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Discs in Stein manifolds containing given discrete sets

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Abstract. Denote by $\triangle$ the open unit disc in $\mathbb{C}$. We prove that given a discrete subset S of a connected Stein manifold M there is a proper holomorphic map $f : \triangle \to M$ such that $S \subset f(\triangle)$; if $\dim M \geq 3$ the map f can be chosen to be an embedding. In addition we prove that we can prescribe higher order contacts of $f(\triangle)$ with given one dimensional submanifolds in M.

1 Introduction and the results

Denote by $\triangle$ the open unit disc in $\mathbb{C}$. It is known that given a discrete subset S of a convex domain $D \subset \mathbb{C}^N$ there is a proper holomorphic map $f : \triangle \to D$ such that $S \subset f(\triangle)$; if $N \geq 3$ the map f can be chosen to be an embedding [G2]. It is also known that given a Stein manifold M, $\dim M \geq 2$, a point $z \in M$ and a direction $X \in T_zM \setminus \{0\}$ there is a proper holomorphic map $f : \triangle \to M$ such that $f(0) = z$ and $f'(0) = \lambda X$ for some $\lambda > 0$ [G1],[FG]. Our main result generalizes both these results.

Theorem 1.1 *Let $\{z_n; n \in \mathbb{N}\}$ be a discrete set of a connected Stein manifold M with $\dim M \geq 2$. There is a proper holomorphic immersion $f : \triangle \to M$ such that $z_n \in f(\triangle)$ $(n \in \mathbb{N})$.*

In addition, if $\dim M \geq 3$ then there is such f which is a proper holomorphic embedding.

In fact we shall prove a stronger result. We shall prescribe higher order contact of $f(\triangle)$ with given one dimensional submanifolds at the points z_n. Before we state our theorem, we explain what we mean by the contact of at least order k:

Let N and P be p-dimensional submanifolds of a complex manifold M. If N and P intersect at a point $z_0 \in M$, we shall say that *N and P have contact of at least order 0 at z_0*. If N and P intersect at a point $z_0 \in M$ and if $T_{z_0}N = T_{z_0}P$ we shall say that *N and P have contact of at least order 1 at z_0*. In this case one can choose a holomorphic coordinate system (U, ϕ) around z_0 and a complex subspace $L \subset \mathbb{C}^{\dim M}$ such that $T_{\phi(z_0)}\phi(N) \oplus L = \mathbb{C}^{\dim M}$. Write $z = (z', z'')$ with $z' \in T_{\phi(z_0)}\phi(N)$ and $z'' \in L$. There are a neighborhood $U \subset T_{\phi(z_0)}\phi(N)$ of 0 and holomorphic maps $g_N : U \to L$, $g_P : U \to L$ satisfying $g_N(0) = g_P(0) = 0$ and $Dg_N(0) = Dg_P(0) = 0$ such that near the point $\phi(z_0)$, $\phi(N)$ is given by $\{z_0+(z', g_N(z')); z' \in U\}$ and $\phi(P)$ is given by $\{z_0+(z', g_P(z')); z' \in U\}$. If the maps g_N and g_P have the same k-jets at 0 we say that *N and P have contact of at least order k at z_0*. It is easy to see that in this way the contact of at least order k is well defined.

Theorem 1.2 *Let $\{z_n; n \in \mathbb{N}\}$ be a discrete set of a connected Stein manifold M, $\dim M \geq 2$. Then there are a sequence $\{\zeta_n\} \subset \triangle$ and a proper holomorphic immersion $f : \triangle \to M$ such that $f(\zeta_n) = z_n$ for each $n \in \mathbb{N}$.*

Moreover, given a sequence $\{X_n \in T_{z_n}M \setminus \{0\}\}$, f and $\zeta_n \in \triangle$ can be chosen so that for each $n \in \mathbb{N}$ there is $\lambda_n > 0$ such that $f'(\zeta_n) = \lambda_n X_n$.

Moreover, given a sequence of local one-dimensional complex submanifolds $\{N_n\}$ in M such that $z_n \in N_n$ and $X_n \in T_{z_n}N_n$ for each $n \in \mathbb{N}$ and given a sequence of positive integers $\{k_n\}$, there are a proper holomorphic immersion $f : \triangle \to M$, $\zeta_n \in \triangle$ and neighborhoods $\mathcal{W}_n$ of ζ_n in $\triangle$ such that for each $n \in \mathbb{N}$, $f(\zeta_n) = z_n$, $f'(\zeta_n) = \lambda_n X_n$ for some $\lambda_n > 0$ and the manifolds $f_n(\mathcal{W}_n)$ and N_n have contact of at least order k_n at z_n.

In addition, if $\dim M \geq 3$ then the maps f can be chosen to be proper holomorphic embeddings.

2 Preliminaries

By the embedding theorem for Stein manifolds [H] we may assume that M is a closed submanifold of $\mathbb{C}^N$ for some $N \in \mathbb{N}$. By the theorem of Docquier and Grauert [GR, pp. 257] there are an open neighborhood E of M in $\mathbb{C}^N$ and a holomorphic map $\pi : E \to M$ such that $\pi(z) = z$ ($z \in M$).

Throughout the paper we denote by B the unit ball in $\mathbb{C}^N$. Let $\rho_a(z) = |z - a|^2$ $(a \in \mathbb{C}^N,\ z \in \mathbb{C}^N)$. Sard's theorem implies that for almost every $a \in B$ the function ρ_a is a Morse function on M. It is easy to see that $\rho_a(z_n)$ is a regular value of $\rho_a|M$ if and only if the sphere $\{z \in \mathbb{C}^N; |z-a| = |z_n-a|\}$ intersects M transversely. Fix $n \in \mathbb{N}$. For almost every $a \in B$ the sphere $\{z \in \mathbb{C}^N; |z - a| = |z_n - a|\}$ intersects M transversely (see [GP, pp. 68]). Therefore for almost every $a \in B$ and for all $n \in \mathbb{N}$ the sphere $\{z \in \mathbb{C}^N; |z - a| = |z_n - a|\}$ intersects M transversely and ρ_a is a Morse function on M. Thus, after translating M for a suitable small a we may assume that the function $\rho = \rho_0$ is a Morse function on M and $\rho(z_n)$ is a regular value of $\rho|M$ for each $n \in \mathbb{N}$.

We shall frequently use the following lemma proved by R. Narasimhan [N].

Lemma 2.1 *Let U be a neighborhood of a compact set K in $\mathbb{C}$.*

If $f : U \to \mathbb{C}^N$ is a holomorphic, regular and one to one map, then there is an $\epsilon > 0$ such that for a holomorphic map $g : U \to \mathbb{C}^N$ with $|g(\zeta)| < \epsilon$ $(\zeta \in U)$ the map $f + g$ is regular and one to one on K.

If $f : U \to \mathbb{C}^N$ is a regular holomorphic map, then there is an $\epsilon > 0$ such that for a holomorphic map $g : U \to \mathbb{C}^N$ with $|g(\zeta)| < \epsilon$ $(\zeta \in U)$ the map $f + g$ is regular on K.

3 Outline of the proof

In the proof we use the following lemma about pushing the boundaries of analytic discs in M to higher levels of the exhaustion function:

Lemma 3.1 *Let $a < b < A < B < \infty$. Assume that ρ has no critical value on $[a, b] \cup [A, B]$.*

Suppose that $f : \bar{\triangle} \to M$ is a continuous map, holomorphic on $\triangle$, such that $a < \rho(f(\zeta)) < b$ $(\zeta \in b\triangle)$. Given $\zeta_1, \dots, \zeta_n \in \triangle$, $K \in \mathbb{N}$, R, $0 < R < 1$, and $\epsilon > 0$ there are r, $R < r < 1$, and a continuous map $g : \bar{\triangle} \to M$, holomorphic on $\triangle$, such that

(i) $A < \rho(g(\zeta)) < B \quad (\zeta \in b\triangle)$
(ii) $\rho(g(t\zeta)) \geq \rho(f(\zeta)) - \epsilon \quad (\zeta \in b\triangle,\ r \leq t \leq 1)$
(iii) $|g(\zeta) - f(\zeta)| < \epsilon \quad (|\zeta| \leq r)$
(iv) $g^{(j)}(\zeta_i) = f^{(j)}(\zeta_i) \quad (0 \leq j \leq K,\ 1 \leq i \leq n)$

Given $\delta > 0$ there is a map g that, in addition, satisfies

(v) $\rho(g(\zeta)) > \rho(f(\zeta)) - \delta \quad (\zeta \in \bar{\triangle})$.

In the proof of our theorems the map will be obtained as the limit of a sequence of maps constructed in an induction process. (iii) above will be necessary for convergence and (i) and (v) will be necessary to obtain a proper map in the limit.

Let $S = \{z_n; n \in \mathbb{N}\}$. We choose an increasing sequence U_n of the components of the sublevel sets of ρ such that their union is M.

We construct the desired map by induction. At each induction step we begin with an analytic disc which hits the points of $S \cap U_n$ and whose boundary is close to the boundary of U_n. We push the boundary of this disc close to the boundary of U_{n+1} and we construct for each point in $S\cap(U_{n+1}\backslash U_n)$ an analytic disc that hits that point and such that its boundary is close to the boundary of U_{n+1}. Then we glue these discs together by paths which are close to the boundary of U_{n+1}. Then we apply the Mergelyan approximation theorem in the ambient space and thus obtain an analytic disc which hits $S \cap U_{n+1}$ and whose boundary is close to the boundary of U_{n+1}.

Additional care in the construction is necessary to insure that the limit map is an immersion and that it hits the prescribed points in the prescribed directions and has given finite order contacts with the prescribed submanifolds in M.

