(1)

(a) Note that λ_a cannot be both injective and nilpotent. If so, nilpotence gives $a^n M = a(a^{n-1}M) \subseteq N$, and injectivity gives $a^{n-1}M \subseteq N$. Inductively, $M \subseteq N$, so $M = N$, contradicting the assumption that N is proper. Thus if N is a primary submodule of M, then $r_M(N)$ is the set of all $a \in R$ such that λ_a is not injective. Since $r_M(N)$ is the radical of an ideal, it is an ideal of R, and in fact it is a prime ideal. For if λ_a and λ_b fail to be injective, so does $\lambda_{ab} = \lambda_a \circ \lambda_b$.

(b) We may assume that $k = 2$; an induction argument takes care of larger values. Let $N = N_1 \cap N_2$ and $r_M(N_1) = r_M(N_2) = P$. Assume for the moment that $r_M(N) = P$. If $a \in R$, $x \in M$, $ax \in N$, and $a \notin r_M(N)$, then since N_1 and N_2 are P-primary, we have $x \in N_1 \cap N_2 = N$. It remains to show that $r_M(N) = P$. If $a \in P$, then there are positive integers n_1 and n_2 such that $a^{n_1}M \subseteq N_1$ and $a^{n_2}M \subseteq N_2$. Therefore $a^{n_1+n_2}M \subseteq N$, so $a \in r_M(N)$. Conversely, if $a \in r_M(N)$ then a belongs to $r_M(N_i)$ for $i = 1, 2$, and therefore $a \in P$.

(c) If not, then for some $a \in R$, $\lambda_a : M/N \to M/N$ is neither injective nor nilpotent. The chain $\ker \lambda_a \subseteq \ker \lambda_a^2 \subseteq \ker \lambda_a^3 \subseteq \cdots$ terminates by the ascending chain condition, say at $\ker \lambda_a^i$. Let $\varphi = \lambda_a^i$; then $\ker \varphi = \ker \varphi^2$ and we claim that $\ker \varphi \cap \operatorname{im} \varphi = 0$. Suppose $x \in \ker \varphi \cap \operatorname{im} \varphi$, and let $x = \varphi(y)$. Then $0 = \varphi(x) = \varphi^2(y)$, so $y \in \ker \varphi^2 = \ker \varphi$, so $x = \varphi(y) = 0$.

Now λ_a is not injective, so $\ker \varphi \neq 0$, and λ_a is not nilpotent, so λ_a^i can't be 0 (because $a^i M \not\subseteq N$). Consequently, $\operatorname{im} \varphi \neq 0$.

Let $p : M \to M/N$ be the canonical epimorphism, and set $N_1 = p^{-1}(\ker \varphi)$, $N_2 = p^{-1}(\operatorname{im} \varphi)$. We will prove that $N = N_1 \cap N_2$. If $x \in N_1 \cap N_2$, then $p(x)$ belongs to both $\ker \varphi$ and $\operatorname{im} \varphi$, so $p(x) = 0$, in other words, $x \in N$. Conversely, if $x \in N$, then $p(x) = 0 \in \ker \varphi \cap \operatorname{im} \varphi$, so $x \in N_1 \cap N_2$.

Finally, we will show that N is properly contained in both N_1 and N_2, so N is reducible, a contradiction. Choose a nonzero element $y \in \ker \varphi$. Since p is surjective, there exists $x \in M$ such that $p(x) = y$. Thus $x \in p^{-1}(\ker \varphi) = N_1$ (because $y = p(x) \in \ker \varphi$), but $x \notin N$ (because $p(x) = y \neq 0$). Similarly, $N \subset N_2$ (with $0 \neq y \in \operatorname{im} \varphi$), and the result follows.

(d) We will show that N can be expressed as a finite intersection of irreducible submodules of M, so that (1.2.4) applies. Let $\mathcal{S}$ be the collection of all submodules of M that cannot be expressed in this form. If $\mathcal{S}$ is nonempty, then $\mathcal{S}$ has a maximal element N (because M is Noetherian). By definition of $\mathcal{S}$, N must be reducible, so we can write $N = N_1 \cap N_2$, $N \subset N_1$, $N \subset N_2$. By maximality of N, N_1 and N_2 can be expressed as finite intersections of irreducible submodules, hence so can N, contradicting $N \in \mathcal{S}$. Thus $\mathcal{S}$ is empty.

