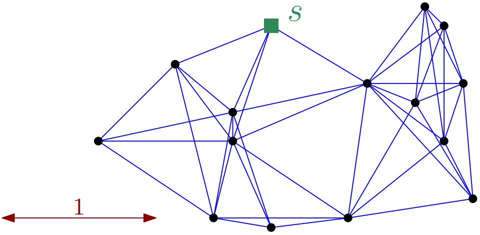
Shortest Paths in Intersection Graphs of Unit Disks

Sergio Cabello
sergio.cabello@fmf.uni-lj.si
University of Ljubljana
Slovenija

Material based on joint work with
Miha Jejčič and Panos Giannopoulos

Setting

P: n points in the plane
$G(P)$: connect two points when distance ≤ 1
intersection graph congruent disks

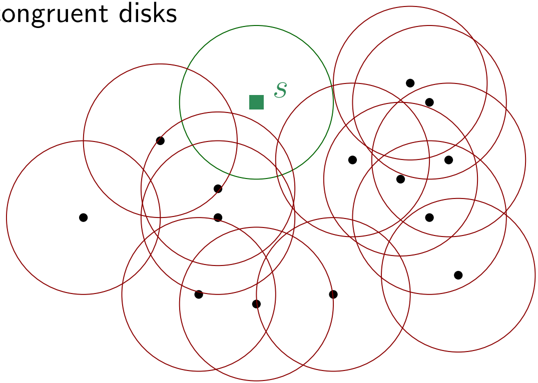


Setting

P: n points in the plane

$G(P)$: connect two points when distance ≤ 1

intersection graph congruent disks



Objective: fast computation of sssp in $G(P)$

Motivation

Bounded communication range:

- minimize hops/links $\rightarrow$ unweighted $G(P)$
- minimize energy $\rightarrow$ weighted $G(P)$

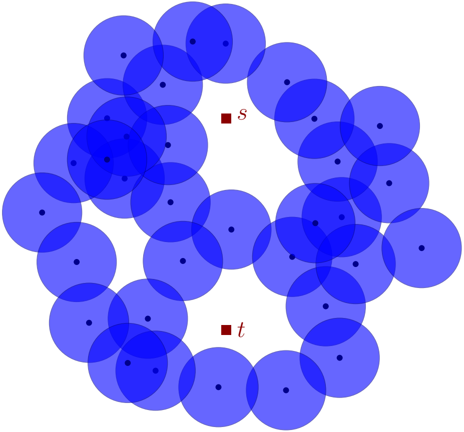
Motivation

Bounded communication range:

- minimize hops/links $\rightarrow$ unweighted $G(P)$
- minimize energy $\rightarrow$ weighted $G(P)$

Separation in the plane:

- set D of unit disks
- s and t in $\mathbb{R}^2 \setminus \bigcup D$
- $\min |D'|$ s.t. $D' \subseteq D$, D' separates s and t



Overview

- Setting/Motivation
- Related work for sssp
- Unweighted
  - $O(n \log n)$ time
  - implementable: Delaunay, Voronoi, point location
- Weighted:
  - $O(n^{1+\varepsilon})$ time
  - unimplementable: dynamic bichromatic closest pair, shallow cuttings
- Separation with unit disks:
  - $O(n^2 \operatorname{polylog} n)$ time
  - Implementable, but many ingredients

Overview

- Setting/Motivation
- Related work for sssp
- Unweighted
  - $O(n \log n)$ time
  - implementable: Delaunay, Voronoi, point location
- Weighted:
  - $O(n^{1+\varepsilon})$ time
  - unimplementable: dynamic bichromatic closest pair, shallow cuttings
- Separation with unit disks:
  - $O(n^2 \operatorname{polylog} n)$ time
  - Implementable, but many ingredients

Related work

exact SSSP

- Roditty and Segal, 2011
  - unweighted: $O(n \log^6 n)$ expected time via Chan's dynamic NN DS
  - weighted: $O(n^{4/3+\varepsilon})$ time
- C. and Jejčič, 2014
  - unweighted: $O(n \log n)$ time; implementable
  - weighted: $O(n^{1+\varepsilon})$ time