4 Pushing the boundaries of the disc to higher levels of ρ

Lemma 3.1 is actually a generalization of Lemma 9.1 in [G1]. The main modification of the proof is the generalization of the construction of the continuous family of analytic discs from the case when $\dim M = 2$ to the case when $\dim M \geq 3$. The construction goes as follows:

Let $m = \dim M$. For each $q \in \mathbb{C}^N \setminus \{0\}$ let $E(q) = \{z \in \mathbb{C}^N; \langle z - q, q\rangle = 0\}$ be the affine complex hyperplane passing through q and tangent to the sphere $b(qB)$, and for each $q \in M$ let $T(q)$ be the affine complex subspace of dimension m passing through q and tangent to M at q.

Assume that $Q \subset M$ is a compact set consisting of regular points of ρ. For each $q \in Q$, $T(q)$ intersects $E(q)$ transversely, so $E(q) \cap T(q) = L(q)$ is an affine complex subspace of dimension $m - 1$ and near q, $E(q) \cap M$ is $m - 1$ dimensional submanifold of M tangential to $L(q)$ at q. Therefore there are $\delta > 0$ and a map $g_q : L(q) \cap (q + \delta B) \to L(q)^{\perp} = \{z \in \mathbb{C}^N;\ \langle z, w\rangle = 0, \forall w \in L(q)\}$ satisfying $g_q(q) = 0$, $Dg_q(q) = 0$ such that $M\cap E(q)\cap(q+\delta B) = \{z+g_q(z); z \in L(q)\cap(q+\delta B)\}\cap(q+\delta B)$. Taking smaller δ if necessary for each r, $0 < r < \delta$, and for each one dimensional affine subspace $N(q)$ of $L(q)$ through q the analytic disc $\{z + g_q(z); z \in N(q) \cap (q + \delta B)\}$ intersects $b(q + rB)$ transversely and the intersection $\{z+g_q(z); z \in N(q)\cap(q+\delta B)\}\cap(q+rB)$ is biholomorphically equivalent to a disc. Since Q is compact, a $\delta > 0$ can be chosen that works for all $q \in Q$.

Since $E(q)$ is orthogonal to q it follows that the spheres in $E(q)$ centered at q are the level sets of the function $z \mapsto |z|^2$ restricted to $E(q)$. In particular $\rho(w) = |q|^2 + r^2 = \rho(q) + r^2$ $(w \in \{z + g_q(z); z \in L(q) \cap (q + \delta B)\} \cap b(q + rB))$.

By transversality everything varies smoothly with $q \in M$ and r, $0 < r < \delta$.

Lemma 4.1 *Given a compact set $Q \subset M$ of regular points of $\rho|M$ there is a $\mu_0 > 0$ such that for every positive continuous function μ on $b\triangle$ that* *satisfies* $\mu(\zeta) < \mu_0$ $(\zeta \in b\triangle)$ *and for every continuous map* $f : b\triangle \to Q$ *there is a continuous map* $F : b\triangle \times \bar{\triangle} \to M$ *such that*

(i) *for each* $\zeta \in b\triangle$ *the function* $\eta \mapsto F(\zeta, \eta)$ *is holomorphic on* $\triangle$
(ii) $F(\zeta, 0) = f(\zeta) \quad (\zeta \in b\triangle)$
(iii) $\rho(F(\zeta, \eta)) > \rho(f(\zeta)) \quad (\zeta \in b\triangle, \eta \in \bar{\triangle} \setminus \{0\})$
(iv) $\rho(F(\zeta, \eta)) = \rho(f(\zeta)) + \mu(\zeta) \quad (\zeta \in b\triangle, \eta \in b\triangle)$.

Proof. Let δ, $L(q)$ and g_q be as in the preceding discussion and put $\mu_0 = \delta^2$. Since $f : b\triangle \to Q$ is continuous, the set $\cup_{\zeta \in b\triangle}\{\zeta\} \times L(f(\zeta))$ is a complex vector bundle of dimension $m - 1$ and there exists an one dimensional subbundle $\cup_{\zeta \in b\triangle}\{\zeta\} \times N(f(\zeta))$. The preceding discussion shows that for each $\zeta \in b\triangle$ the sphere $b(f(\zeta) + \mu(\zeta)^{\frac{1}{2}}B)$ intersects $\{z + g_q(z); z \in N(q) \cap (q + \delta B)\}$ transversely and $D(\zeta) = \{z + g_q(z); z \in N(q) \cap (q + \delta B)\} \cap (f(\zeta) + \mu(\zeta)^{\frac{1}{2}}B)$ is biholomorphically equivalent to a disc. If w belongs to the boundary of this disc, that is, if $w \in \{z + g_q(z); z \in N(q) \cap (q + \delta B)\} \cap b(f(\zeta) + \mu(\zeta)^{\frac{1}{2}}B)$ then $\rho(w) = \rho(f(\zeta)) + \mu(\zeta)$. By the transversality and by the continuity of f and μ the discs $D(\zeta)$ change continuously with ζ.

The rest of the proof is the same as the proof of Lemma 4.1 in [G1].

5 Construction of a disc through a given point

In this section we show how to construct a disc through a prescribed point tangent to a given submanifold in M at this point. In the proof of Theorem 1.2 we shall glue these discs together.

Lemma 5.1 *Let* N *be a local one dimensional complex submanifold in* M*,* p *a point in* N *such that* $\rho(p)$ *is a regular value of* $\rho|M$*,* X *a tangent vector to* N *at* p *and* $K \in \mathbb{N} \cup \{0\}$. *Given* $\eta > 0$*,* $\delta > 0$ *and a regular value* a *of* $\rho|M$ *such that* $a > \rho(p)$ *there exists a continuous map* $f : \bar{\triangle} \to M$*, holomorphic on* $\triangle$*, such that*

(i) $f(0) = p$, $f'(0) = \lambda X$ *for some* $\lambda > 0$ *and there is a neighborhood* $\mathcal{W}$ *of* 0 *such that* $f(\mathcal{W})$ *and* N *have contact of at least order* K *at* p.
(ii) $a - \eta < \rho(f(\zeta)) < a \quad (\zeta \in b\triangle)$
(iii) $\rho(f(\zeta)) \geq \rho(p) - \delta \quad (\zeta \in \bar{\triangle})$.

Proof. Since N is a one dimensional complex submanifold of M through p, near p, N is a graph over its (complex) tangent space at p. In this way we obtain a small holomorphic disc $g : \bar{\triangle} \to M$ such that

(i) $g(0) = p$, $g'(0) = \lambda X$ for some $\lambda > 0$ and there is a neighborhood $\mathcal{U}$ of 0 such that $g(\mathcal{U})$ and N have contact of at least order K at p

(ii) $\rho(g(\zeta))$ is a regular value of $\rho|M \quad (\zeta \in b\triangle)$
(iii) $\rho(g(\zeta)) \geq \rho(p) - \frac{\delta}{2} \quad (\zeta \in \bar{\triangle})$.

We now use Lemma 3.1 to obtain a continuous map $f : \bar{\triangle} \to M$, holomorphic on $\triangle$, such that

(i) $f^{(j)}(0) = g^{(j)}(0) \quad (0 \leq j \leq K)$
(ii) $a - \eta < \rho(f(\zeta)) < a \quad (\zeta \in b\triangle)$
(iii) $\rho(f(\zeta)) \geq \rho(g(\zeta)) - \frac{\delta}{2} \quad (\zeta \in \bar{\triangle})$.

This f meets all the conditions in the lemma. The proof is complete.

6 Perturbing *f* to get a regular map

Recall that E is a neighborhood of M in $\mathbb{C}^N$ and $\pi : E \to M$ is a holomorphic retraction.

As described in the outline we shall prove Theorem 1.2 inductively. At each inductive step our map will be regular on a certain compact subset of $\triangle$. To get such a map we perform a small perturbation.

Lemma 6.1 *Let $f : \bar{\triangle} \to M$ be a nonconstant continuous map, holomorphic on $\triangle$. Suppose that $\zeta_1, \ldots, \zeta_n \in \triangle$ are regular points of f. Given $U \subset\subset \triangle$, $K \in \mathbb{N}$ and $\epsilon > 0$ there is a continuous map $g : \bar{\triangle} \to \mathbb{C}^N$, holomorphic on $\triangle$, such that*

(i) $|g(\zeta)| < \epsilon \quad (\zeta \in \bar{\triangle})$
(ii) $(f + g)(\bar{\triangle}) \subset M$
(iii) $f + g$ *is regular on* U
(iv) $(f + g)^{(i)}(\zeta_j) = f^{(i)}(\zeta_j) \quad (1 \leq i \leq K,\ 1 \leq j \leq n)$.

Proof. Since f is nonconstant there are only finitely many points in U at which f' vanishes. Let $\{\zeta \in U; f'(\zeta) = 0\} = \{\eta_1, \ldots, \eta_s\}$ and $z_j = f(\eta_j)$, $1 \leq j \leq s$.

