Geometric clustering: fixed-parameter tractability and lower bounds with respect to the dimension

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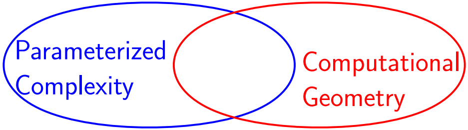
Motivation

Computational geometry parameterized by the dimension



k-center problem

- k-center optimization problem
  Input: a set of n points S in $\mathbb{R}^d$
  Task: find the smallest k congruent balls that cover S
- k-center decision problem
  Input: a set of n points S in $\mathbb{R}^d$
  Question: can S be covered with k unit balls?

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- most discussion about decision problem
- we will consider L_2 and L_∞ metrics
- d is not a constant

k-center problem in L_2

- $k = 1$
  - linear programming in $d + 1$ dimensions
  - solvable in $O(f(d)n) = O(3^{d^2} n)$ time
- $k = 2$
  - easily solvable in $O(n^{2d+2})$ time using arrangements
  - NP-hard [Megiddo 90]

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  - W[1]-hard

New results: 2-center problem in L_2

Theorem

2-center problem parameterized by the dimension is W[1]-hard.

- if there is an algorithm solving 2-center in $O(f(d)n^c)$ time
  - we can find k-cliques in graphs in $O(g(k)n^{c'})$ time
  - we can solve 3-SAT in $O(2^{o(n)})$ time
  - some hierarchy collapses

New results: 2-center problem in L_2

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  - some hierarchy collapses
- an algorithm solving 2-center in $O(f(d)n^{100})$ time is unlikely
- an algorithm solving 2-center in $O(f(d)n^{o(d)})$ time is unlikely

k-center problem in L_∞

- $k = 1$
  - trivial $O(dn)$ time
- $k = 2$
  - solvable in $O(dn^2)$ time [Megiddo 90]
- $k = 3$
  - solvable in $O(n^{\lfloor d/3 \rfloor} \log n)$ time [Assa, Katz 99]
  - NP-hard [Megiddo 90]

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  - $O(6^d \cdot dn \log(dn))$ time
- $k = 4$
  - W[1]-hard
  - an algorithm solving 4-center in $O(f(d)n^{100})$ is unlikely

What is new?

Finer classification of k-center for unbounded dimension

- L_2
  - easy for $k = 1$
  - W[1]-hard for $k = 2$
- L_∞
  - easy for $k = 1, 2$
  - NP-hard, but fixed-parameter tractable for $k = 3$
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Other related work:

- k-center problem parameterized by k is W[1]-hard for $d \geq 2$

  [Marx 05]

Outline

- Introduction
- What is new?
- Ideas
  - Solving 3-center in L_∞ $\leftarrow$
  - W[1]-hardness of 2-center in L_2
- Conclusions

Solving 3-center in L_∞ – Frame

- consider decision problem
  Input: a set of n points S in $\mathbb{R}^d$
  Question: can S be covered with 3 unit cubes?
- the points are denoted 1, 2, . . . , n
  - u a generic point
- the cubes are denoted A, B, C
  - X a generic cube

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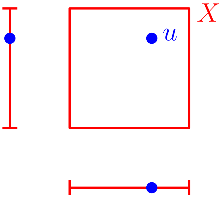
- decision $\rightarrow$ optimization
  - easy using [Frederickson, Johnson '84]

Solving 3-center in L_∞ – General Idea

- cube X covers point u iff
  $\pi_j(u) \in \pi_j(X)$ for each coordinate projection π_j



- classify possible solutions according to certain patterns
- for each pattern
  - reduce the problem to 2-SAT

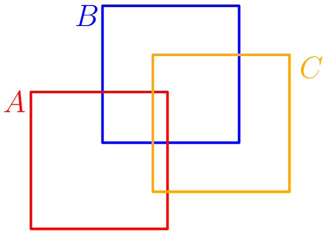
Solving 3-center in L_∞ – Patterns

- the pattern of 3 cubes A, B, C is

$$(L_1, M_1, R_1), (L_2, M_2, R_2), \dots, (L_d, M_d, R_d),$$

  where
  - (L_j, M_j, R_j) a permutation of (A, B, C)
  - $\pi_j(L_j)$ left of $\pi_j(M_j)$ left of $\pi_j(R_j)$

- example with pattern $(A, B, C), (B, C, A)$ in $d = 2$



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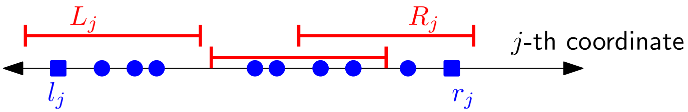
- there are 6^d possible patterns
- each pattern explored independently
- each pattern, one 2-SAT problem

Solving 3-center in L_∞ – A pattern

- consider a pattern $(L_1, M_1, R_1), (L_2, M_2, R_2), \ldots, (L_d, M_d, R_d)$
- we can fix the position of $\pi_j(L_j)$ using l_j



