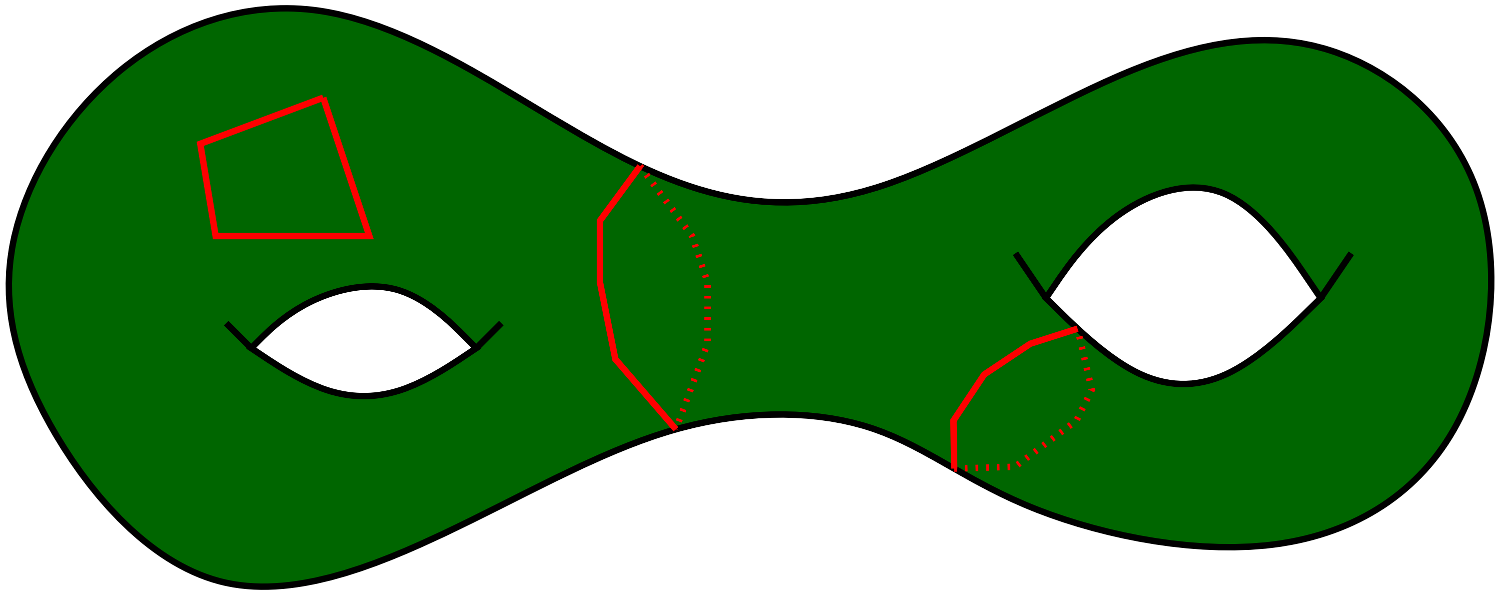
Surface: compact set, locally like the plane



Genus g of Σ: nb of holes $=$ nb of merged torus

Surfaces and graphs

Contractibe, non-contracible, and non-separating loops.