Choose j, $1 \leq j \leq s$. Since f is nonconstant there are $m_j \in \mathbb{N}$ and a holomorphic map $h_j : \triangle \to \mathbb{C}^N$ such that $f'(\zeta) = (\zeta - \eta_j)^{m_j} h_j(\zeta)$ $(\zeta \in \triangle)$ and $h_j(\eta_j) \neq 0$. Since $\dim M \geq 2$ there is a vector $B(j) \in T_{z_j}M$ such that $h_j(\eta_j)$ and $B(j)$ are linearly independent. Since $T_{z_j}M \cap T_{z_j}\pi^{-1}(z_j) = \{0\}$ there are a neighborhood (in a Grassmann manifold) $\mathcal{U}_j$ of $T_{z_j}\pi^{-1}(z_j)$ and $\nu_j > 0$ such that

$$\begin{aligned}&\text{if } A, B \in \mathbb{C}^N,\ |A - h_j(\eta_j)| < \nu_j,\ |B - B(j)| < \nu_j,\ U \in \mathcal{U}_j \\ &\text{then } U \cap \operatorname{Span}\{A, B\} = \{0\}.\end{aligned} \tag{1}$$

As the map $\pi : E \to M$ is a holomorphic retraction the rank of π on M is maximal and constant. Therefore the rank of π is constant in the neighborhood of M and the rank theorem implies that locally in the neighborhood of each point $z \in M$ in $\mathbb{C}^N$ the map π is a holomorphic projection. So there is $\delta_j > 0$ such that

$$z \in \mathbb{C}^N,\ |z - z_j| < \delta_j \text{ implies that } T_z\pi^{-1}(\pi(z)) \in \mathcal{U}_j\ . \tag{2}$$

Choose a holomorphic polynomial $P : \mathbb{C} \to \mathbb{C}^N$ such that $P'(\eta_j) = B(j)$ $(1 \le j \le s)$ and $P^{(k)}(\zeta_i) = 0$ $(0 \le k \le K,\ 1 \le i \le n)$.

Choose $\lambda > 0$ so small that for j, $1 \le j \le s$, and for ζ, $|\zeta - \eta_j| < \lambda$, we have

$$|h_j(\zeta) - h_j(\eta_j)| < \nu_j \text{ and } |P'(\zeta) - B(j)| < \nu_j. \tag{3}$$

Taking smaller λ if necessary, there is an $\alpha_0 > 0$ such that for each j, $1 \le j \le s$, for each ζ, $|\zeta - \eta_j| < \lambda$ and for each α, $0 < \alpha < \alpha_0$, we have

$$|f(\zeta) + \alpha P(\zeta) - z_j| < \delta_j. \tag{4}$$

Since f is regular on $U \setminus \cup_{i=1}^{s}\{\eta_i\}$ there is ϵ_1, $0 < \epsilon_1 < \epsilon$, such that for any holomorphic map $g : \triangle \to \mathbb{C}^N$ with $|g(\zeta)| < \epsilon_1$ $(\zeta \in \triangle)$ the map $f + g$ is regular on $U \setminus \cup_{i=1}^{s}\{\zeta; |\zeta - \eta_i| < \lambda\}$. One can choose ϵ_2, $\epsilon_2 > 0$, such that for a map $h : \bar{\triangle} \to \mathbb{C}^N$, with $|h(\zeta)| < \epsilon_2$ $(\zeta \in \bar{\triangle})$ we have $f(\zeta) + h(\zeta) \in E$ and $|\pi(f(\zeta) + h(\zeta)) - f(\zeta)| < \epsilon_1$ $(\zeta \in \bar{\triangle})$.

Take α, $0 < \alpha < \alpha_0$, so small that $|\alpha P(\zeta)| < \epsilon_2$ $(\zeta \in \bar{\triangle})$ and let $g(\zeta) = \pi(f(\zeta) + \alpha P(\zeta)) - f(\zeta)$. Then (ii) is satisfied. According to the choice of ϵ_2, we have $|g(\zeta)| < \epsilon_1$ $(\zeta \in \bar{\triangle})$, which proves (i), and proves that $f + g$ is regular on $U \setminus \cup_{i=1}^{s}\{\zeta; |\zeta - \eta_i| < \lambda\}$. Further, let $1 \le j \le s$ and $|\zeta - \eta_j| < \lambda$. We have $(f+g)'(\zeta) = D\pi(f(\zeta) + \alpha P(\zeta))(f'(\zeta) + \alpha P'(\zeta))$. Since $\ker D\pi(z) = T_z\pi^{-1}(\pi(z))$ $(z \in E)$ it follows by (4), (2), (3) and (1) that $f'(\zeta) + \alpha P'(\zeta) \notin \ker D\pi(f(\zeta) + \alpha P(\zeta))$. This proves (iii). (iv) follows from the fact that $P^{(k)}(\zeta_i) = 0$ $(0 \le k \le K,\ 1 \le i \le n)$ and that $\pi|M = id$. This completes the proof.

7 Removing the selfintersection points of properly immersed discs

Lemma 7.1 *Let P be a domain in $\mathbb{C}^N$ and $m = \dim M \ge 3$. Let $f : \bar{\triangle} \to M$ be a continuous map, holomorphic on $\triangle$, and $\mathcal{U} \subset\subset \triangle$ conformally equivalent to the disc, such that $f|\mathcal{U} : \mathcal{U} \to P$ is a proper map, regular on $U \subset\subset \mathcal{U}$ and a normalization map for the variety $f(\mathcal{U}) \subset P$. Let $W \subset\subset U$ be a domain and suppose that $\zeta_1, \ldots, \zeta_n \in \triangle$, $f(\zeta_i) \ne f(\zeta_j) (i \ne j, 1 \le i \le n, 1 \le j \le n)$. Given $K \in \mathbb{N}$ and $\epsilon > 0$ there is a continuous map $g : \bar{\triangle} \to \mathbb{C}^N$, holomorphic in $\triangle$, such that*

(i) $|g(\zeta)| < \epsilon \quad (\zeta \in \bar{\triangle})$
(ii) $(f + g)(\bar{\triangle}) \subset M$
(iii) *$f + g$ is regular and one to one on W*

(iv) $(f+g)^{(i)}(\zeta_j) = f^{(i)}(\zeta_j) \quad (1 \le i \le K,\ 1 \le j \le n).$

In the proof of Lemma 7.1 we need the following lemma

Lemma 7.2 *Let $f, g : \triangle \to M$, $m = \dim M \ge 2$, be holomorphic maps such that $f(0) = g(0)$, $f'(0) \ne 0$, $g'(0) \ne 0$. Let $P_j : \mathbb{C}^N \to \mathbb{C}^N$ $(1 \le j \le m-1)$ be holomorphic polynomial maps such that $P_1(f(0)), \dots P_{m-1}(f(0))$ are linearly independent, $P_j(f(0)) \in T_{f(0)}M$ $(1 \le j \le m-1)$ and $f'(0), g'(0) \notin \mathrm{Span}\{P_1(f(0)), \dots P_{m-1}(f(0))\}$. Assume that ϕ and ψ are holomorphic functions on $\triangle$ such that $\phi(0) \ne \psi(0)$. There are $\mu > 0$ and $\tau > 0$ with the following property: The set of all $\lambda = (\lambda_1, \dots, \lambda_{m-1}) \in \mathbb{C}^{m-1}$, $|\lambda| < \mu$, such that*

$$\begin{aligned}&\{\pi(f(\zeta) + \phi(\zeta) \textstyle\sum_{j=1}^{m-1} \lambda_j P_j(f(\zeta))); |\zeta| < \tau\}\\ &\cap\{\pi(g(\zeta) + \psi(\zeta) \textstyle\sum_{j=1}^{m-1} \lambda_j P_j(g(\zeta))); |\zeta| < \tau\} \ne \emptyset\end{aligned}$$

has three dimensional Hausdorff measure zero.

Proof. Choose $\alpha > 0$ so small that for each λ, $|\lambda| < \alpha$, and for each $\zeta \in \triangle$ we have $f(\zeta) + \phi(\zeta) \sum_{j=1}^{m-1} \lambda_j P_j(f(\zeta)) \in E$ and $g(\zeta) + \phi(\zeta) \sum_{j=1}^{m-1} \lambda_j P_j(g(\zeta)) \in E$. Let

$$A = \Bigg\{(\zeta, \eta, \lambda) \in \triangle \times \triangle \times \{z \in \mathbb{C}^{m-1}; |z| < \alpha\};$$
$$\pi(f(\zeta) + \phi(\zeta) \sum_{j=1}^{m-1} \lambda_j P_j(f(\zeta))) = \pi\Big(g(\eta) + \psi(\eta) \sum_{j=1}^{m-1} \lambda_j P_j(g(\eta))\Big)\Bigg\}.$$

The set A is analytic set in $\triangle \times \triangle \times \{z \in \mathbb{C}^{m-1}; |z| < \alpha\}$. We will show that $0 \in \mathbb{C}^{m+1}$ is an isolated point of $A \cap \{(\zeta, 0, \lambda)\}$.