More related work

- Roditty and Segal, 2011
  - $(1+\varepsilon)$-approximate distance oracles, improving Bose, Maheshwari, Narasimhan, Smid, and Zeh, 2004.
- Gao and Zhang, 2005
  - WSPD of size $O(n \log n)$ for unit-disk metric
  - $(1+\varepsilon)$-approximate sssp distance in $O(n \log n)$ time
- Chan and Efrat, 2001 (Fuel consumption)
  - distances $\ell : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_{>0}$
  - $O(n \log n)$ time when $\ell(p,q) = f(|pq|) \cdot |pq|^2$, f increasing.
  - $O(n^{4/3+\varepsilon})$ time when ℓ has constant size description.

More related work

- Roditty and Segal, 2011
  - $(1+\varepsilon)$-approximate distance oracles, improving Bose, Maheshwari, Narasimhan, Smid, and Zeh, 2004.
- Gao and Zhang, 2005
  - WSPD of size $O(n \log n)$ for unit-disk metric
  - $(1+\varepsilon)$-approximate sssp distance in $O(n \log n)$ time
- Chan and Efrat, 2001 (Fuel consumption)
  - distances $\ell : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_{>0}$
  - $O(n \log n)$ time when $\ell(p,q) = f(|pq|) \cdot |pq|^2$, f increasing.
  - $O(n^{4/3+\varepsilon})$ time when ℓ has constant size description.

- Faster algorihtms for geometric intersection graphs

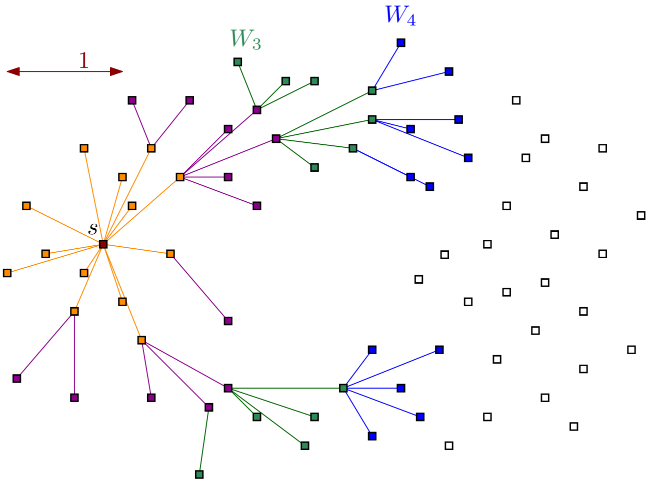
Overview

- Setting/Motivation
- Related work for sssp
- Unweighted
  - $O(n \log n)$ time
  - implementable: Delaunay, Voronoi, point location
- Weighted:
  - $O(n^{1+\varepsilon})$ time
  - unimplementable: dynamic bichromatic closest pair, shallow cuttings
- Separation with unit disks:
  - $O(n^2 \operatorname{polylog} n)$ time
  - Implementable, but many ingredients