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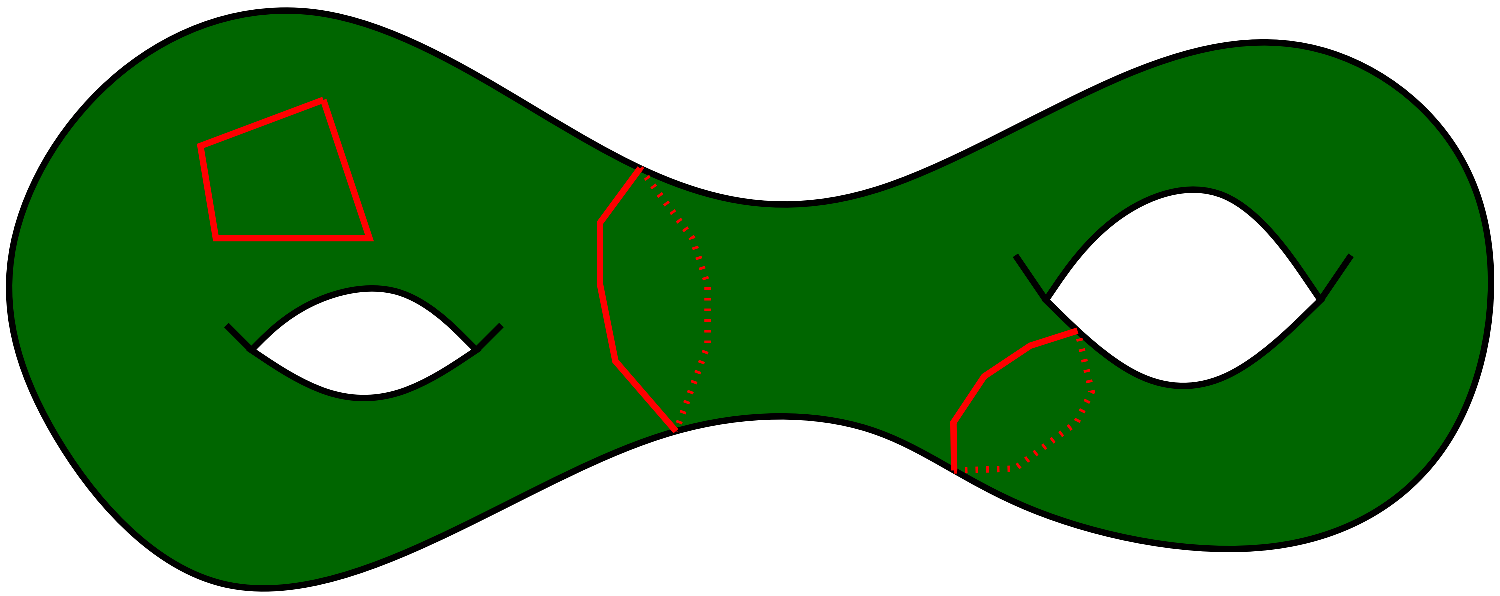
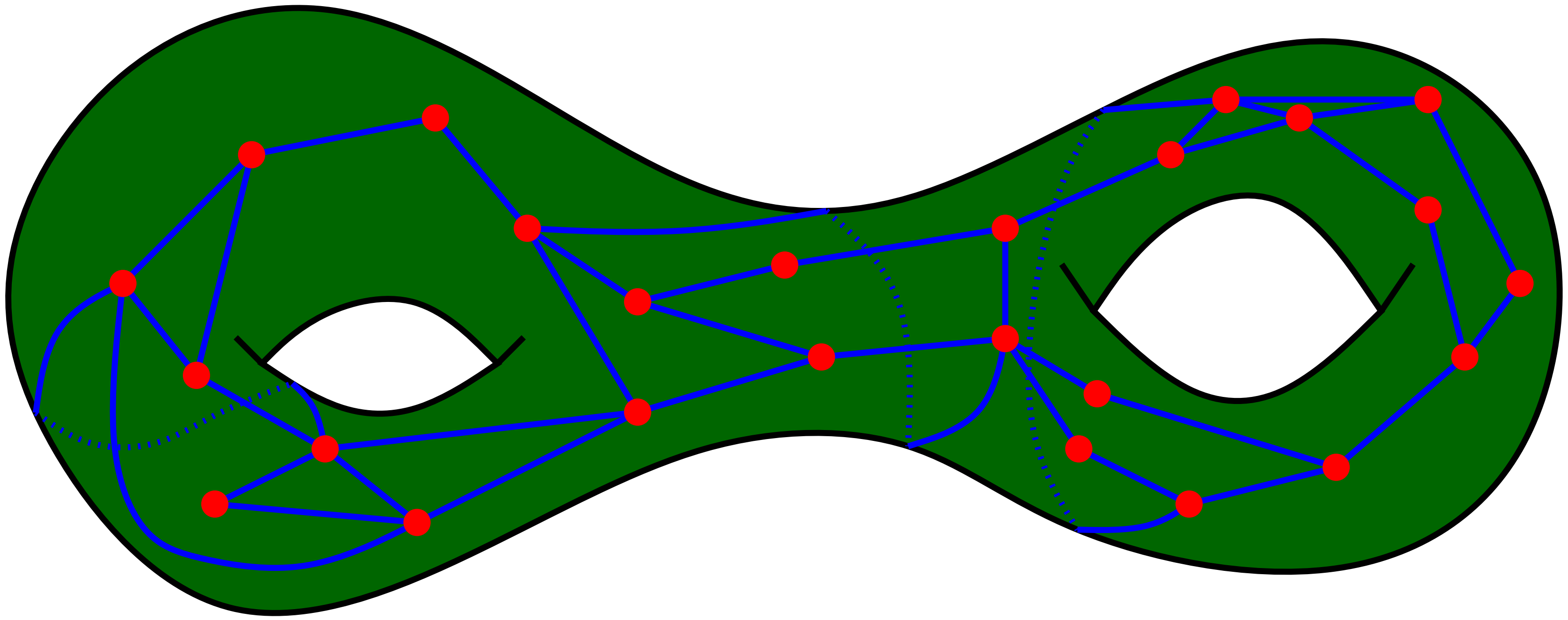