Let

$$\begin{aligned}H(\zeta, \lambda) = \pi &\left(f(\zeta) + \phi(\zeta) \sum_{j=1}^{m-1} \lambda_j P_j(f(\zeta))\right)\\ &-\pi\left(g(0) + \psi(0) \sum_{j=1}^{m-1} \lambda_j P_j(g(0))\right) \quad (|\zeta| < 1,\ |\lambda| < \alpha).\end{aligned}$$

For $\zeta \in \triangle$ write $P_j(f(\zeta)) = Q_j(\zeta) + R_j(\zeta)$, $1 \le j \le m-1$, where Q_j is orthogonal projection of $P_j(f(\zeta))$ onto $T_{f(\zeta)}M$. The functions Q_j and R_j

are smooth on $\triangle$. Then

$$H(\zeta,\lambda) = \pi\left(f(\zeta) + \phi(\zeta)\sum_{j=1}^{m-1}\lambda_j Q_j(\zeta) + \phi(\zeta)\sum_{j=1}^{m-1}\lambda_j R_j(\zeta)\right) \\ -\pi\left(g(0) + \psi(0)\sum_{j=1}^{m-1}\lambda_j P_j(g(0))\right).$$

As $f(0) = g(0)$, $D\pi(f(\zeta))|T_{f(\zeta)}M = I$, and $\pi(f(\zeta)+h) = f(\zeta) + D\pi(f(\zeta))h + O(|h|^2)$ we have

$$H(\zeta,\lambda) = f(\zeta) + \phi(\zeta)\sum_{j=1}^{m-1}\lambda_j Q_j(\zeta) + D\pi(f(\zeta))\left(\phi(\zeta)\sum_{j=1}^{m-1}\lambda_j R_j(\zeta)\right) \\ +O\left(\left|\phi(\zeta)\sum_{j=1}^{m-1}\lambda_j P_j(f(\zeta))\right|^2\right) - f(0) - \psi(0)\sum_{j=1}^{m-1}\lambda_j P_j(f(0)) \\ -O\left(\left|\psi(0)\sum_{j=1}^{m-1}\lambda_j P_j(f(0))\right|^2\right).$$

By rearranging we get

$$H(\zeta,\lambda) = \left[f(\zeta) - f(0) + (\phi(\zeta)-\phi(0))\sum_{j=1}^{m-1}\lambda_j Q_j(\zeta)\right. \\ \left. +\ \phi(0)\sum_{j=1}^{m-1}\lambda_j (Q_j(\zeta) - Q_j(0)) + D\pi(f(\zeta))\left(\phi(\zeta)\sum_{j=1}^{m-1}\lambda_j R_j(\zeta)\right)\right] \\ +\ \left[(\phi(0)-\psi(0))\sum_{j=1}^{m-1}\lambda_j Q_j(0) + O\left(\left|\phi(\zeta)\sum_{j=1}^{m-1}\lambda_j P_j(f(\zeta))\right|^2\right)\right. \\ \left. -\ O\left(\left|\psi(0)\sum_{j=1}^{m-1}\lambda_j P_j(f(0))\right|^2\right)\right].$$

It is easy to see that

$$f(\zeta) - f(0) = \zeta f'(0) + \zeta^2 O(1) \ \ (\zeta \to 0),$$
$$(\phi(\zeta)-\phi(0))\sum_{j=1}^{m-1}\lambda_j Q_j(\zeta) = \zeta|\lambda|O(1) \ \ ((\zeta,\lambda)\to 0),$$

$$\phi(0)\sum_{j=1}^{m-1}\lambda_j(Q_j(\zeta)-Q_j(0))=\zeta|\lambda|O(1)\ \ ((\zeta,\lambda)\to 0),$$

$$D\pi(f(\zeta))\left(\phi(\zeta)\sum_{j=1}^{m-1}\lambda_j R_j(\zeta)\right)=\zeta|\lambda|O(1)\ \ ((\zeta,\lambda)\to 0),$$

$$O\left(\left|\phi(\zeta)\sum_{j=1}^{m-1}\lambda_j P_j(f(\zeta))\right|^2\right)=|\lambda|^2O(1)\ \ ((\zeta,\lambda)\to 0),$$

$$O\left(\left|\psi(0)\sum_{j=1}^{m-1}\lambda_j P_j(f(0))\right|^2\right)=|\lambda|^2O(1)\ \ (\lambda\to 0).$$

This implies that

$$H(\zeta,\lambda)=\zeta\left[f'(0)+\zeta O(1)+|\lambda|O(1)\right]$$
$$+|\lambda|\left[(\phi(0)-\psi(0))\sum_{j=1}^{m-1}\lambda_j|\lambda|^{-1}P_j(f(0))+|\lambda|O(1)\right]\ \ ((\zeta,\lambda)\to 0).$$

Since $f'(0)\notin \mathrm{Span}\{P_1(f(0)),\dots P_{m-1}(f(0))\}$, there is $\delta>0$ small enough such that for each ζ, $|\zeta|<\delta$, and for each λ, $0<|\lambda|<\delta$, the vectors in the brackets are linearly independent. Therefore for each ζ, $|\zeta|<\delta$, and for each λ, $0<|\lambda|<\delta$ we have $H(\zeta,\lambda)\neq 0$. This, together with the fact that $H(\zeta,0)\neq 0$ for $0<|\zeta|<\delta$, implies that $A\cap\{(\zeta,0,\lambda)\in\mathbb{C}^{m+1}; |\zeta|<\delta,\ \ |\lambda|<\delta\}=\{0\}$, that is, 0 is an isolated point of $A\cap\{(\zeta,0,\lambda)\in\mathbb{C}^{m+1}; |\zeta|<\delta,\ \ |\lambda|<\delta\}$. Therefore by [C, page 34] $\dim_0 A\le 1$. So there is a neighborhood U of 0 in $\mathbb{C}^{m+1}$ such that $\dim(A\cap U)\le 1$. Then the three dimensional Hausdorff measure of $A\cap U$ is zero. Let $\Pi:\mathbb{C}^{m+1}\to\mathbb{C}^{m-1}$ be the projection $(z_1,z_2,z')\mapsto z'$. So the set $\Pi(A\cap U)$ has three dimensional Hausdorff measure zero as well. Choose $\tau>0$ and $\mu>0$ small enough such that $\{(\zeta,\eta,\lambda); |\zeta|<\tau,\ |\eta|<\tau,\ |\lambda|<\mu\}\subset U$ and the lemma follows.

Proof of Lemma 7.1 . The proof of the lemma is similar to the proof of Lemma 6.1 in [G2]. Let S be the set of singular points of $V=f(\mathcal{U})$ and $T=f^{-1}(S)$. Since f is a normalization map for V and since f is regular on U the map $f|[(\mathcal{U}\setminus T)\cup\{\zeta\}]\to(V\setminus S)\cup\{f(\zeta)\}$ is regular and one to one for $\zeta\in U\cap T$ (see Appendix).

Let $U\cap T=\{\eta_1,\dots,\eta_s\}$ and $f(U\cap T)=\{z_1,\dots,z_j\}$ where $z_1,\dots,z_j$ are distinct. With no loss of generality we may assume that there are integers m_i $(1\le i\le j+1)$ such that $f(\eta_l)=z_i$ $(m_i\le l<m_{i+1},\ 1\le i\le j)$

and if $\zeta_l \in \{\eta_{m_i}, \eta_{m_i+1}, \ldots, \eta_{m_{i+1}-1}\}$ then $\zeta_l = \eta_{m_i}$. Choose holomorphic polynomial maps $P_1, \ldots, P_{m-1} : \mathbb{C}^N \to \mathbb{C}^N$ such that for i, $1 \le i \le j$,

(i) $P_1(z_i), \ldots, P_{m-1}(z_i)$ are linearly independent and $\mathrm{Span}\{P_1(z_i), \ldots, P_{m-1}(z_i)\} \subset T_{z_i}M$,
(ii) $f'(\eta_l) \notin \mathrm{Span}\{P_1(z_i), \ldots, P_{m-1}(z_i)\}$ $(m_i \le l < m_{i+1})$.

Let ϕ be a polynomial such that $\phi(\eta_{m_i+l}) = l$ $(0 \le l < m_{i+1} - m_i,\ 1 \le i \le j)$, $\phi^{(k)}(\zeta_i) = 0$ $(0 \le k \le K, 1 \le i \le n)$.

By Lemma 7.2 there are $\mu > 0$ and $\tau > 0$ with the following property: The set of all $\lambda \in \mathbb{C}^{m-1}$, $|\lambda| < \mu$, such that $\{\pi(f(\zeta) + \phi(\zeta) \sum_{i=1}^{m-1} \lambda_i P_i(f(\zeta))); |\zeta - \eta_k| < \tau\} \cap \{\pi(f(\zeta) + \phi(\zeta) \sum_{i=1}^{m-1} \lambda_i P_i(f(\zeta)));$ $|\zeta - \eta_l| < \tau\} \ne \emptyset$ for at least one pair $k,\ l,\ k \ne l,\ 1 \le k,\ l \le s$ has three dimensional Hausdorff measure zero.

With no loss of generality assume that τ is so small that $\eta_i + \tau\triangle \subset U$, $1 \le i \le s$, are pairwise disjoint and that W is so large that $\eta_i + \tau\triangle \subset W$, $1 \le i \le s$. Since $m \ge 3$ it follows that for each $\epsilon > 0$ one can choose $\lambda \in \mathbb{C}^{m-1}$, $|\lambda| < \epsilon$, such that

$$\left\{\pi(f(\zeta) + \phi(\zeta) \sum_{i=1}^{m-1} \lambda_i P_i(f(\zeta))); |\zeta - \eta_k| < \tau\right\} \cap \left\{\pi(f(\zeta)\right.$$
$$\left. + \phi(\zeta) \sum_{i=1}^{m-1} \lambda_i P_i(f(\zeta))); |\zeta - \eta_l| < \tau\right\} = \emptyset \ \ (1 \le k, l \le s,\ k \ne l) \quad (5)$$

Fix i, $1 \le i \le s$. f is one to one and regular on $U \setminus \{\eta_k; 1 \le k \le s, k \ne i\}$. By Lemma 2.1 it follows that there is an ϵ_1, $0 < \epsilon_1 < \epsilon$, such that for each holomorphic map $g : \triangle \to \mathbb{C}^N$ with $|g(\zeta)| < \epsilon_1$ $(\zeta \in \triangle)$ and for each i, $1 \le i \le s$, the map $f + g$ is regular and one to one on $W \setminus \cup_{k=1,k\ne i}^{s} (\eta_k + \tau\triangle)$. One can choose $\lambda \in \mathbb{C}^{m-1}$ such that (5) holds and such that $g = \pi(f + \phi \sum_{i=1}^{m-1} \lambda_i P_i(f)) - f$ satisfies $|g(\zeta)| < \epsilon_1$ $(\zeta \in \triangle)$, which proves (i). (ii) is satisfied by definition of the map g. In the same way as in the proof of Lemma 6.1 in [G2] we see that $f + g$ is regular and injective on W, which gives (iii). (iv) follows from the fact that $\phi^{(k)}(\zeta_i) = 0$ $(0 \le k \le K, 1 \le i \le n)$ and that $\pi|M = id$. This completes the proof.