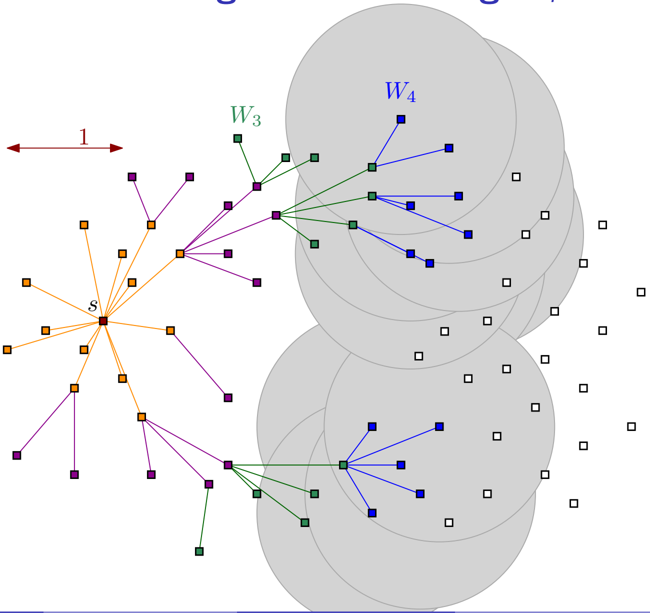
Unweighted

- BFS in $G(P)$ without building $G(P)$
- $W_i = \{p \in P \mid d_{G(P)}(s, p) = i\}$
- Build $W_0 = \{s\}$
- Iteratively build W_i from W_{i-1}
- Edge connecting p to $NN(p, W_{i-1})$ for all $p \in W_i$
- Until W_i empty