Contractibe $\Rightarrow$separating
Non-separating $\Rightarrow$non-contractible

Contractibe $\Leftrightarrow$ Zero in the homotopy group
Separating $\Leftrightarrow$ Zero in the $\mathbb{Z}_2$-homology group

Surfaces and graphs

(Weighted) graph G on Σ:



Cycles/loops in G are curves in Σ.

Surfaces and graphs

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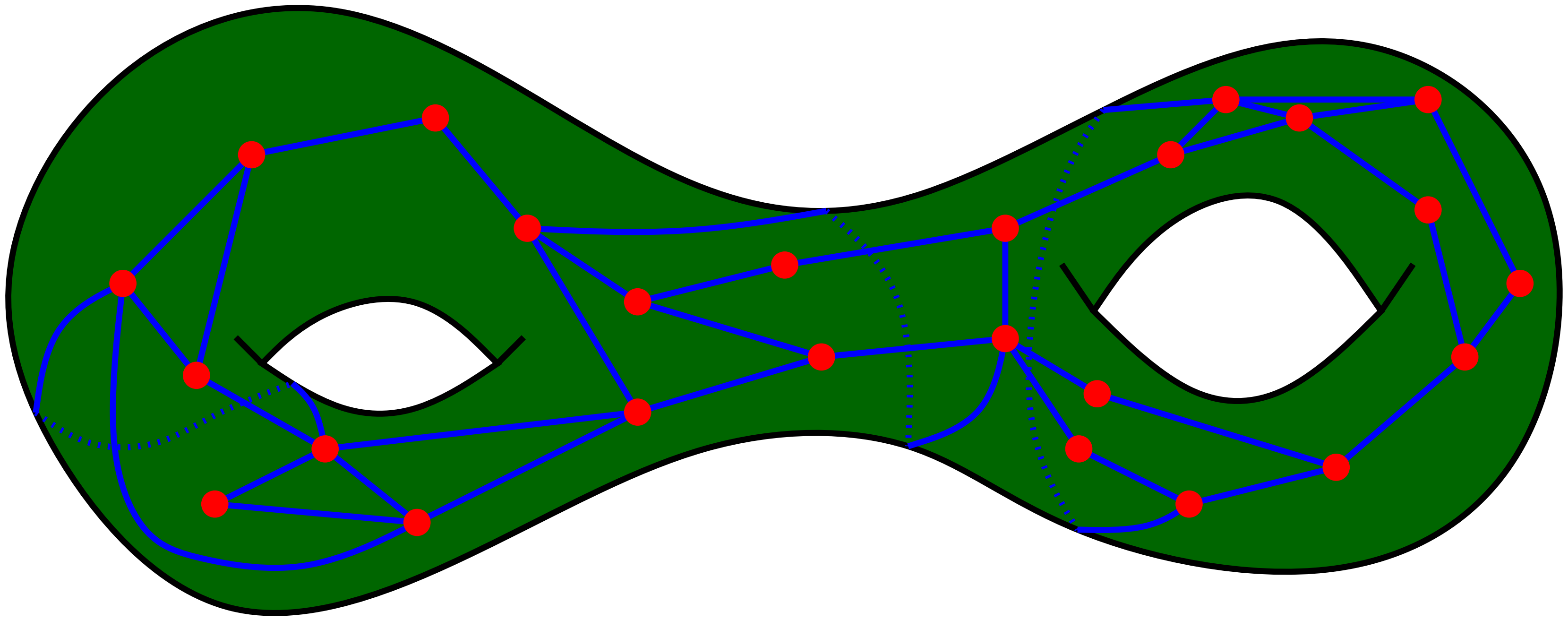

Cycles/loops in G are curves in Σ.

Problem: Find shortest non-contractible cycle.

Problem: Find shortest non-separating cycle.