8 Proof of Theorem 1.2

We prove Theorem 1.2 in the case $\dim M \ge 3$ and postpone the simpler proof of the case $\dim M = 2$ until the end of this section.

Part 1. We shall rearrange the sequence $\{z_n\}$. We may assume that the sequence $\{\rho(z_n)\}$ is nondecreasing. Since M is connected one can choose an increasing sequence $\{a_n\}$ of regular values of $\rho|M$ converging to infinity with the property that if U_n is the component of the sublevel set $\{z \in M; \rho(z) < a_n\}$ containing z_1 then for each $n \in \mathbb{N}$, U_{n+1} contains the first term in the sequence $\{z_n\}$ that is not contained in U_n and that the boundary of U_n does not contain any point of the sequence $\{z_n\}$. Let $a_{-1} = -\infty$.

Let $S = \{z_n; n \in \mathbb{N}\}$. For each n, let $S_n = S \cap U_n$. Since the sequence $\{a_n\}$ is increasing and converges to infinity and since ρ is an exhaustion function for the connected manifold M it follows that S_n is an increasing sequence of finite sets whose union is S. Thus, if $m(n)$ is the number of points in S_n, $n \in \mathbb{N}$, one can renumber the sequence $\{z_n\}$ so that $S_n = \{z_1, \ldots, z_{m(n)}\}$ and $\rho(z_{m(n)+1}) = \min\{\rho(z_{m(n)+1}), \ldots, \rho(z_{m(n+1)-1})\}$. Let z_0 be a minimum of ρ on U_1 and let $m(0) = 0$, $k_0 = 0$.

Part 2. We shall obtain a regular and injective holomorphic map f_0 to begin the construction. Let $\triangle_0$ be the unit disc centered at 0. Since $\rho|M$ is a Morse function and as z_0 is a minimum of ρ, z_0 is an isolated singular point of $\rho|M$. Locally near z_0, M is a graph over its tangent space at z_0. Therefore there is a regular, one to one holomorphic map $\phi : \triangle \to M$ such that $\phi(0) = z_0$ and $\rho(\phi(\zeta)) > \rho(\phi(0))$ $(\zeta \in \triangle \setminus \{0\})$. There is a regular value a_0 of $\rho|M$, such that $\rho(z_0) < a_0 < \rho(z_1)$ and such that $\{\zeta \in \triangle; \rho(\phi(\zeta)) < a_0\}$ is relatively compact in $\triangle$. By the maximum principle applied to the subharmonic function $\rho\circ\phi$, each connected component of $\{\zeta \in \triangle; \rho(\phi(\zeta)) < a_0\}$ is simply connected, therefore conformally equivalent to the disc. Therefore there are a continuous map $f_0 : \bar{\triangle}_0 \to M$, holomorphic on $\triangle_0$, and $\gamma > 0$ such that

(i) $f_0(0) = z_0$
(ii) $\rho(f_0(\zeta)) = a_0$ $(\zeta \in b\triangle_0)$
(iii) f_0 is one to one and regular on $\triangle_0$
(iv) $\rho(z_0) < a_0 - 4\gamma$.

By Lemma 2.1 there is an $\epsilon_0 > 0$ such that

if $g : \triangle_0 \to \mathbb{C}^N$ is a holomorphic map with $|f_0(\zeta) - g(\zeta)| < 2\epsilon_0$ $(\zeta \in \{\xi \in \triangle_0;\ \rho(f_0(\xi)) < a_0 - \gamma\})$ then g is regular and one to one on $\{\xi \in \triangle_0;\ \rho(f_0(\xi)) < a_0 - 2\gamma\}$. (6)

Taking smaller $\epsilon_0 > 0$ we may assume that

$$\text{if } z, w \in M,\ \rho(z) < a_0 \text{ and } |z - w| < 2\epsilon_0 \text{ then } |\rho(z) - \rho(w)| < \gamma. \tag{7}$$

Part 3. Now we shall construct a sequence of holomorphic maps whose limit will satisfy the conditions of the Theorem 1.2.

Choose a decreasing sequence δ_j of positive numbers converging to 0, $\delta_0 \le \frac{\gamma}{5}$, such that $\rho|M$ has no critical value on the interval $(a_j - 3\delta_j, a_j + \delta_j)$

$(j \in \mathbb{N} \cup \{0\})$, $\rho(z_k) \notin (a_j - 3\delta_j, a_j + \delta_j)$ $(j, k \in \mathbb{N} \cup \{0\})$ and the intervals $(a_j - 3\delta_j, a_j + \delta_j)$ $(j \in \mathbb{N} \cup \{0\})$ are pairwise disjoint.

We will construct

(A) a sequence β_k, $0 < \beta_k < 1$, and a sequence of domains $\triangle_j \subset \mathbb{C}$ such that if D_k is the open disc of radius 1 centered at $3k$ then for given $j \in \mathbb{N}$ the set $\triangle_j$ will be the union of $m(j)$ discs $D_1, \ldots, D_{m(j)}$ and $m(j) - 1$ strips $(3k, 3(k+1)) \times (-\beta_k, \beta_k)$, $1 \le k \le m(j) - 1$,

(B) an increasing sequence Ω_j of connected domains, $\Omega_{-4} = \Omega_{-3} = \Omega_{-2} = \Omega_{-1} = \emptyset$ such that $\{\xi \in \triangle_0; \rho(f_0(\xi)) < a_0 - \gamma\} \subset \Omega_0$, $\Omega_{j-1} \subset\subset \Omega_j$ $(j \ge 1)$, $\Omega_j \subset\subset \triangle_j$ $(j \ge 0)$ and

$$\left\{\xi \in \triangle_j; \mathrm{dist}(\xi, b\triangle_j) > \frac{1}{j}\right\} \subset \Omega_j \quad (j \in \mathbb{N}),$$

(C) a sequence f_j of maps such that for each $j \in \mathbb{N} \cup \{0\}$

(i) $f_j : \bar{\triangle}_j \to M$ is continuous, holomorphic on $\triangle_j$ and such that $\rho(f_j(\zeta)) \in (a_j - 2\delta_j, a_j]$ $(\zeta \in b\triangle_j)$

(ii) f_j is regular on Ω_{j-1} and one to one on Ω_{j-3}

(iii) $\rho(f_j(\zeta)) < a_{j-1} + \delta_{j-1}$ $(\zeta \in \Omega_{j-1})$

(iv) $\rho(f_{j+1}(\zeta)) \ge \min\{\rho(z_{m(j)+1}), a_j\} - \gamma$ $(\zeta \in \triangle_{j+1} \setminus \Omega_j)$

(v) $f_{j+1}(\zeta_i) = z_i$, $f'_{j+1}(\zeta_i) = \mu_i X_i$ for some $\mu_i > 0$ and there is a neighborhood $\mathcal{V}_i$ of $3i$ in D_i such that $f_{j+1}(\mathcal{V}_i)$ and N_i have contact of at least order k_i at z_i $(m(j) + 1 \le i \le m(j+1))$ and $f_{j+1}^{(l)}(\zeta_i) = f_j^{(l)}(\zeta_i)$ $(0 \le l \le k_i,\ 0 \le i \le m(j))$

(vi) $|f_{j+1}(\zeta) - f_j(\zeta)| < \frac{\epsilon_j}{2^j}$ $(\zeta \in \Omega_j)$,

(D) a decreasing sequence ϵ_j of positive numbers converging to 0 such that for each $j \in \mathbb{N}$

(a) if $g : \Omega_{j-3} \to \mathbb{C}^N$ is a holomorphic map such that $|g(\zeta)| < \epsilon_j$ $(\zeta \in \Omega_{j-3})$ then $f_j + g$ is regular and one to one on Ω_{j-4}

(b) if $z, w \in M$, $\min\{\rho(z_{m(j-1)+1}), a_{j-1}\} - \gamma < \rho(z) \le a_j$ and $|z - w| < \frac{\epsilon_j}{2^{j-1}}$ then $\rho(w) > \min\{\rho(z_{m(j-1)+1}), a_{j-1}\} - 2\gamma$,

(E) a sequence of positive numbers α_j, a decreasing sequence of positive numbers λ_j converging to 0, $\lambda_1 = 1$, and a decreasing sequence of positive numbers η_j converging to 0, such that for each $j \in \mathbb{N}$, $0 < \lambda_j < \min\{\frac{\epsilon_j}{4 \cdot 2^j}, \frac{\alpha_j}{4}\}$, $0 < \eta_j < \frac{\lambda_j}{2}$ and

$$z, w \in M,\ \rho(z) \le a_{j+1},\ |z - w| < \lambda_j \text{ implies that}$$
$$|\rho(z) - \rho(w)| < \frac{\delta_{j+1}}{4} \tag{8}$$
$$z \in M,\ \rho(z) \le a_{j+1},\ w \in \mathbb{C}^N \text{ and } |w - z| < \eta_j \text{ implies that}$$
$$w \in E,\ |\pi(w) - w| < \frac{\lambda_j}{2}. \tag{9}$$

Now we shall briefly explain the inductive construction. In (A) we describe domains where the maps f_j are defined. In (B) we define subdomains $\Omega_j \subset \triangle_j$ where we approximate f_{j+1} by f_j. In (C) we describe the properties of the maps f_j: (iv) and (vi) together with (D) will be necessary to get a proper holomorphic map in the limit. (ii) and (vi) together with (D) will guarantee that the limit map is regular and one to one, (v) will imply that the range of the limit map hits the prescribed points in the prescribed directions and has given finite order contacts with the prescribed submanifolds in M. (iii) together with (D) will be used in the inductive construction of f_{j+1} to obtain an one to one map on a subset of Ω_{j+1} in Step 1. (E) will imply that at each step of the inductive construction the constructed disc remains below a_{j+1} level of the exhaustion function and that it does not fall out of the retraction neighborhood E.