(2)

(a) Let P be an associated prime of M, so that $P = \mathrm{ann}(x)$, $x \neq 0$, $x \in M$. Renumber the N_i so that $x \notin N_i$ for $1 \leq i \leq j$ and $x \in N_i$ for $j+1 \leq i \leq r$. Since N_i is P_i-primary, we have $P_i = r_M(N_i)$ (see (1.1.1)). Since P_i is finitely generated, $P_i^{n_i} M \subseteq N_i$ for some $n_i \geq 1$. Therefore

$$(\bigcap_{i=1}^{j} P_i^{n_i})x \subseteq \bigcap_{i=1}^{r} N_i = (0)$$

so $\cap_{i=1}^{j} P_i^{n_i} \subseteq \mathrm{ann}(x) = P$. (By our renumbering, there is a j rather than an r on the left side of the inclusion.) Since P is prime, $P_i \subseteq P$ for some $i \leq j$. We claim that $P_i = P$, so that every associated prime must be one of the P_i. To verify this, let $a \in P$. Then $ax = 0$ and $x \notin N_i$, so λ_a is not injective and therefore must be nilpotent. Consequently, $a \in r_M(N_i) = P_i$, as claimed.

Conversely, we show that each P_i is an associated prime. Without loss of generality, we may take $i = 1$. Since the decomposition is reduced, N_1 does not contain the intersection of the other N_i's, so we can choose $x \in N_2 \cap \cdots \cap N_r$ with $x \notin N_1$. Now N_1 is P_1-primary, so as in the preceding paragraph, for some $n \geq 1$ we have $P_1^n x \subseteq N_1$ but $P_1^{n-1} x \not\subseteq N_1$. (Take $P_1^0 x = Rx$ and recall that $x \notin N_1$.) If we choose $y \in P_1^{n-1} x \setminus N_1$ (hence $y \neq 0$), the proof will be complete upon showing that P_1 is the annihilator of y. We have $P_1 y \subseteq P_1^n x \subseteq N_1$ and $x \in \cap_{i=2}^{r} N_i$, so $P_1^n x \subseteq \cap_{i=2}^{r} N_i$. Thus $P_1 y \subseteq \cap_{i=1}^{r} N_i = (0)$, so $P_1 \subseteq \mathrm{ann}\, y$. On the other hand, if $a \in R$ and $ay = 0$, then $ay \in N_1$ but $y \notin N_1$, so $\lambda_a : M/N_1 \to M/N_1$ is not injective and is therefore nilpotent. Thus $a \in r_M(N_1) = P_1$.

(b) By the correspondence theorem, a reduced primary decomposition of (0) in M/N is given by $(0) = \cap_{i=1}^{r} N_i/N$, and N_i/N is P_i-primary, $1 \leq i \leq r$. By (a) the associated primes of M/N are $\{P_1, \ldots, P_r\}$ and are determined by N.

(3) Za začetek opazimo, da množice $U(f)$ tvorijo bazo za $\operatorname{Spec} R$, saj je $U(f) \cap U(g) = U(fg)$. Dalje definirajmo še $Z(f) := \operatorname{Spec} R \setminus U(f) = \{\mathfrak{p} \in \operatorname{Spec} R \mid f \in \mathfrak{p}\}$ in $Z(S) := \cap_{f\in S} Z(f)$ za $S \subseteq R$.

(a) Vzemimo poljubna različna praideala $\mathfrak{p}_1, \mathfrak{p}_2 \in \operatorname{Spec} R$. Velja $\mathfrak{p}_1 \neq \mathfrak{p}_2$, brez škode za splošnost $\mathfrak{p}_1 \not\subseteq \mathfrak{p}_2$. Za vsak $f \in \mathfrak{p}_1 \setminus \mathfrak{p}_2$ je $U(f)$ okolica $\mathfrak{p}_2$, ki ne vsebuje $\mathfrak{p}_1$. S tem smo dokazali, da je $\operatorname{Spec} R \in T_0$.