Old and new results

G a graph with V vertices in a surface of genus g

	Older results
Shortest non-contractible cycle	$O^*(V(V+g)^2)$ [Thomassen]
Shortest non-separating cycle	$O^*(V(V+g))$ [Erickson, Har-Peled]

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Better if $g = O(1)$

Better if $g = o(V^{1/3})$

Old and new results

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	Older results	New result
Shortest non-contractible cycle	$O^*(V(V+g)^2)$ [Thomassen]	~~$O(g^{O(g)}V^{3/2})$~~ $O(g^{O(g)}V^{4/3})$
Shortest non-separating cycle	$O^*(V(V+g))$ [Erickson, Har-Peled]	$O^*(g^{3/2}V^{3/2} + g^{5/2}V^{1/2})$

SODA'06

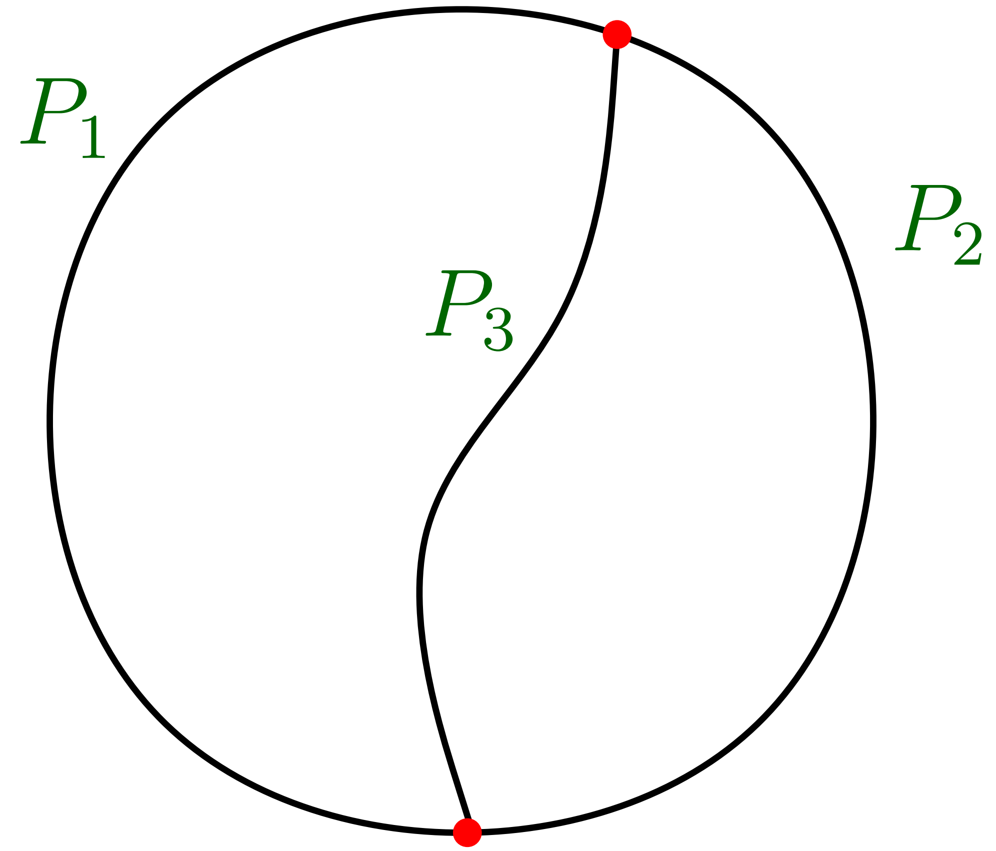
Why non-contractible/non-separating?

3-path property

$P_1 + P_2$ non-separating

$\Downarrow$

$P_1 + P_3$ or $P_2 + P_3$ non-separating



Are there polynomial time algorithms for shortest separating?
More difficult for shortest contractible.

Other similar work

- Erickson and Har-Peled (2004,2005)
  find minimum-length cut subgraph C s.t. $\Sigma \setminus C$ planar.

- Colin de Verdière and Lazarus (2002, 2004)
  find shortest loop/cycle in a homotopy class.

- Eppstein (2003)
  tree-cotree decomposition.

- Erickson and Whittlesey (2005)
  find shortest system of loops with given basepoint.

- Colin de Verdière and Erickson (SODA'06)
  find shortest loop/cycle/path in a homotopy class.