- idem for $\pi_j(R_j)$ using r_j
- the position of $\pi_j(M_j)$ is unclear
- Boolean variable $y_{Xu} \equiv$ point u covered by cube X

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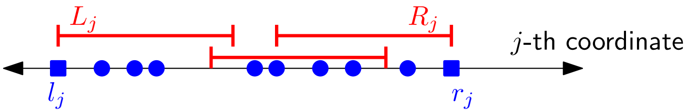

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Solving 3-center in L_∞ – SAT

- each point u is covered

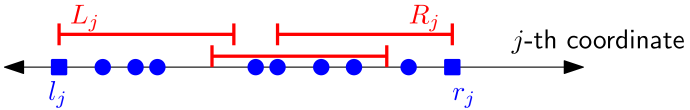
$$y_{Au} \vee y_{Bu} \vee y_{Cu} \quad \text{for each point } u$$

- incompatible pairs; for each dimension

$$\neg y_{L_j u} \quad \text{for each point } u \text{ with } \pi_j(u) > l_j + 1$$

$$\neg y_{R_j u} \quad \text{for each point } u \text{ with } \pi_j(u) < r_j - 1$$

$$\neg y_{M_j u} \vee \neg y_{M_j v} \quad \text{for each points } u, v \text{ with } |\pi_j(u) - \pi_j(v)| > 1$$



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- there are 3 cubes covering with the given pattern iff all clauses satisfiable simultaneously
- 3-SAT instance with $O(dn^2)$ clauses

Solving 3-center in L_∞ – 2-SAT

$$\begin{aligned}
y_{Au} \vee y_{Bu} \vee y_{Cu} & \quad \forall \text{ points } u \\
\neg y_{L_j u} & \quad \forall j, \forall \text{ points } u \text{ with } \pi_j(u) > l_j + 1 \\
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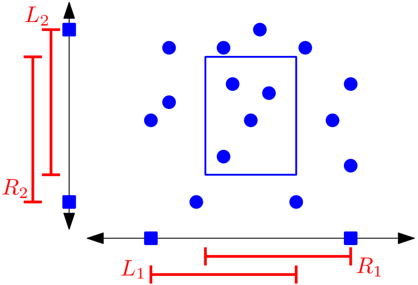
For each point u either

- $y_{Au} \vee y_{Bu} \vee y_{Cu}$ reducible to 2-SAT clause, or
- point u always covered



Solving 3-center in L_∞ – SAT

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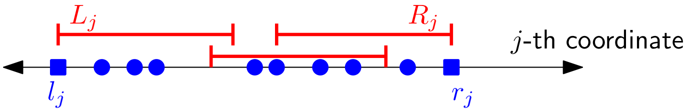
- deciding for a pattern → 2-SAT with $O(dn^2)$ clauses
- deciding for a pattern takes $O(dn^2)$ time
- can be reduced to $O(dn)$ time per pattern



- $O(dn \log n + 6^d dn)$ time for decision 3-center

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  - W[1]-hardness of 2-center in L_2 $\leftarrow$
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Hardness 2-center in L_2 – Idea

- consider the decision 2-center
- assumption: we cannot find k-cliques in $O(f(k)n^c)$
- polynomial-time reduction from clique to 2-center

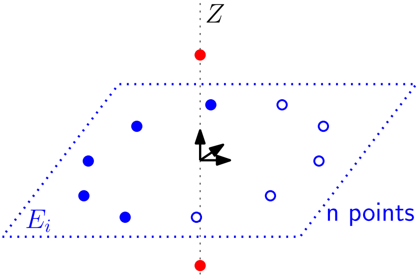
$$\text{Clique } (G,k) \xrightarrow{\text{reduction}} \text{2-center } S{=}S(G,k) \subset \mathbb{R}^{g(k)}$$

where G has k-clique iff S can be 2-covered

- if 2-center solvable in $O(f(d)n^c)$ time
  $\Rightarrow$ (G,k) solvable in $O(f(g(k))n^{c'})$ time

Hardness 2-center in L_2 – Point set

- k orthogonal planes $E_1, \dots E_k$ and one Z axis
- point set in $\mathbb{R}^{2k+1}$
- in Z 2 points with $z = 2$ and $z = -2$



- in each E_i a point set like
- choose appropriate radius
- bijection k-tuples of $V(G)$ and 2-coverings of S
- add extra points killing k-tuples with non-adjacent vertices

Conclusions

Finer classification of k-center problem for unbounded dimension

- L_2
  - easy for $k = 1$
  - W[1]-hard for $k = 2$
    - reduction from parameterized-clique
    - lots of symmetry
- L_∞
  - easy for $k = 1, 2$
  - fixed parameter tractable for $k = 3$
    - reduction to 2-SAT
    - simple
  - W[1]-hard for $k = 4$