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Proper holomorphic discs avoiding closed convex sets

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Abstract. Denote by $\triangle$ the open unit disc in $\mathbb{C}$. Let C be a closed convex subset of $\mathbb{C}^2$. We prove that for each $p \in \mathbb{C}^2 \setminus C$ there is a proper holomorphic map $\varphi : \triangle \to \mathbb{C}^2$ such that $\varphi(0) = p$ and $\varphi(\triangle) \cap C = \emptyset$ if and only if either C is a complex line or C does not contain any complex line.

Let $\triangle$ denote the open unit disc in $\mathbb{C}$. Let X be a Stein manifold with $\dim X \geq 2$ and $\rho : X \to \mathbb{R}$ a (strongly) plurisubharmonic function whose sub-level sets are not necessarily relatively compact. This paper was motivated by the question what conditions on X and ρ imply that for each $M \in \mathbb{R}$ and for each $p \in X$ with $\rho(p) > M$ there exists a proper holomorphic disc $\varphi : \triangle \to X$ such that $\varphi(0) = p$ and $\rho(\varphi(\zeta)) > M$ for each $\zeta \in \triangle$.

In the case of a strongly plurisubharmonic exhaustion function ρ Globevnik [Glo] proved that such discs exist. Further, Forstnerič and Globevnik [FG] constructed such discs in the case when $X = \mathbb{C}^2$ and $\rho(z_1, z_2) = \rho(x_1 + iy_1, x_2 + iy_2) = x_1^2 + x_2^2 - c(y_1^2 + y_2^2)$ for $c < 1$; here ρ is strongly plurisubharmonic which, for $0 \leq c < 1$, is not an exhaustion function on $\mathbb{C}^2$. Our main result is the following:

Theorem 1 *Let C be a closed convex subset of $\mathbb{C}^2$. The following are equivalent:*
(a) For each $p \in \mathbb{C}^2 \setminus C$ there is a proper holomorphic map $\varphi : \triangle \to \mathbb{C}^2$ such that $\varphi(0) = p$ and $\varphi(\triangle) \cap C = \emptyset$.
(b) Either C is a complex line or C does not contain any complex line.

It is known that a proper holomorphic disc $\triangle \to \mathbb{C}^2$ cannot avoid a non-polar set of parallel complex lines (see [Jul],[Tsu],[Ale] and [FG]). If a closed convex set C contains a complex line L and if C is not equal to L then C contains an interval of parallel complex lines. Since a segment is not a polar set it follows that for such a set C we have $f(\triangle) \cap C \neq \emptyset$ for every proper holomorphic map $f : \triangle \to \mathbb{C}^2$.

Alexander [Ale] proved that if $E \subset \mathbb{C}$ is a closed polar set containing at least two points, there exists a proper holomorphic map $\varphi = (\varphi_1, \varphi_2) : \triangle \to \mathbb{C}^2$ such that $\varphi_1 : \triangle \to \mathbb{C} \setminus E$ is a universal covering map of the disc onto $\mathbb{C} \setminus E$. This implies that if C is a complex line and $p \notin C$ then there exists a proper holomorphic map such that $\varphi(0) = p$ and $\varphi(\triangle) \cap C = \emptyset$.

In view of these remarks Theorem 1 follows from the following theorem.

Theorem 2 *Let C be a closed convex subset of $\mathbb{C}^2$ which does not contain any complex line and let $p \in \mathbb{C}^2 \setminus C$. Then there exists a proper holomorphic map $\varphi : \triangle \to \mathbb{C}^2$ such that $\varphi(0) = p$ and $\varphi(\triangle) \cap C = \emptyset$.*

We shall first reduce Theorem 2 to the special case when $C = \{(z_1, z_2) \in \mathbb{C}^2; \operatorname{Re} z_1 \leq 0, \operatorname{Re} z_2 \leq 0\}$:

Lemma 3 *Let C be a closed convex subset of $\mathbb{C}^2$ which does not contain any complex line. Then there is an complex affine change of coordinates such that in the new coordinates*

$$C \subset \{(z_1, z_2) \in \mathbb{C}^2; \operatorname{Re} z_1 \leq 0, \operatorname{Re} z_2 \leq 0\}.$$

Moreover, if $p \notin C$ and $M > 0$ we can make the above change of coordinates so that in the new coordinates $p = (M, M)$.

Proof. There is a point in C that is the closest to p; we can assume that it is the origin. Let Φ be the complex linear functional defined by $\Phi(z) = \langle z, p \rangle$ where $\langle\ ,\ \rangle$ denotes the standard Hermitian product on $\mathbb{C}^2$. Then $\Phi(p) > 0$ while $\operatorname{Re} \Phi \leq 0$ on C.

Since C does not contain any complex line the kernel of Φ is not included in C and so there is a point q in the kernel of Φ such that $q \notin C$. As above there exists a complex linear functional Ψ such that $\operatorname{Re} \Psi(q) > \sup_C \operatorname{Re} \Psi$. Since $0 \in C$ this inequality implies that $\Psi(q) \neq 0$ and therefore Ψ is not a multiple of Φ. Then the pair of functions $\Phi, \Psi - \sup_C \operatorname{Re} \Psi$ provides complex affine change of coordinates so that in the new coordinates $p = (p_1, p_2)$ with $p_1 > 0$ and $C \subset \{(z_1, z_2) \in \mathbb{C}^2; \operatorname{Re} z_1 \leq 0, \operatorname{Re} z_2 \leq 0\}$.