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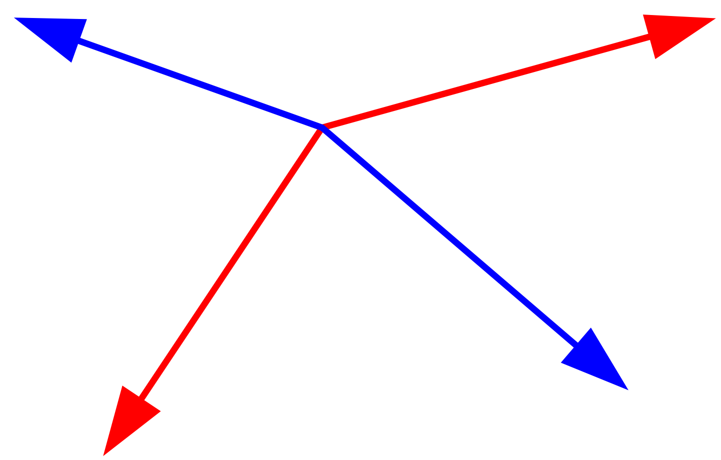
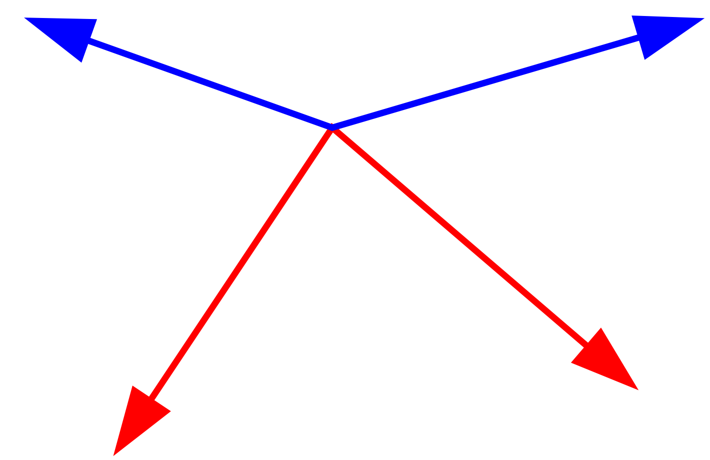
- ~~surfaces and graphs~~
- ~~old and new results~~
- ~~other similar work~~
- key points for the non-separating case
- key point for the non-contractible case

Key points for the non-separating cycle

Crossings



crossing



intersection without crossing

Key points for the non-separating cycle

$\mathcal{C} = \{C_1, \ldots, C_{\Theta(g)}\}$ system of fundamental loops.

$C_1, \ldots, C_{\Theta(g)}$ through a common vertex.
Surface cut along $C_1, \ldots, C_{\Theta(g)}$ is a disk.

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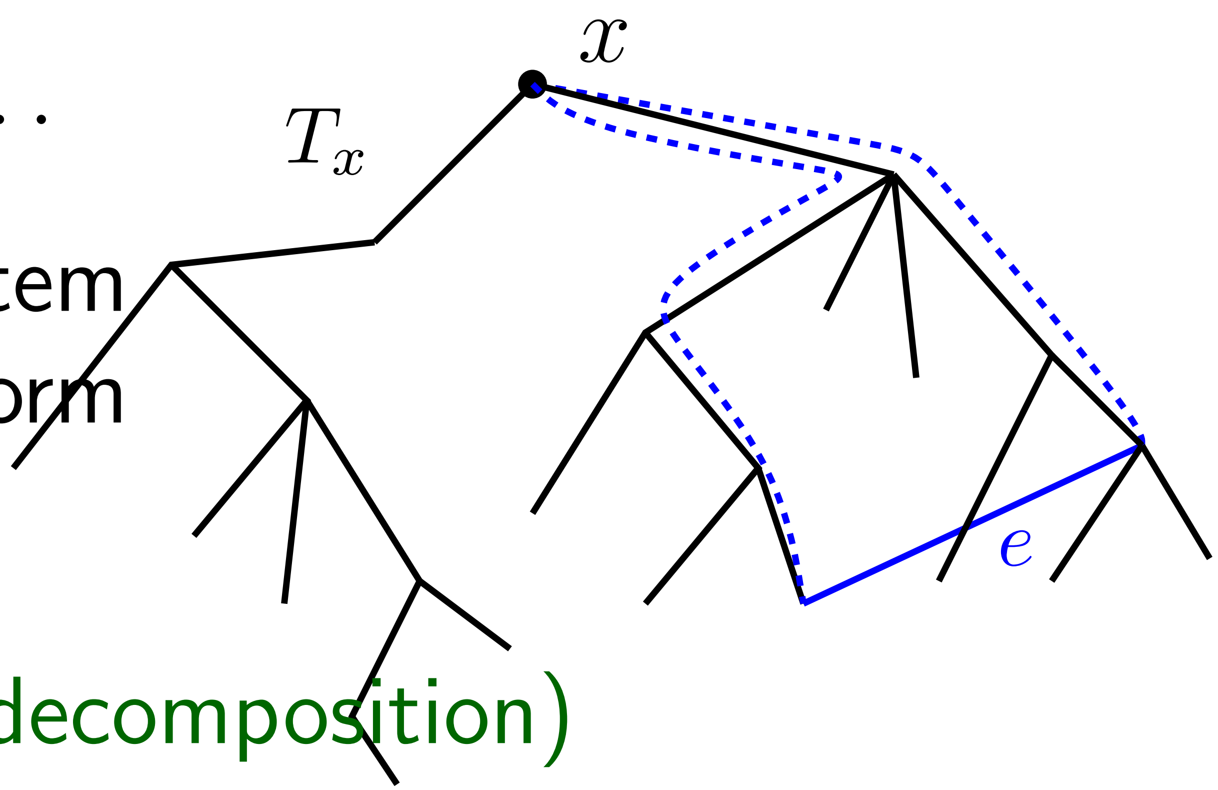
Surface cut along $C_1, \ldots, C_{\Theta(g)}$ is a disk.