Ostala nam je torej še kompaktnost prostora $\operatorname{Spec} R$. Dokazali bomo nekoliko več, in sicer, da so vse množice $U(f)$ za $f \in R$ kompaktne (to zadošča, saj je $\operatorname{Spec} R = U(1)$). Velja:

$$U(f) \subseteq \cup_{\lambda\in\Lambda} U(g_\lambda) \iff Z(f) \supseteq \cap_{\lambda\in\Lambda} Z(g_\lambda) = Z\big(\{g_\lambda \mid \lambda \in \Lambda\}\big) = Z(\mathfrak{a}),$$

kjer smo z $\mathfrak{a}$ označili ideal $(g_\lambda \mid \lambda \in \Lambda)$. Sedaj uporabimo, da je $\operatorname{Rad}(\mathfrak{a}) = \bigcap\{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{a} \subseteq \mathfrak{p}\}$ in dobimo

$$Z(f) \supseteq Z(\mathfrak{a}) \iff \operatorname{Rad}(f) \subseteq \operatorname{Rad}(\mathfrak{a}) \iff f \in \operatorname{Rad}(\mathfrak{a}).$$

Očitno f leži v radikalu ideala $\mathfrak{a}$, če $f^n \in \mathfrak{a}$ za neko naravno število n. Ker je $\mathfrak{a}$ generiran z g_λ, lahko zapišemo $f^n = a_1 g_{\lambda_1} + \cdots + a_\ell g_{\lambda_\ell}$ za primerne $a_i \in R$ in $\lambda_i \in \Lambda$. Torej

$$f \in \operatorname{Rad}(\mathfrak{a}) \iff f \in \operatorname{Rad}(g_{\lambda_1}, \dots, g_{\lambda_\ell}) \iff U(f) \subseteq U(g_{\lambda_1}) \cup \cdots \cup U(g_{\lambda_\ell}).$$

(b) Vsaka zaprta množica spektra je oblike $Z(T)$ za neko podmnožico $T \subseteq R$. O"citno lahko predpostavimo, da je $T = \mathfrak{a} \subset R$ ideal. Enostavno je videti, da je $Z(\mathfrak{a}) = Z\big(\operatorname{Rad}(\mathfrak{a})\big)$. Predpostavimo sedaj, da je $Z(\mathfrak{a})$ nerazcepna in je $\mathfrak{a} = \operatorname{Rad}(\mathfrak{a})$. Trdimo, da je $\mathfrak{a}$ praideal. Denimo nasprotno. Tedaj obstajata $f, g \in R \setminus \mathfrak{a}$ z lastnostjo $fg \in \mathfrak{a}$. Sledi $Z(\mathfrak{a}) \subseteq Z(f) \cup Z(g)$. Zaradi nerazcepnosti lahko predpostavimo, da je $Z(\mathfrak{a}) \subseteq Z(f)$, torej $f \in \mathfrak{a}$. Prišli smo v protislovje, torej je $\mathfrak{a}$ praideal. Očitno pa v tem primeru velja $Z(\mathfrak{a}) = \overline{\{\mathfrak{a}\}}$.

(c) Zgoraj smo dokazali, da so vse množice $U(f)$ kompaktne in odprte. Po drugi strani pa je vsaka odprta množica unija teh množic. Torej je $\mathring{\mathcal{K}}(\operatorname{Spec} R) \subseteq \{U(f_1) \cup \cdots \cup U(f_\ell) \mid \ell \in \mathbb{N},\ f_i \in R\}$. Vendar so tudi vse množice na desni strani kompaktne. Torej velja $\mathring{\mathcal{K}}(\operatorname{Spec} R) = \{U(f_1) \cup \cdots \cup U(f_\ell) \mid \ell \in \mathbb{N},\ f_i \in R\}$. Očitno je $\mathring{\mathcal{K}}(\operatorname{Spec} R)$ baza topologije na $\operatorname{Spec} R$, ki je zaprta za končne preseke.

(d) Postavimo $Z := \{0,1\}^R$, kjer $\{0,1\}$ opremimo z diskretno topologijo, Z pa s produktno. Z je očitno povsem nepovezan in T_2, po izreku Tihonova pa je tudi kompakten. Kot ponavadi, identificiramo funkcijo $f : R \to \{0,1\}$ iz Z s podmnožico $\{r \in R \mid f(r) = 0\}$ množice R; torej je Z podmnožica potenčne množice od R. Zato je $j(\mathfrak{a}) := \mathfrak{a}$ injektivna preslikava $j\colon \operatorname{Spec} R \hookrightarrow Z$. Iz definicije produktne topologije takoj dobimo, da je inducirana topologija na $\operatorname{Spec} R$ natanko konstruktibilna topologija. Preostane nam le še, da dokažemo, da je $j(\operatorname{Spec} R)$ zaprta podmnožica Z. Naj bo $S \subseteq R$ poljubna množica, ki ni v $j(\operatorname{Spec} R)$. Torej S ni praideal kolobarja R. V posebnem S ne izpolnjuje vsaj enega od naslednjih aksiomov:

