HOLOMORPHIC CURVES IN COMPLEX SPACES

BARBARA DRINOVEC DRNOVŠEK and FRANC FORSTNERIČ

To Josip Globevnik

Abstract

We study the existence of topologically closed complex curves normalized by bordered Riemann surfaces in complex spaces. Our main result is that such curves abound in any noncompact complex space admitting an exhaustion function whose Levi form has at least two positive eigenvalues at every point outside a compact set, and this condition is essential. We also construct a Stein neighborhood basis of any compact complex curve with $\mathscr{C}^2$-boundary in a complex space.

Contents

1. Introduction

Let X be an irreducible (reduced, paracompact) complex space of dimension greater than 1. For every topologically closed complex curve C in X, we have a sequence of holomorphic maps

$$\{\mathbb{CP}^1, \mathbb{C}, \triangle\} \ni \widetilde{D} \to D \to C \hookrightarrow X,$$

where $C \hookrightarrow X$ is the inclusion, $D \to C$ is a normalization of C by a Riemann surface D, and $\widetilde{D} \to D$ is a universal covering combined with a uniformization map.

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Here $\triangle = \{z \in \mathbb{C}\colon |z| < 1\}$. Thus C is the image of a generically one-to-one proper holomorphic map $D \to X$; hence it is natural to ask which Riemann surfaces D admit any proper holomorphic maps to a given complex space and how plentiful they are. This question has been investigated most intensively for compact complex curves that form a part of the *Douady space* and of the *cycle space* of X (see [3], [8], [18]).

In this article, we obtain essentially optimal existence and approximation results when D is a *finite bordered Riemann surface*, that is, a one-dimensional complex manifold with compact closure $\bar{D} = D \cup bD$ whose boundary bD consists of finitely many closed Jordan curves; such a D is uniformized by the disc $\triangle$. The existence of a proper holomorphic map $D \to X$ implies that X is noncompact, but additional conditions are needed in general since there exist open complex manifolds without any topologically closed complex curves; an example is obtained by removing a point from a compact complex manifold that admits no closed complex curves (a condition satisfied, e.g., by certain complex tori of dimension greater than 1).

We begin by a brief survey of the known results. Every open Riemann surface admits a proper holomorphic immersion in $\mathbb{C}^2$ and a proper holomorphic embedding in $\mathbb{C}^3$ (see [7], [61]). Some open Riemann surfaces also embed in $\mathbb{C}^2$, but it is unknown whether all of them do; impressive results on this subject have been obtained recently by Wold in [77], [78], [79], where the reader can find references to older works on the subject.

Turning to more general target spaces, we note that the Kobayashi hyperbolicity of X excludes curves uniformized by $\mathbb{C}$ but imposes fewer restrictions on those uniformized by the disc $\triangle$ (see [50], [51]). There are other, less tangible obstructions: Dor [17] found a bounded domain with nonsmooth boundary in $\mathbb{C}^n$ without any proper holomorphic images of $\triangle$; even in smoothly bounded (non-pseudoconvex) domains in $\mathbb{C}^n$, the union of images of all proper analytic discs can omit a nonempty open subset (see [27]). On the positive side, every point in a Stein manifold X of dimension greater than 1 is contained in the image of a proper holomorphic map $\triangle \to X$ (see Globevnik [35]; see also [16], [19], [20], [21], [27], [28], [29]). The same holds for discs in any connected complex manifold X that is q-complete for some $q < \dim X$ (see [21]). The first cases of interest, inaccessible with the existing techniques, are Stein spaces with singularities.

Recall that a smooth function $\rho\colon X \to \mathbb{R}$ on a complex space X is said to be *q-convex* on an open subset $U \subset X$ (in the sense of Andreotti and Grauert [2] and [38, Definition 1.4, page 263]) if there is a covering of U by open sets $V_j \subset U$, biholomorphic to closed analytic subsets of open sets $\Omega_j \subset \mathbb{C}^{n_j}$, such that for each j the restriction $\rho|_{V_j}$ admits an extension $\widetilde{\rho}_j\colon \Omega_j \to \mathbb{R}$ whose Levi form $i\partial\bar{\partial}\widetilde{\rho}_j$ has at most $q-1$ negative or zero eigenvalues at each point of Ω_j. The space X is *q-complete* (resp., *q-convex*) if it admits a smooth exhaustion function $\rho\colon X \to \mathbb{R}$ which is q-convex on X (resp., on $\{x \in X\colon \rho(x) > c\}$ for some $c \in \mathbb{R}$). A 1-complete complex space is just a Stein space, and a 1-convex space is a proper modification of a Stein space. We denote by X_{reg} (resp., by X_{sing}) the set of regular (resp., singular) points of X.

We are now ready to state our first main result; it is proved in §6.

THEOREM 1.1

Let X be an irreducible complex space of $\dim X > 1$, and let $\rho\colon X \to \mathbb{R}$ be a smooth exhaustion function that is $(n-1)$-convex on $X_c = \{x \in X\colon \rho(x) > c\}$ for some $c \in \mathbb{R}$. Given a bordered Riemann surface D and a $\mathscr{C}^2$-map $f\colon \bar{D} \to X$ which is holomorphic in D and satisfies $f(D) \not\subset X_{\mathrm{sing}}$ and $f(bD) \subset X_c$, there is a sequence of proper holomorphic maps $g_\nu\colon D \to X$ homotopic to $f|_D$ and converging to f uniformly on compacts in D as $\nu \to \infty$. Given an integer $k \in \mathbb{N}$ and finitely many points $\{z_j\} \subset D$, each g_ν can be chosen to have the same k-jet as f at each of the points z_j.