Fix $x \in V(G)$ and construct from-x-shortest-path tree T_x.

For edge $e \notin T_x$, $loop(T_x, e)$ is ...



Theorem: There is always a system of fundamental loops of the form $loop(T_x, e_1), \ldots, loop(T_x, e_{\Theta(g)})$.

Easy to compute it (tree-cotree decomposition)

Key points for the non-separating cycle

$loop(T_x, e_1), \ldots, loop(T_x, e_{\Theta(g)})$ system of fund. loops.

Lem: $\exists$ shortest non-separ cycle crossing ≤ 2 each $loop(T_x, e_i)$.

Lem: each non-sep cycle crosses some $loop(T_x, e_i)$ odd times.

$\Rightarrow$ $\exists$ shortest non-sep cycle C^* and $loop(T_x, e_i)$ holding $cr(C^*, loop(T_x, e_i)) = 1$

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Algorithm:

for each cycle $C_i = loop(T_x, e_i)$ in the system
find a shortest cycle crossing C_i exactly once;
report the shortest one

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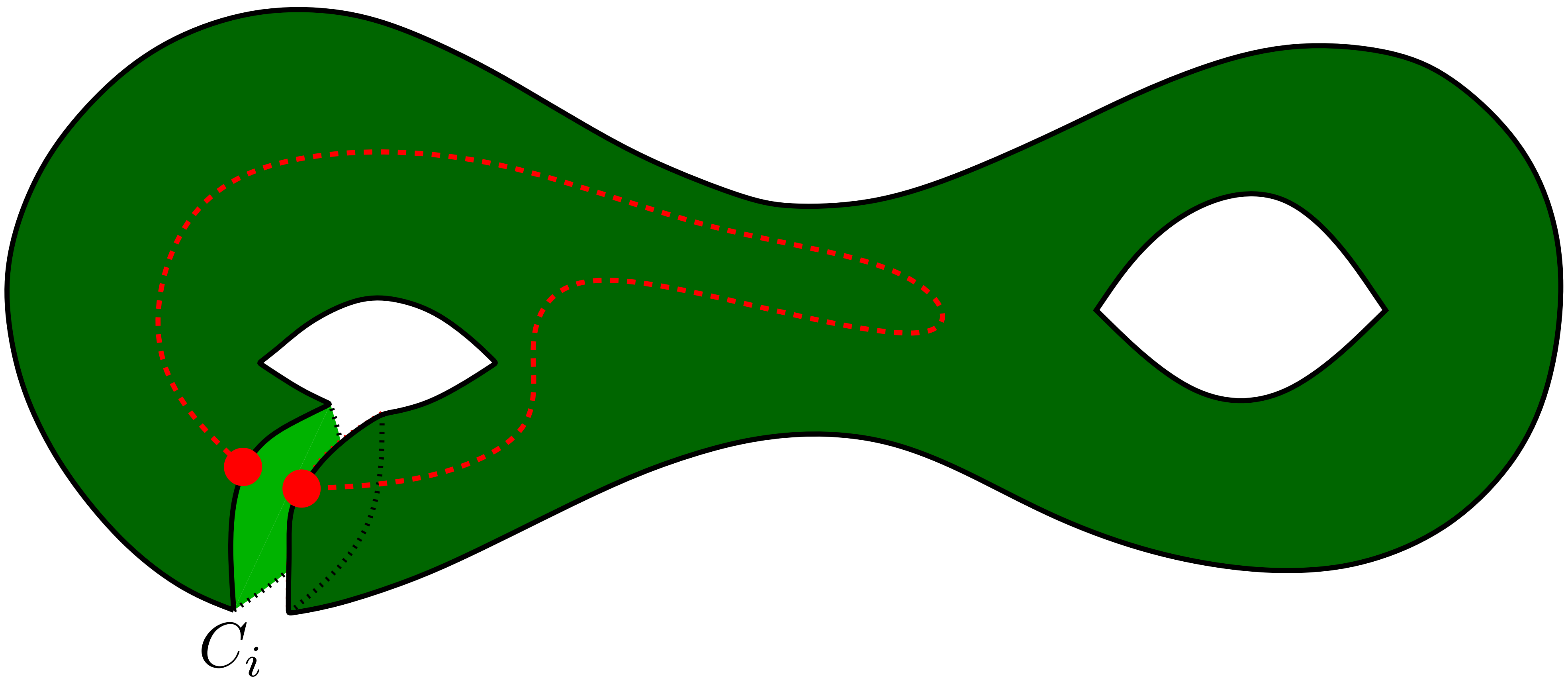