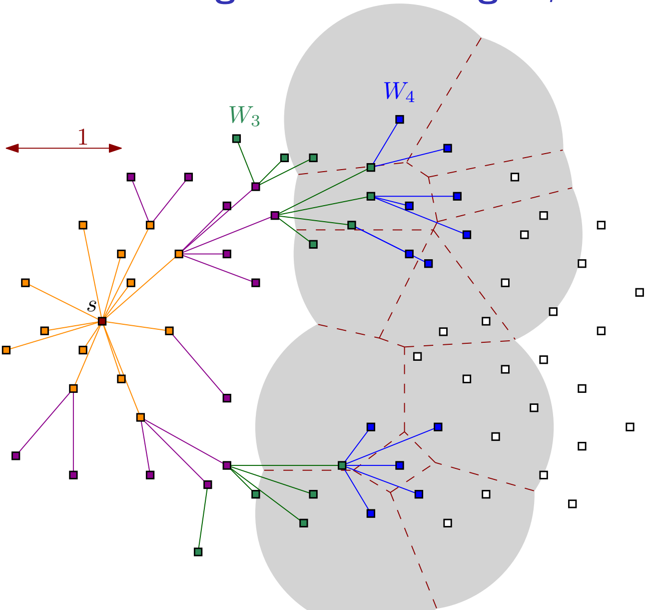
Unweighted - Growing W_i



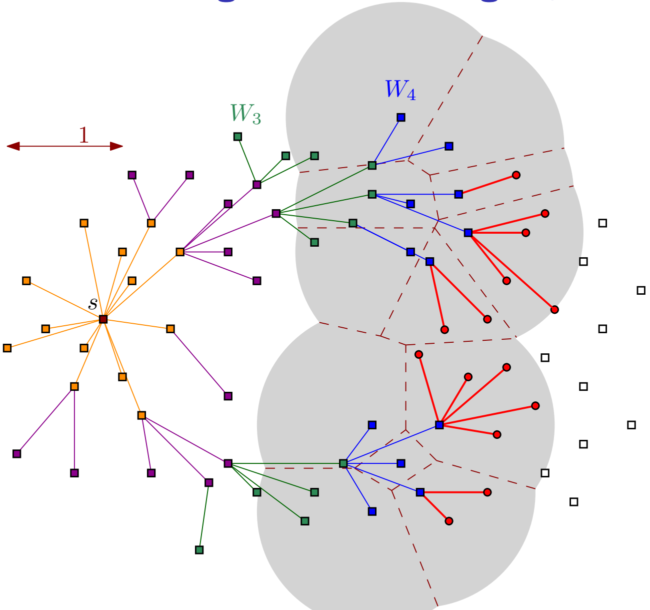
Unweighted - Growing W_i



Unweighted - Growing W_i



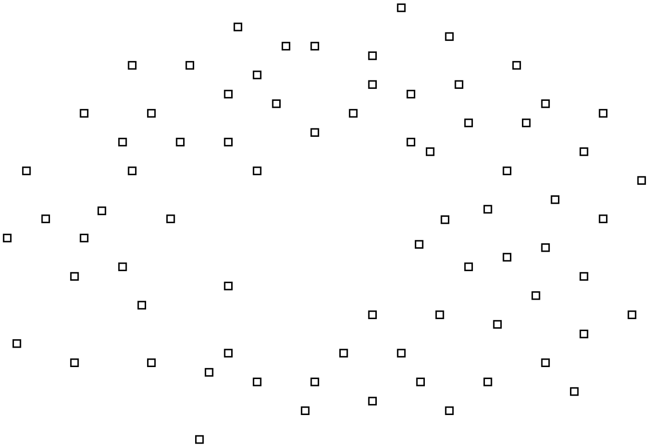
Unweighted - Growing W_i



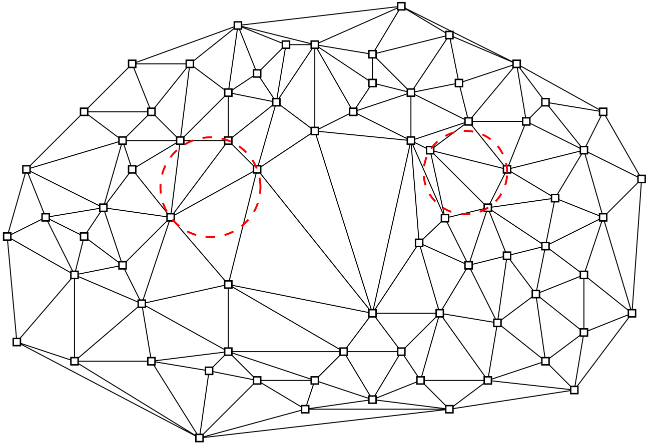
Unweighted

- BFS in $G(P)$ without building $G(P)$
- $W_i = \{p \in P \mid d_{G(P)}(s,p) = i\}$
- Build $W_0 = \{s\}$
- Iteratively build W_i from W_{i-1}
- edge connecting p to $NN(p, W_{i-1})$ for all $p \in W_i$
- Until W_i empty
- Use $DT(P)$ to guide the search of candidate points for W_i
- Candidate points for W_i:
  - points adjacent to W_{i-1} in $DT(P)$
  - points adjacent to W_i in $DT(P)$
- Is this good enough?

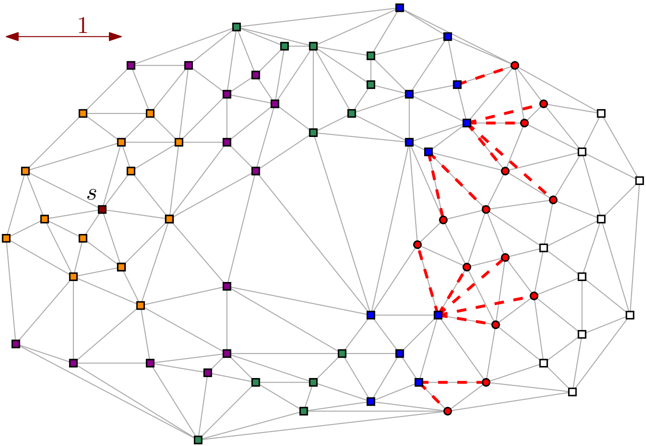
Unweighted - Growing W_i



Unweighted - Growing W_i



Unweighted - Growing W_i



Unweighted - Growing W_i

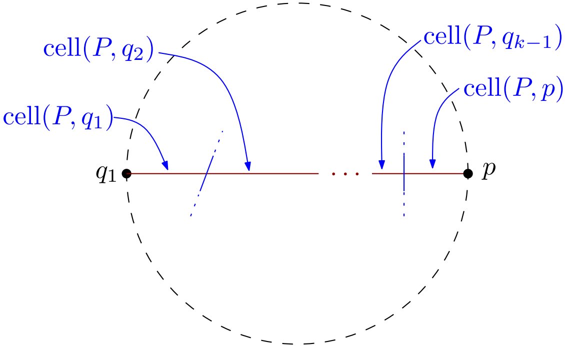
Lemma

Let $p \in W_i$.
There exists a path $q_1, \dots, q_k = p$ *in* $G(P) \cap DT(P)$ *with* $q_1 \in W_{i-1}$ *and* $q_2, \dots, q_k \in W_i$.