Part 4. Assume for a moment that we have finished the construction in part 3. To prove the theorem we proceed in a way similar to the one in [G2]. Let $\Omega = \cup_{n=1}^{\infty} \triangle_n$. It is easy to see that Ω is simply connected. Therefore there is a biholomorphic map $\Phi : \triangle \to \Omega$ such that $\Phi(0) = 0$ and $\Phi'(0) > 0$. Since Ω is symmetric with respect to the real axis we have $\Phi(\mathbb{R} \cap \triangle) = \mathbb{R} \cap \Omega$ and $\Phi'(\zeta) > 0$ $(\zeta \in \mathbb{R} \cap \triangle)$.

By (B) $\Omega = \cup_{n=1}^{\infty} \Omega_n$ which, by (vi), implies that for each $\zeta \in \Omega$, $f(\zeta) = \lim_{n\to\infty} f_n(\zeta)$ exists and that the map f is holomorphic on Ω. Since $f_n(\triangle_n) \subset M$ and M is closed in $\mathbb{C}^N$ we have $f(\Omega) \subset M$. We show that f is regular and one to one on Ω. Fix $n \in \mathbb{N}$. By (vi), $|f_n(\zeta) - f(\zeta)| \le |f_n(\zeta) - f_{n+1}(\zeta)| + |f_{n+1}(\zeta) - f_{n+2}(\zeta)| + \cdots < \frac{\epsilon_n}{2^n} + \frac{\epsilon_{n+1}}{2^{n+1}} + \cdots < \epsilon_n$ $(\zeta \in \Omega_n)$. Since $\Omega_{n-3} \subset \Omega_n$ it follows by (a) that f is regular and one to one on Ω_{n-4}. So for each $n \in \mathbb{N}$, f is regular and one to one on Ω_{n-4} and this implies that f is regular and one to one on Ω.

Next we show that $f : \Omega \to M$ is a proper map. Fix $n \in \mathbb{N}$ and let $\zeta \in \Omega_{n+1} \setminus \Omega_n$. It follows by (vi) that $|f_{n+1}(\zeta) - f(\zeta)| \le |f_{n+1}(\zeta) - f_{n+2}(\zeta)| + |f_{n+2}(\zeta) - f_{n+3}(\zeta)| + \cdots < \frac{\epsilon_{n+1}}{2^{n+1}} + \frac{\epsilon_{n+2}}{2^{n+2}} + \cdots < \frac{\epsilon_{n+1}}{2^n}$. By (iv) we have $\rho(f_{n+1}(\zeta)) \ge \min\{\rho(z_{m(n)+1}), a_n\} - \gamma$ and (b) implies that $\rho(f(\zeta)) \ge \min\{\rho(z_{m(n)+1}), a_n\} - 2\gamma$. As $\min\{\rho(z_{m(n)+1}), a_n\} \ge \rho(z_{m(n-1)+1})$ we obtain $\rho(f(\zeta)) \ge \rho(z_{m(n-1)+1}) - 2\gamma$. Since $\rho(z_{m(n-1)+1}) - 2\gamma$ is nondecreasing and $\Omega = \cup_{k=n+1}^{\infty} \Omega_k$ it follows that $\rho(f(\zeta)) \ge \rho(z_{m(n-1)+1}) - 2\gamma$ for $\zeta \in \Omega \setminus \Omega_n$ and as $\rho(z_{m(n-1)+1}) - 2\gamma \to \infty$ $(n \to \infty)$ it follows that $f : \Omega \to M$ is a proper map.

Thus, f is one to one, regular and proper, i.e. an embedding.

We have to show that the range of f hits the prescribed points in the prescribed directions and has given finite order contacts with the prescribed submanifolds in M. Fix $n \in \mathbb{N}$ and i, $m(n-1) + 1 \le i \le m(n)$. By (v) we have $f_n^{(l)}(\zeta_i) = f_{n+l}^{(l)}(\zeta_i)$ for $0 \le l \le k_i,\ l \in \mathbb{N}$ and this implies that $f_n^{(l)}(\zeta_i) = f^{(l)}(\zeta_i)$ for $0 \le l \le k_i$. Since by (v) we have $f_n(\zeta_i) = z_i$,

$f_n'(\zeta_i) = \mu_i X_i$ and $f_n(\mathcal{V}_i)$ and N_i have contact of at least order k_i at z_i, it follows that $f(\zeta_i) = z_i$ and $f'(\zeta_i) = \mu_i X_i$ and that there is a neighborhood of $\mathcal{W}_i \subset \mathcal{V}_i$ of ζ_i such that $f(\mathcal{W}_i)$ and N_i have contact of at least order k_i at z_i.

Since $\Phi'(\zeta) > 0$ $(\zeta \in \mathbb{R} \cap \triangle)$, $f \circ \Phi$ is a map which has all required properties of Theorem 1.2.

Part 5. f_0, $\triangle_0$ and ϵ_0 constructed in part 2 satisfy (A), (C)(i)-(iii) and (D). Suppose that $n \in \mathbb{N} \cup \{0\}$ and that we have constructed f_j, $\triangle_j$, ϵ_j, $\beta_{m(j-1)}, \ldots, \beta_{m(j)-1}$ $0 \le j \le n$, and Ω_j, α_j, λ_j and η_j, $0 \le j \le n-1$, such that (A), (C)(i)-(iii) and (D) hold for $0 \le j \le n$ and (B), (C)(iv)-(vi) and (E) hold for $0 \le j \le n-1$.

Step 1. We shall perturb the map f_n slightly to get a map $g_1 : \bar{\triangle}_n \to M$ that is one to one in a neighborhood of Ω_{n-2}. As $\Omega_{-2} = \Omega_{-1} = \emptyset$, for $n = 0, 1$ we define $g_1 = f_n$ and $U = \emptyset$. We now assume that $n \ge 2$. Let U be an open set such that $\Omega_{n-2} \subset U \subset \Omega_{n-1}$. Choose $c \in (a_{n-1} + \delta_{n-1}, a_n - 2\delta_n)$. By (iii) there is a component $\mathcal{U}$ of the set $\{\zeta \in \triangle; \rho(f_n(\zeta)) < c\}$ which contains $\bar{\Omega}_{n-1}$. It follows by (i) that $\mathcal{U} \subset\subset \triangle$. By the maximum principle applied to the subharmonic function $\rho \circ f_n$ the set $\mathcal{U}$ is conformally equivalent to the disc.

Take k, $1 \le k \le n$, and $\zeta \in \Omega_k \setminus \Omega_{k-1}$. It follows by (i) and (iv) that $\min\{\rho(z_{m(k-1)+1}), a_{k-1}\} - \gamma \le \rho(f_k(\zeta)) \le a_k$ and it follows by (vi) that $|f_k(\zeta) - f_n(\zeta)| \le |f_k(\zeta) - f_{k+1}(\zeta)| + |f_{k+1}(\zeta) - f_{k+2}(\zeta)| + \cdots + |f_{n-1}(\zeta) - f_n(\zeta)| \le \frac{\epsilon_k}{2^k} + \frac{\epsilon_{k+1}}{2^{k+1}} + \cdots + \frac{\epsilon_{n-1}}{2^{n-1}} \le \frac{\epsilon_k}{2^{k-1}}$. By (b) this implies that $\rho(f_n(\zeta)) \ge \min\{\rho(z_{m(k-1)+1}), a_{k-1}\} - 2\gamma \ge a_0 - 2\gamma$. This together with (iv) implies that $\rho(f_n(\zeta)) \ge a_0 - 2\gamma$ $(\zeta \in \triangle_n \setminus \Omega_0)$.

By (vi), $|f_0(\zeta) - f_n(\zeta)| \le |f_0(\zeta) - f_1(\zeta)| + |f_1(\zeta) - f_2(\zeta)| + \cdots + |f_{n-1}(\zeta) - f_n(\zeta)| \le \epsilon_0 + \frac{\epsilon_1}{2} + \cdots + \frac{\epsilon_{n-1}}{2^{n-1}} < 2\epsilon_0$ $(\zeta \in \Omega_0)$. By (B) and (6) this implies that f_n is regular and one to one on $\{\xi \in \triangle_0;\ \rho(f_0(\xi)) < a_0 - 2\gamma\}$ and it follows by (7) that $\{\xi \in \Omega_0;\ \rho(f_n(\xi)) < a_0 - 3\gamma\} \subset \{\xi \in \triangle_0;\ \rho(f_0(\xi)) < a_0 - 2\gamma\}$. As $\rho(f_n(\zeta)) > a_0 - 2\gamma$ $(\zeta \in \triangle_n \setminus \Omega_0)$, it follows that f_n is regular and one to one on the nonempty set $\{\xi \in \triangle_n;\ \rho(f_n(\xi)) < a_0 - 3\gamma\}$ and therefore by Lemma A.2, $f_n|\mathcal{U} : \mathcal{U} \to \{z \in M; \rho(z) < c\}$ is a normalization map. By Lemma 7.1 we obtain a continuous map $g_1 : \bar{\triangle}_n \to M$, holomorphic on $\triangle_n$, such that

(1i) $|g_1(\zeta) - f_n(\zeta)| < \min\{\frac{\epsilon_n}{4 \cdot 2^n}, \lambda_{n-1}\}$ $(\zeta \in \bar{\triangle}_n)$
(1ii) g_1 is regular and one to one in U
(1iii) $g_1^{(j)}(\zeta_i) = f^{(j)}(\zeta_i)$ $(0 \le j \le k_i,\ 0 \le i \le m(n))$.