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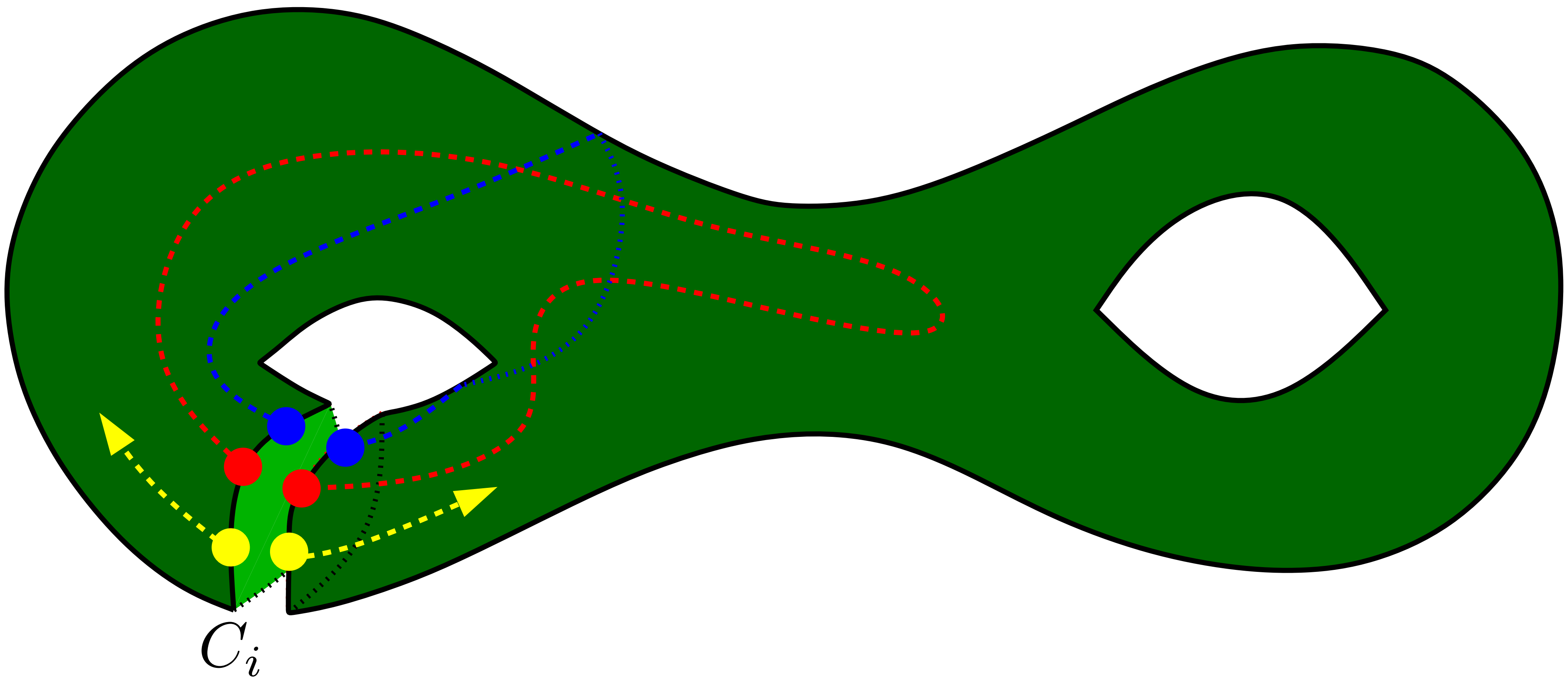