Unweighted - Growing W_i

Lemma

Let $p \in W_i$.
There exists a path $q_1, \ldots, q_k = p$ in $G(P) \cap DT(P)$ with $q_1 \in W_{i-1}$ and $q_2, \ldots, q_k \in W_i$.



Unweighted - Growing W_i

Lemma

Let $p \in W_i$.
There exists a path $q_1, \ldots, q_k = p$ in $G(P) \cap DT(P)$ with $q_1 \in W_{i-1}$ and $q_2, \ldots, q_k \in W_i$.

- Data structure to decide whether candidate q is $\in W_i$
  - DS for $NN(q, W_{i-1})$
  - check if distance ≤ 1
- each edge of $DT(P)$ explored twice
- building W_i takes time

$$O\Big(\big(|W_{i-1}| + |W_i| + \sum_{p \in W_{i-1} \cup W_i} \deg_{DT(P)}(p)\big) \log n\Big)$$

```
for p ∈ P do
    dist[p] ← ∞;
dist[s] ← 0
build the Delaunay triangulation DT(P)
W_0 ← {s}
i ← 1
while W_{i−1} ≠ ∅ do
    build data structure for nearest neighbour queries in W_{i−1}
    Q ← W_{i−1}     (* generator of candidate points *)
    W_i ← ∅
    while Q ≠ ∅ do
        q an arbitrary point of Q
        remove q from Q
        for qp edge in DT(P) do
            w ← NN(W_{i−1}, q)
            if dist[p] = ∞ and |pw| ≤ 1 then
                dist[p] ← i
                add p to Q and to W_i
    i ← i + 1
return dist[·]
```

Overview

- Setting/Motivation
- Related work for sssp
- Unweighted $\longleftarrow$ Done
  - $O(n \log n)$ time
  - implementable: Delaunay, Voronoi, point location
- Weighted:
  - $O(n^{1+\varepsilon})$ time
  - unimplementable: dynamic bichromatic closest pair, shallow cuttings
- Separation with unit disks:
  - $O(n^2 \operatorname{polylog} n)$ time
  - Implementable, but many ingredients

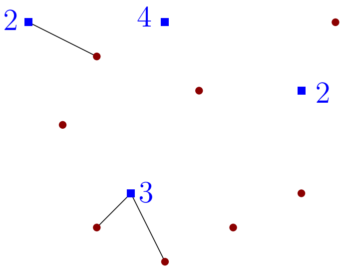
Overview

- Setting/Motivation
- Related work for sssp
- Unweighted
  - $O(n \log n)$ time
  - implementable: Delaunay, Voronoi, point location
- Weighted:
  - $O(n^{1+\varepsilon})$ time
  - unimplementable: dynamic bichromatic closest pair, shallow cuttings
- Separation with unit disks:
  - $O(n^2 \operatorname{polylog} n)$ time
  - Implementable, but many ingredients

Weighted - Ingredient - BCP

Bichromatic closest pair (BCP)

- weighted Euclidean
- red points R
- blue points B
- weights w_b for each $b \in B$
- $\delta\colon B \times R \to \mathbb{R} \qquad \delta(b,r) = w_b + |br|$



Weighted - Ingredient - BCP

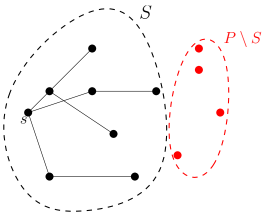
Bichromatic closest pair (BCP)

- weighted Euclidean
- red points R
- blue points B
- weights w_b for each $b \in B$
- $\delta\colon B \times R \to \mathbb{R} \qquad \delta(b, r) = w_b + |br|$

- Eppstein 1995 + Agarwal, Efrat, Sharir 1999: dynamic BCP in $O(n^{\varepsilon})$ amortized per operation
  - insertion/deletion
  - query for minima $\min_{r,b} \delta(r, b)$