Step 2. We shall push the boundary of the disc $g_1 : \bar{\triangle}_n \to M$ to higher levels of $\rho|M$.

Let $\alpha_n > 0$ be so small that for a holomorphic map $h : U \to \mathbb{C}^N$ such that $|g_1(\zeta) - h(\zeta)| < \alpha_n$ $(\zeta \in U)$ it follows that h is regular and one to one on Ω_{n-2}. Choose $\lambda_n < \min\{\lambda_{n-1}, \frac{\alpha_n}{4}, \frac{\epsilon_n}{4\cdot 2^n}\}$ such that (8) holds for $j = n$. Let $\eta_n < \frac{\lambda_n}{2}$ be small enough that (9) holds for $j = n$.

Since $|g_1(\zeta) - f_n(\zeta)| < \lambda_{n-1}$ $(\zeta \in \bar{\triangle}_n)$ and $\rho(f_n(\zeta)) \in (a_n - 2\delta_n, a_n)$ $(\zeta \in b\triangle_n)$ it follows by (8) that $\rho(g_1(\zeta)) \in (a_n - 3\delta_n, a_n + \frac{\delta_n}{4})$ $(\zeta \in b\triangle)$ and therefore $(\rho\circ g_1)(b\triangle)$ contains only regular values of $\rho|M$. Let $K \subset \triangle_n$ be a compact set such that $\Omega_{n-1} \cup \{z; \mathrm{dist}(z, b\triangle_n) > \frac{1}{n}\} \subset K$. By Lemma 3.1 there are a continuous map $g_2 : \bar{\triangle}_n \to M$, holomorphic on $\triangle_n$ and an open set Ω_n, $K \subset \Omega_n \subset\subset \triangle_n$, such that

(2i) $a_{n+1} - \delta_{n+1} < \rho(g_2(\zeta)) < a_{n+1} - \frac{\delta_{n+1}}{2}$ $(\zeta \in b\triangle_n)$
(2ii) $\rho(g_2(\zeta)) \geq a_n - 4\delta_n$ $(\zeta \in \triangle_n \setminus \Omega_n)$
(2iii) $|g_2(\zeta) - g_1(\zeta)| < \min\{\lambda_n, \frac{\delta_{n+1}}{8}\}$ $(\zeta \in \Omega_n)$
(2iv) $g_2^{(j)}(\zeta_i) = g_1^{(j)}(\zeta_i)$ $(0 \leq j \leq k_i,\ 0 \leq i \leq m(n))$.

Step 3. For each j, $m(n) + 1 \leq j \leq m(n+1)$, we construct an analytic disc that hits the point z_j in the prescribed direction, which has at z_j a given finite order contact with the given submanifold of M and its boundary is close to the a_{n+1} level of the exhaustion function $\rho|M$. Then we glue these discs and the map g_2 together.

By Lemma 5.1 we obtain the continuous maps $h_j : \bar{D}_j \to M$ $(m(n) + 1 \leq j \leq m(n+1))$ holomorphic on D_j such that

(hi) $h_j(3j) = z_j$, $h'_j(3j) = \mu_j X_j$ for some $\mu_j > 0$ and there is a neighborhood $\mathcal{V}_j$ of $3j$ in D_j such that $f(\mathcal{V}_j)$ and N_j have contact of at least order k_j at z_j
(hii) $\rho(h_j(\zeta))) \in (a_{n+1} - \delta_{n+1}, a_{n+1} - \frac{\delta_{n+1}}{2})$ $(\zeta \in bD_j)$
(hiii) $\rho(h_j(\zeta)) \geq \rho(z_j) - \frac{\gamma}{4}$ $(\zeta \in \bar{D}_j)$.

A consequence of [GR, pp. 227, Theorem 2] is the fact that the boundary of any connected component of the sublevel set of $\rho|M$ is connected. Therefore one can connect $f_n(3m(n) + 1)$ with $h_{m(n)+1}(3(m(n) + 1) - 1)$ by a path contained in $U_{n+1} \cap \rho^{-1}((a_{n+1} - \delta_{n+1}, a_{n+1} - \frac{\delta_{n+1}}{2})$ and similarly, for each j, $m(n)+1 \leq j \leq m(n+1)-1$ one can connect the points $h_j(3j + 1)$ and $h_{j+1}(3(j + 1) - 1)$, by a path contained in $U_{n+1} \cap \rho^{-1}((a_{n+1} - \delta_{n+1}, a_{n+1} - \frac{\delta_{n+1}}{2}))$.

Thus, if L_{n+1} is the union of $\triangle_n$, the discs $D_{m(n)+1}, \ldots, D_{m(n+1)}$ and the segments $I_j = [3j + 1, 3(j + 1) - 1]$, $m(n) \leq j \leq m(n + 1) - 1$, it follows that there is a continuous map $g_3 : \bar{L}_{n+1} \to U_{n+1}$ which extends all the maps $f_n, h_{m(n)+1}, \ldots, h_{m(n+1)}$ and such that

$$g_3|bL_{n+1} \subset \left(a_{n+1} - \delta_{n+1}, a_{n+1} - \frac{\delta_{n+1}}{2}\right) . \tag{10}$$

The map g_3 is continuous on L_{n+1} and holomorphic in the interior of L_{n+1}.

Step 4. We use a version of Mergelyan's theorem to approximate the map g_3 by a polynomial in the ambient space. In this way we obtain a map from a neighborhood of L_{n+1} to the retraction neighborhood E and then we compose this map with the holomorphic retraction π.

By Proposition A.3 there is a holomorphic polynomial $P : \mathbb{C} \to \mathbb{C}^N$ such that

(3i) $|P(\zeta) - g_3(\zeta)| < \eta_n \ (\zeta \in \bar{L}_{n+1})$
(3ii) $P^{(j)}(\zeta_i) = g_3^{(j)}(\zeta_i) \ (0 \leq j \leq k_i,\ 0 \leq i \leq m(n+1))$.

Take $\zeta \in \bar{L}_{n+1}$. By (9) we have $P(\zeta) \in E$ and $|\pi(P(\zeta)) - P(\zeta)| < \frac{\lambda_n}{2}$ and therefore by (3i)

$$|\pi(P(\zeta)) - g_3(\zeta)| < \eta_n + \frac{\lambda_n}{2} < \lambda_n \ \ (\zeta \in \bar{L}_{n+1})$$

and by (8) we have $|\rho(\pi(P(\zeta))) - \rho(g_3(\zeta))| < \frac{\delta_{n+1}}{4} \ (\zeta \in \bar{L}_{n+1})$. This, together with (10), implies that $\rho(\pi(P(\zeta))) \in (a_{n+1} - \frac{5\delta_{n+1}}{4}, a_{n+1} - \frac{\delta_{n+1}}{4})$ $(\zeta \in bL_{n+1})$. The last condition is fullfilled for ζ in the neighborhood of bL_{n+1} in $\mathbb{C}$ as well. Thus we can choose a β, $0 < \beta < 1$, such that $\rho(\pi(P(\zeta))) \in (a_{n+1} - \frac{5\delta_{n+1}}{4}, a_{n+1} - \frac{\delta_{n+1}}{4})\ (\zeta \in ([3j, 3(j+1)] \times (-\beta, \beta)) \setminus (D_j \cup D_{j+1}),\ m(n) \leq j \leq m(n+1) - 1)$. Put $\beta_j = \beta\ (m(n) \leq j \leq m(n+1) - 1)$. This defines $\triangle_{n+1}$ as described in (A).

Let $g_4(\zeta) = \pi(P(\zeta))$ for $\zeta \in \triangle_{n+1}$. The map $g_4 : \bar{\triangle}_{n+1} \to M$ is continuous, holomorphic on $\triangle_{n+1}$, and

(4i) $\rho(g_4(\zeta)) \in \left(a_{n+1} - \frac{5\delta_{n+1}}{4}, a_{n+1} - \frac{\delta_{n+1}}{4}\right) (\zeta \in b\triangle_{n+1})$
(4ii) $\rho(g_4(\zeta)) \in \left(a_{n+1} - \frac{5\delta_{n+1}}{4}, a_{n+1} - \frac{\delta_{n+1}}{4}\right) (\zeta \in ([3j, 3(j+1)] \times (-\beta_j, \beta_j)) \setminus (D_{jsave} \cup D_{j+1}),\ m(n) \leq j \leq m(n+1) - 1)$
(4iii) $|g_4(\zeta) - g_3(\zeta)| < \lambda_n \ (\zeta \in L_{n+1})$
(4iv) $g_4^{(j)}(\zeta_i) = g_3^{(j)}(\zeta_i) \ (0 \leq j \leq k_i,\ 0 \leq i \leq m(n+1))$.

Step 5. We perturb the map g_4 to get a regular map on Ω_n.