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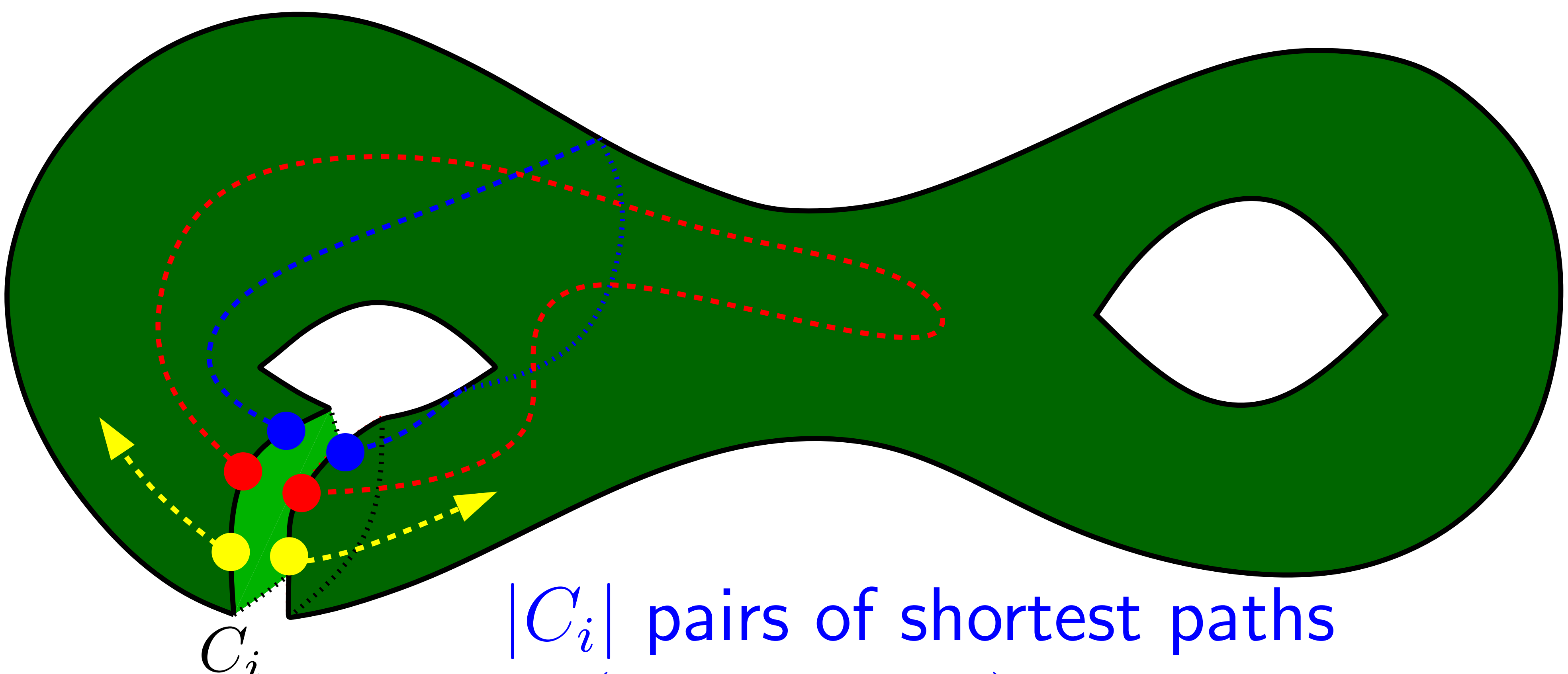

$|C_i|$ pairs of shortest paths
in (G-cut-by-C_i)

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The algorithm takes $O^*(g^{3/2}V^{3/2} + g^{5/2}V^{1/2})$ time.

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Solvable in $O^*(\tilde{V}^{4/3})$ [SODA'06].

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- shortest non-separating cycle: $O^*(g^{3/2}V^{3/2} + g^{5/2}V^{1/2})$;
- shortest non-contractible cycle: $O^*(g^{O(g)}V^{3/2})$. $4/3$

Main techniques:

- system of fundamental loops made of 2 geodesics;
- reduce the problem to computing V distances in graphs.

Skipped:

- better results for face-width in $\mathbb{P}^2$ and $\mathbb{T}$

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THANKS!