Weighted - Idea

- Modification of Dijkstra's algorithm
- Standard Dijsktra's algorithm
  - keep an array dist[$\cdot$]
  - dist[v] is an (over)estimate of $d_{G(P)}(s, v)$
  - keep partition P into S and $P \setminus S$
  - S contains vertices with dist[s] $= d_{G(P)}(s, v)$



Weighted - Idea

- Modification of Dijkstra's algorithm
- Standard Dijsktra's algorithm
  - keep an array dist[$\cdot$]
  - dist[v] is an (over)estimate of $d_{G(P)}(s, v)$
  - keep partition P into S and $P \setminus S$
  - S contains vertices with dist[s] $= d_{G(P)}(s, v)$
  - an iteration: find a vertex

$$q^* \in \arg \min_{q \in P \setminus S} \min_{p \in S,, |pq| \leq 1} \text{dist}[p] + |pq|$$

  - move q^* from $P \setminus S$ to S
  - usually we keep dist[q] $= \min_{p \in S} \text{dist}[p] + |pq|$

Weighted - Idea

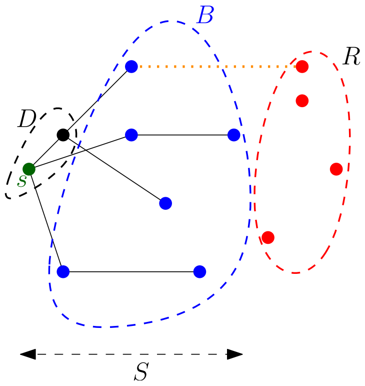
- Modification of Dijkstra's algorithm
  - array dist[$\cdot$], dist[v] is an (over)estimate of $d_{G(P)}(s, v)$
  - keep partition P into S and $R = P \setminus S$
  - partition S into D and B
  - D are "dead" points, irrelevant when min dist[p] + $|pq|$
  - an iteration: find a pair

$$(b^*, r^*) \ \in \ \arg \min_{(b,r)\in\in B\times R} \ \mathrm{dist}[b] + |br|$$

  - if $|b^* r^*| > 1$, move b^* from B to D
  - else normal Dijsktra's step

Weighted - Idea

- Modification of Dijkstra's algorithm



for $p \in P$ **do**
    $\text{dist}[p] \leftarrow \infty$
$\text{dist}[s] \leftarrow 0$
$B \leftarrow \{s\}$
$D \leftarrow \emptyset$
$R \leftarrow P \setminus \{s\}$
store $R \cup B$ in a BCP dynamic DS wrt $\delta(b, r) = \text{dist}[b] + |br|$
while $R \neq \emptyset$ **do**
    $(b^*, r^*) \leftarrow \text{BCP}(B, R)$
    if $|b^* r^*| > 1$ **then**
        $\text{delete}(B, b^*)$
        $D \leftarrow D \cup \{b^*\}$
    else
        $\text{dist}[r^*] \leftarrow \text{dist}[b^*] + |b^* r^*|$
        $\text{delete}(R, r^*)$
        $\text{insert}(B, r^*)$
return $\text{dist}[\cdot]$

Overview

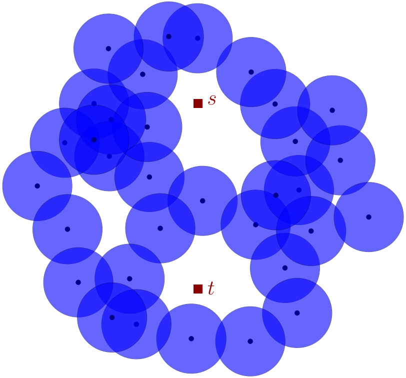
- Setting/Motivation
- Related work for sssp
- Unweighted
  - $O(n \log n)$ time
  - implementable: Delaunay, Voronoi, point location
- Weighted $\longleftarrow$ Done
  - $O(n^{1+\varepsilon})$ time
  - unimplementable: dynamic bichromatic closest pair, shallow cuttings
- Separation with unit disks:
  - $O(n^2 \operatorname{polylog} n)$ time
  - Implementable, but many ingredients