There is $\epsilon > 0$ so small that after an additional change of coordinates of the kind

$$(z_1, z_2) \mapsto (z_1, z_1 + \epsilon(z_2 - i\operatorname{Im} p_2))$$

we have that in the new coordinates $p = (p_1, p_2)$ with $p_1 > 0$ and $p_2 > 0$ and $C \subset \{(z_1, z_2) \in \mathbb{C}^2; \operatorname{Re} z_1 \leq 0, \operatorname{Re} z_2 \leq 0\}$.

After the dilation

$$(z_1, z_2) \mapsto (M z_1 / p_1, M z_2 / p_2)$$

we obtain in the new coordinates $p = (M, M)$ and $C \subset \{(z_1, z_2) \in \mathbb{C}^2; \operatorname{Re} z_1 \leq 0, \operatorname{Re} z_2 \leq 0\}$ which concludes the proof. □

Proof of Theorem 2. By [FG, Theorem 1.3] there is a proper holomorphic map $\psi = (\psi_1, \psi_2) : \triangle \to \mathbb{C}^2$ whose image $\psi(\triangle)$ is contained in $\mathbb{C}^2 \setminus \{z_1 z_2 = 0\}$. So we can define $(\lambda_1, \lambda_2) = (\log \psi_1, \log \psi_2)$. The map (λ_1, λ_2) is a proper holomorphic map from the unit disc to $\mathbb{C}^2$. Note that

$$\max\{\operatorname{Re} \lambda_1(\zeta), \operatorname{Re} \lambda_2(\zeta)\} \to \infty \text{ when } |\zeta| \to 1\,.$$

After adding a suitable large constant we get a proper holomorphic map whose image avoids the set $\{(z_1, z_2) \in \mathbb{C}^2; \operatorname{Re} z_1 \leq 0, \operatorname{Re} z_2 \leq 0\}$ and hits the point (M, M) for some $M > 0$. By Lemma 3 this proves the theorem. □

Remark. We do not know whether Theorem 1 holds for proper holomorphic embeddings. From Lemma 4 below and from the proof of Theorem 2 we conclude that this question has an affirmative answer if [FG, Theorem 1.3] can be extended to proper holomorphic embeddings, that is, if there is a proper holomorphic embedding of the unit disc into $\mathbb{C}^2$ whose image does not intersect coordinate axes.

Lemma 4 *Let C be a complex line in $\mathbb{C}^2$ and $p \notin C$. Then there is a proper holomorphic embedding $\varphi : \triangle \to \mathbb{C}^2$ such that $\varphi(0) = p$ and $\varphi(\triangle) \cap C = \emptyset$.*

Proof. It is enough to prove that there is a proper holomorphic embedding $\varphi = (\varphi_1, \varphi_2) : \triangle \to \mathbb{C}^2$ such that $\varphi_2(\zeta) \neq 0$ $(\zeta \in \triangle)$.

By [RR, p. 78] there is a biholomorphic map $\Phi = (\Phi_1, \Phi_2) : \mathbb{C}^2 \to \mathbb{C}^2$ such that Φ fixes both coordinate axes and such that $\Omega = \Phi(\mathbb{C}^2)$ is not dense in $\mathbb{C}^2$. Domain Ω is a basin of attraction and thus a Runge domain.

We can assume that $(1, 1) \notin \Omega$. Let D be the connected component of the intersection of Ω with the complex line $z_2 = 1$ which contains $(0, 1)$. The fact that Ω is Runge implies that D is biholomorphically equivalent to the unit disc; let $h : \triangle \to D$ be a biholomorphic map. Denote by Σ the z_2-axis and by Ψ the inverse of Φ. Since Ψ is one to one and $D \cap \Sigma = \emptyset$, it follows that $\Psi(D) \cap \Psi(\Sigma) = \emptyset$ and so $\Psi(D) \cap \Sigma = \emptyset$ as $\Psi(\Sigma) = \Sigma$. Then the map $\varphi = \Psi \circ h$ is a proper holomorphic embedding of the unit disc into $\mathbb{C}^2$ such that $\varphi_2(\zeta) \neq 0$ $(\zeta \in \triangle)$. This completes the proof. □

Remark. Let C be a closed convex set in $\mathbb{C}^N$, $N \geq 3$, and $p \notin C$. Then there is a proper holomorphic embedding $\varphi : \triangle \to \mathbb{C}^N$ such that $\varphi(0) = p$ and $\varphi(\triangle) \cap C = \emptyset$. To prove this, note first that there is a real affine hyperplane H through p such that $C \cap H = \emptyset$ and denote by $H^C \subset \mathbb{C}^N$ the unique complex affine hyperplane through p contained in H. Then the complex dimension of H^C is at least 2 and so there exists a proper holomorphic embedding $\varphi : \triangle \to H^C$ such that $\varphi(0) = p$.

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References

[Ale] H. Alexander, On a problem of Julia, Duke Math. J. **42** (1975), 327–332.

[FG1] F. Forstnerič, J. Globevnik, Discs in pseudoconvex domains, Comment. Math. Helvetici **67** (1992), 129–145.

[FG] F. Forstnerič, J. Globevnik, Proper holomorphic discs in $\mathbb{C}^2$, Math. Res. Lett. **8** (2001), 257–274.

[Glo] J. Globevnik, Discs in Stein manifolds, Indiana Univ. Math. J. **49** (2000), 553–574.

[Jul] G. Julia, Sur le domaine d'existence d'une fonction implicite définie par une relation entière $G(x, y) = 0$, Bull. Soc. Math. France **54** (1926), 26–37.

[RR] J.P. Rosay, W. Rudin, Holomorphic maps from $\mathbb{C}^N$ to $\mathbb{C}^N$, Trans. Amer. Math. Soc. **310** (1988), 47–86.

[Tsu] M. Tsuji, Theory of meromorphic functions in a neighborhood of a closed set of capacity zero, Japanese J. Math. **19** (1944), 139–154.