By Lemma 6.1 we get a continuous map $g_5 : \bar{\triangle}_{n+1} \to M$, holomorphic in $\triangle_{n+1}$, such that

(5i) g_5 is regular in Ω_n
(5ii) $|g_5(\zeta) - g_4(\zeta)| < \lambda_n \ (\zeta \in \bar{\triangle}_{n+1})$
(5iii) $g_5^{(j)}(\zeta_i) = g_4^{(j)}(\zeta_i) \ (0 \leq j \leq k_i,\ 0 \leq i \leq m(n+1))$.

It follows by (8), (4i) and (5ii) that

(5iv) $\rho(g_5(\zeta)) \in \left(a_{n+1} - \frac{3\delta_{n+1}}{2}, a_{n+1}\right) (\zeta \in b\triangle_{n+1})$.

Step 6. Put $f_{n+1} = g_5$. Choose $\epsilon_{n+1} < \min\{\frac{1}{n}, \epsilon_n\}$ so small that (D) holds for $j = n + 1$. We shall prove that the map f_{n+1} has all the required properties.

By (5iv), (i) is satisfied for $j = n + 1$. Let $\zeta \in U$. By (5ii), (4iii) and (2iii) we get $|f_{n+1}(\zeta) - g_1(\zeta)| < \alpha_n$. The map g_1 is regular and one to one on U and from definition of α_n we get that f_{n+1} is regular and one to one on Ω_{n-2}. By (5i) f_{n+1} is regular on Ω_n, so (ii) follows for $j = n + 1$.

Take $\zeta \in \Omega_n$. By (i),(8) and (1i) we get $\rho(g_1(\zeta)) < a_n + \frac{\delta_n}{4}$. By (2iii),(4iii),(5ii) and (8) we get $\rho(f_{n+1}(\zeta)) < \rho(g_1(\zeta)) + \frac{3\delta_{n+1}}{4}$. Therefore $\rho(f_{n+1}(\zeta)) < a_n + \frac{\delta_n}{4} + \frac{3\delta_{n+1}}{4}$ and since the sequence $\{\delta_n\}$ is decreasing (iii) follows for $j = n + 1$.

Recall that the sequence δ_n is decreasing with $\delta_0 \leq \frac{\gamma}{5}$ and that $\rho(z_{m(n)+1}) = \min\{\rho(z_{m(n)+1}), \rho(z_{m(n)+2}), \ldots, \rho(z_{m(n+1)-1})\}$. For $\zeta \in \triangle_n \setminus \Omega_n$ by (5ii), (4i), (8), (4iii) and (2ii) it follows that $\rho(f_{n+1}(\zeta)) \geq a_n - \gamma$. Take $\zeta \in \triangle_{n+1} \setminus \triangle_n$. By (hiii), (4iii), (4i), (8) and (4ii) we have $\rho(g_4(\zeta)) \geq \min\{\rho(z_{m(n)+1}) - \frac{\gamma}{2}, a_n - \frac{\gamma}{2}\}$ and by (5ii), (4i), (8) it follows that $\rho(g_5(\zeta)) \geq \rho(g_4(\zeta)) - \frac{\gamma}{2} \geq \min\{\rho(z_{m(n)+1}), a_n\} - \gamma$. Therefore (iv) is satisfied for $j = n$.

Let $\zeta \in \Omega_n$. By (5ii), (4iii), (2iii), (1i) and (E) we get $|f_{n+1}(\zeta) - f_n(\zeta)| \leq |g_5(\zeta) - g_4(\zeta)| + |g_4(\zeta) - g_2(\zeta)| + |g_2(\zeta) - g_1(\zeta)| + |g_1(\zeta) - f_n(\zeta)| < \frac{\varepsilon_n}{2^n}$ therefore (vi) is satisfied for $j = n$.

By the construction of the map g_3 and by (1iii), (2iv), (3ii), (4iv), (5iii) we get (v) for $j = n$.

The proof is complete for $\dim M \geq 3$.

In the case $\dim M = 2$ omit one to one everywhere and put $g_1 = f_n$ in step 1. The rest of the proof remains unchanged. This completes the proof of Theorem 1.2.

A Appendix

A.1. Normalization maps

Let P be a domain in $\mathbb{C}^N$, $N \geq 2$, and let $\Phi : \triangle \to P$ be a proper holomorphic map. Then $V = \Phi(\triangle)$ is a variety in P [C]. Let S be the singular set of $\Phi(\triangle)$. We say that the map Φ is *a normalization map* for the variety $\Phi(\triangle)$ if $\Phi|(\triangle \setminus \Phi^{-1}(S)) \to \Phi(\triangle) \setminus S$ is biholomorphic. Assume that Φ is regular at a point $\zeta \in \Phi^{-1}(S)$. Then $\Phi|(\triangle \setminus \Phi^{-1}(S)) \cup \{\zeta\} \to (\Phi(\triangle) \setminus S) \cup \Phi(\zeta)$ is regular and one to one.

We shall need the following result on normalization maps proved in [S].

Lemma A.1 *Let P be a domain in $\mathbb{C}^N$, $N \geq 2$, and let $\Phi : \triangle \to P$ be a proper holomorphic map. Then $\Phi = \Psi \circ B$ where B is a finite Blaschke product and Ψ is a normalization map for $\Phi(\triangle)$.*

Let $M(a) = \{z \in M; \rho(z) < a\}$ be a sublevel set of $\rho|M$.

Lemma A.2 *Let $a < A$ and let $\Phi : \triangle \to M(A)$ be a proper holomorphic map. Suppose that the set $\omega = \{\zeta \in \triangle; \rho(\Phi(\zeta)) < a\}$ is nonempty and that Φ is one to one on ω. Then Φ is a normalization map for the variety $\Phi(\triangle)$.*

Proof. Since $\rho(z) = |z|^2$ the map Φ is a proper holomorphic map of $\triangle$ to $\{z \in \mathbb{C}^N; |z|^2 < A\}$ and the lemma is a consequence of Lemma 3.2 in [G2].

A.2. Mergelyan's theorem

In the proof of Theorem 1.2 we need approximation by polynomials and interpolation of values and finitely many derivatives at a finite set of points. We use the following consequence of Mergelyan's theorem

Proposition A.3 *Let $k \in \mathbb{N} \cup \{0\}$. Let K be a compact set in $\mathbb{C}$ whose complement is connected. Suppose that $\zeta_1, \dots, \zeta_n$ are in the interior of K and let f be a continuous complex function on K which is holomorphic in the interior of K. Given $\epsilon > 0$ there is a polynomial P such that $|f(\zeta) - P(\zeta)| < \epsilon$ for all $\zeta \in K$ and $P^{(j)}(\zeta_i) = f^{(j)}(\zeta_i)$ $(0 \le j \le k, 1 \le i \le n)$.*

Proof. Put $Q_i^j(\zeta) = \frac{(\zeta-\zeta_i)^j \prod_{l\neq i,\, l=1}^{n}(\zeta-\zeta_l)^{j+1}}{j! \prod_{l\neq i,\, l=1}^{n}(\zeta_i-\zeta_l)^{j+1}}$ $(j \in \mathbb{N} \cup \{0\},\ 1 \le i \le n)$. Then

$$(Q_i^j)^{(l)}(\zeta_s) = 0 \ (0 \le l \le j-1) \text{ and } (Q_i^j)^{(j)}(\zeta_s) = \delta_{is} \tag{11}$$

Put $M_j = \sup\{Q_i^j(\zeta);\ 1 \le i \le n,\ \zeta \in K\}$ $(j \in \mathbb{N} \cup \{0\})$.

We proceed by induction on k. For $k = 0$ put $\eta = \min\{\frac{\epsilon}{3}, \frac{\epsilon}{3M_0 n}\}$. By Mergelyan's theorem there is a polynomial P_0 such that $|f(\zeta) - P_0(\zeta)| < \eta$ $(\zeta \in K)$. Put $P_1(\zeta) = \sum_{i=1}^{n}(f(\zeta_i) - P_0(\zeta_i))Q_i^0(\zeta)$. It follows that $|f(\zeta) - P_0(\zeta) - P_1(\zeta)| \le |f(\zeta) - P_0(\zeta)| + |P_1(\zeta)| < \eta + M_0 n\eta < \epsilon$ and by (11) it follows that $f(\zeta_i) - P_0(\zeta_i) - P_1(\zeta_i) = 0$ $(1 \le i \le n)$. Therefore $P = P_0 + P_1$ satisfies the conditions in the proposition.

Suppose the proposition holds for k. Let $\eta = \min\{\frac{\epsilon}{3}, \frac{\epsilon}{3M_{k+1}n}\}$. Since the proposition holds for k there is a polynomial P_k such that $|f(\zeta) - P_k(\zeta)| < \eta$ $(\zeta \in K)$ and $P_k^{(j)}(\zeta_i) = f^{(j)}(\zeta_i)$ $(0 \le j \le k, 1 \le i \le n)$. Put $P_{k+1}(\zeta) = \sum_{i=1}^{n}(f^{(k+1)}(\zeta_i) - P_k^{(k+1)}(\zeta_i))Q_i^{k+1}(\zeta)$. It follows that $|f(\zeta) - P_k(\zeta) - P_{k+1}(\zeta)| \le |f(\zeta) - P_k(\zeta)| + |P_{k+1}(\zeta)| < \eta + M_{k+1}n\eta < \epsilon$ and by (11) it follows that $f^{(j)}(\zeta_i) - P_k^{(j)}(\zeta_i) - P_{k+1}^{(j)}(\zeta_i) = 0$ $(0 \le j \le k+1,\ 1 \le i \le n)$. Therefore $P = P_k + P_{k+1}$ satisfies the conditions in the proposition and this finishes the proof.

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