Overview

- Setting/Motivation
- Related work for sssp
- Unweighted
  - $O(n \log n)$ time
  - implementable: Delaunay, Voronoi, point location
- Weighted
  - $O(n^{1+\varepsilon})$ time
  - unimplementable: dynamic bichromatic closest pair, shallow cuttings
- Separation with unit disks:
  - $O(n^2 \operatorname{polylog} n)$ time
  - Implementable, but many ingredients

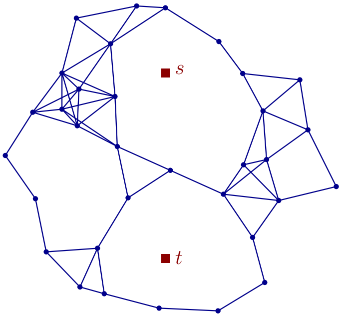
Setting

- set D of unit disks
- s, t points in $\mathbb{R}^2 \setminus \bigcup D$
- P centers of the disks
- $G(P)$ as before, with distance 2
- C. and Giannopoulos
  - $O(n^2 + n \cdot |E(G(D))|)$
  - general objects



Setting

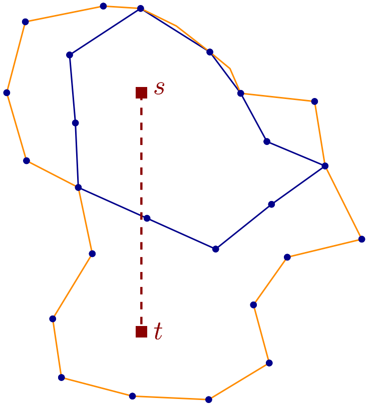
- set D of unit disks
- s, t points in $\mathbb{R}^2 \setminus \bigcup D$
- P centers of the disks
- $G(P)$ as before, with distance 2
- C. and Giannopoulos
  - $O(n^2 + n \cdot |E(G(D))|)$
  - general objects
- today: $O(n^2 \log^4 n)$ for unit disks
- also easier to explain & understand



Algorithm of C. & Giannopoulos

- for a closed walk $\pi = p_1 \dots p_k p_1$ in $G(P)$

$$N(\pi) = \pi \cap \overline{st} \pmod 2$$



Algorithm of C. & Giannopoulos

- for a closed walk $\pi = p_1 \dots p_k p_1$ in $G(P)$

$$N(\pi) = \pi \cap \overline{st} \pmod 2$$

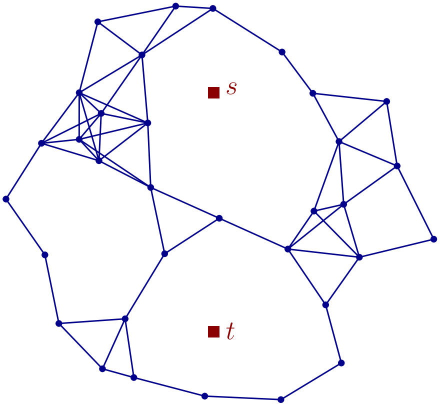
- if $N(\pi) = 1$ then $\bigcup_{p \in V(\pi)} disk(p, 1)$ separates s and t
- shortest closed walk π with $N(\pi) = 1$ gives an optimal solution
- shortest closed walk π with $N(\pi) = 1$ is actually a cycle
- enough to restrict the search to fundamental cycles: defined by a BFS-tree and an additional edge

$$\begin{aligned} \min\ & |V(cycle(T_r, e))| \\ \text{s.t.}\ & r \in P,\ T_r \text{ BFS cycle from } r \\ & e \in E(G(P)) \setminus E(T_r) \\ & N(cycle(T_r, e)) = 1 \end{aligned}$$