(1) $S + S \subseteq S$,
(2) $RS \subseteq S$,
(3) $a \notin S,\ b \notin S \Rightarrow ab \notin S$ $(a, b \in R)$.

Trdimo, da ta aksiom ni izpolnjen tudi v kakšni okolici S v Z. Pokažimo to npr. za (3). Obstajata $a, b \notin S$ z lastnostjo $ab \in S$. Potem je $U := \{T \subseteq R \mid a \notin T,\ b \notin T, ab \in T\}$ okolica S v Z, in aksiom (3) ni izpolnjen za noben $T \in U$.

(4)

(a) Ker sta $F, G \in K[X_1, X_2]$ paroma tuja, sta paroma tuja tudi kot polinoma $F, G \in K(X_1)[X_2]$ (Gaussov izrek). Ker je $K(X_1)$ obseg, je $K(X_1)[X_2]$ glavni kolobar. Torej obstajata $a_1, a_2 \in K(X_1)[X_2]$ z lastnostjo $1 = a_1 F + a_2 G$. To enačbo pomnožimo s primernim $0 \neq d \in K[X_1]$, da $a_1 d, a_2 d \in K[X_1][X_2]$. Dobimo $d = b_1 F + b_2 G$, kjer $b_i := a_i d$ $(i = 1, 2)$ in $d \in K[X_1]$.

Ker iščemo rešitve sistema $F = 0 = G$ v K^2, smemo predpostaviti, da je K neskončen. Če je (x_0, y_0) rešitev našega sistema, je $d(x_0) = 0$. Ker je $d \neq 0$, ima (kot polinom 1 spremenljivke) le končno mnogo ničel. Skupaj z dejstvom, da pri fiksnem x_0 obstaja kvečjemu končno točk oblike (x_0, y), ki rešijo naš sistem (tukaj uporabimo, da sta F in G paroma tuja in je obseg K neskončen), to implicira, da ima sistem $F = 0 = G$ v K^2 kvečjemu končno mnogo rešitev.

(b) Najprej opazimo, da so vsi ideali v (i), (ii) in (iii) res maksideali. Naj bo sedaj $\mathfrak{g}$ polj. pravi, netrivialni praideal v $K[X,Y]$. Po Basissatzu je ta kolobar noetherski, torej je $\mathfrak{g} = (f_1, \dots, f_r)$ za primerne $f_j \in K[X,Y]$. Najprej dokažimo, da smemo privzeti, da so vsi f_j nerazcepni. Če to ni res, potem je npr. $f_j = g_1 \cdots g_p$ razcep f_j na nerazcepne faktorje in $p > 1$. Ker $f_j \in \mathfrak{g}$, obstaja k, da je $g_k \in \mathfrak{g}$. Tedaj pa je $\mathfrak{g} = (f_1, \dots, f_{j-1}, g_k, f_{j+1}, \dots, f_r)$.

Če je $r = 1$, potem je $\mathfrak{g}$ oblike (ii). Če je $r > 1$, potem sta f_1 in f_2 paroma tuja (saj smo predpostavili, da so vsi f_i nerazcepni). Zdaj uporabimo dejstvo, da je obseg K algebraično zaprt in V–J korespondenco. Po (a) je alg. množica, ki pripada $\mathfrak{g}$, končna. Ker je tudi nerazcepna (saj je $\mathfrak{g}$ praideal), je singleton, npr. $\{(\varphi_1, \varphi_2)\}$. Ponovno uporabimo V–J korespondenco in dobimo $\mathfrak{g} = (X - \varphi_1, Y - \varphi_2)$.

(c) Tak maksideal obstaja, npr. $\mathfrak{g} = (X^2 + 1, Y)$. $\mathfrak{g}$ je maksideal, saj je $\mathbb{R}[X,Y]/\mathfrak{g} \cong \mathbb{C}$.