Algorithm of C. & Giannopoulos

- for a closed walk $\pi = p_1 \dots p_k p_1$ in $G(P)$

$$N(\pi) = \pi \cap \overline{st} \pmod 2$$



Adaptation

for each r in P

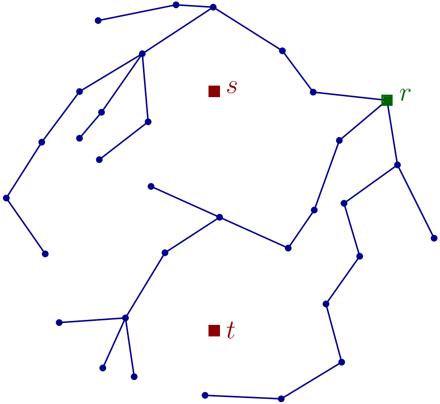
- construct BFS tree T_r from r
- attach to each $p \in P$ the label $d[p] = d_{G(P)}(s, p)$
- solve

$$\begin{aligned} \text{min} \ & d[p] + d[q] \\ \text{s.t.} \ & |pq| \leq 1 \\ & N(\mathit{cycle}(T_r, pq)) = 1 \end{aligned}$$

- break P into groups depending on $N(T_r[r, p])$
- use range searching & vertical shooting
  to solve the resulting problems

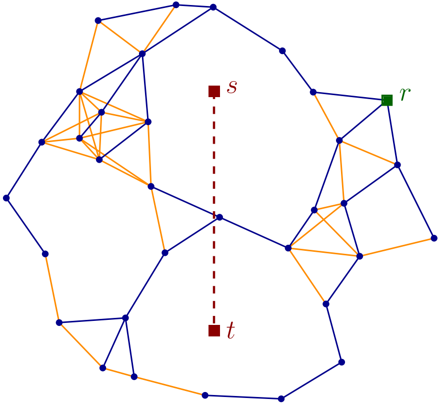
Adaptation

for each r in P



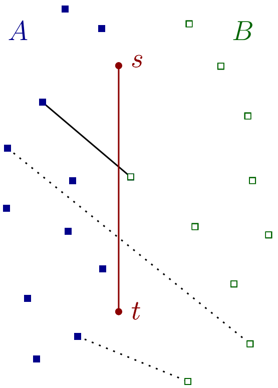
Adaptation

for each r in P



Resulting problem - Example

- vertical segment st
- points A and B with weights $(w_p)_{p \in A \cup B}$



$$\begin{aligned} \min\ & w_a + w_b \\ \text{s.t.}\ & a \in A, b \in B \\ & |ab| \le 1 \\ & ab \cap st \ne \emptyset \end{aligned}$$

Solvable in $O(n \log^4 n)$

Conclusions

- shortest paths in unit disk graphs
  - $O(n \log n)$ for unweighted
  - $O(n^{1+\varepsilon})$ for weighted
- Improvement for separation with unit disks

Conclusions

- shortest paths in unit disk graphs
  - $O(n \log n)$ for unweighted
  - $O(n^{1+\varepsilon})$ for weighted
- Improvement for separation with unit disks

- Open problems:
  - Can we compute efficiently a compact representation of the distances in all the graphs $G_{\leq\lambda}(P)$?
  - Given $s, t \in P$ and $k \in \mathbb{N}$,
    find minimum λ such that $d_{G_{\leq\lambda}(P)}(s, t) \leq k$.
    Easy in $\tilde{O}(n^{4/3})$.
  - Dual to separation problem – barrier resilience:
    find (s, t)-curve that touches as few disks as possible.
    Polynomial? Hard?

Conclusions

- shortest paths in unit disk graphs
  - $O(n \log n)$ for unweighted
  - $O(n^{1+\varepsilon})$ for weighted
- Improvement for separation with unit disks

- Open problems:
  - Can we compute efficiently a compact representation of the distances in all the graphs $G_{\leq\lambda}(P)$?
  - Given $s, t \in P$ and $k \in \mathbb{N}$,
    find minimum λ such that $d_{G_{\leq\lambda}(P)}(s, t) \leq k$.
    Easy in $\tilde{O}(n^{4/3})$.
  - Dual to separation problem – barrier resilience:
    find (s, t)-curve that touches as few disks as possible.
    Polynomial? Hard? Or both?