We now show by examples that the conditions in Theorem 1.1 are essentially optimal. The assumption on ρ means that its Levi form has at least two positive eigenvalues at every point of $X_c = \{\rho > c\}$. One positive eigenvalue does not suffice in view of Dor's example of a domain in $\mathbb{C}^n$ without any proper analytic discs (see [17]) and the fact that every domain in $\mathbb{C}^n$ is n-complete (see [39], [64]). Necessity of the hypothesis $f(D) \not\subset X_{\mathrm{sing}}$ is seen by [34, Proposition 3] (based on an example of Kaliman and Zaidenberg [48]): an analytic disc contained in X_{sing} may be forced to remain there under analytic perturbations, and it need not be approximable by proper holomorphic maps $\triangle \to X$. The only possible improvement is a reduction of the boundary regularity assumption on the initial map. If D is a planar domain bounded by finitely many Jordan curves and X is a manifold, it suffices to assume that f is continuous on $\bar{D}$ by appealing to [9, Theorem 1.1.4] in order to approximate f by a more regular map.

If $f\colon \bar{D} \to X$ in Theorem 1.1 is generically injective, then so is any proper holomorphic map $g_\nu\colon D \to X$ approximating f sufficiently closely; its image $g_\nu(D)$ is then a closed complex curve in X normalized by D. Assuming that $f(\bar{D}) \subset X_{\mathrm{reg}}$, one can choose each g_ν to be an immersion, and even an embedding when $n \geq 3$. Each map g_ν is a locally uniform limit in D of a sequence of $\mathscr{C}^2$-maps $f_j\colon \bar{D} \to X$ which are holomorphic in D and satisfy

$$\lim_{j\to\infty} \inf\bigl\{\rho \circ f_j(z)\colon z \in bD\bigr\} \to +\infty; \tag{1.1}$$

that is, their boundaries $f_j(bD)$ tend to infinity in X. Embedding $\bar{D}$ as a domain in an open Riemann surface S, we can choose each f_j to be holomorphic in open set $U_j \subset S$ containing $\bar{D}$.

Theorem 1.1 also gives new information on *algebraic curves* in $(n-1)$-convex *quasi-projective algebraic spaces* $X = Y\backslash Z$, where $Y, Z \subset \mathbb{CP}^N$ are closed complex (i.e., algebraic) subvarieties in a complex projective space. We embed our bordered Riemann surface D as a domain with smooth real analytic boundary in its *double* $\hat{S}$, a compact Riemann surface obtained by gluing two copies of $\bar{D}$ along their boundaries (see [5, page 581], [74, page 217]). There is a meromorphic embedding $\hat{S} \hookrightarrow \mathbb{CP}^3$ with poles outside of $\bar{D}$; the subset $S \subset \hat{S}$ which is mapped to the affine part $\mathbb{C}^3 \subset \mathbb{CP}^3$ is a smooth affine algebraic curve, and D is Runge in S. A holomorphic map $f: U \to X$ from an open set $U \subset S$ to a quasi-projective algebraic space X is said to be *Nash algebraic* (see Nash [63]) if the graph

$$G_f = \left\{\left(z, f(z)\right) \in S \times X : z \in U\right\}$$

is contained in a one-dimensional algebraic subvariety of $S \times X$.

COROLLARY 1.2
Let X be an irreducible quasi-projective algebraic space of $\dim X > 1$*, and let $D \Subset S$ be a smoothly bounded Runge domain in an affine algebraic curve S. Assume that $\rho: X \to \mathbb{R}$ and $f: \bar{D} \to X$ satisfy the hypotheses of Theorem 1.1. Then there is a sequence of Nash algebraic maps $f_j: U_j \to X$ in open sets $U_j \supset \bar{D}$ satisfying (1.1) such that the sequence $f_j|_D$ converges to a proper holomorphic map $g: D \to X$.*

Corollary 1.2 is obtained by approximating each of the holomorphic maps $f_j: U_j \to X$, obtained in the proof of Theorem 1.1, uniformly on $\bar{D}$ by a Nash algebraic map, appealing to theorems of Demailly, Lempert, and Shiffman [15, Theorem 1.1] and Lempert [54, Theorem 1.1, page 335]. Their results give Nash algebraic approximations of any holomorphic map from an open Runge domain in an affine algebraic variety to a quasi-projective algebraic space. Of course, g can be chosen to also satisfy the additional properties in Theorem 1.1. If $\Gamma_j \subset S \times X$ is an algebraic curve containing the graph of the Nash algebraic map $f_j: U_j \to X$, then its projection $C_j \subset X$ under the map $(z, x) \to x$ is an algebraic curve in X containing $f_j(U_j)$; as $j \to \infty$, the domains $f_j(D) \subset C_j$ converge to the closed transcendental curve $g(D) \subset X$, while their boundaries $f_j(bD)$ leave any compact subset of X.

Corollary 1.2 applies, for example, to $X = \mathbb{CP}^n\backslash A$, where A is a closed complex submanifold of dimension $d \in \{[(n+1)/2], \ldots, n-1\}$. Indeed, $\mathbb{CP}^n\backslash A$ is then $(2(n-d)-1)$-complete by a result of Peternell [65] (improving an earlier result of Barth [4]) and hence is $(n-1)$-complete if $n \leq 2d$.

Another interesting and relevant example is due to Schneider [71], who proved that for a compact complex manifold X and a complex submanifold $A \subset X$ of codimension q whose normal bundle $N_{A|X}$ is (Griffiths) positive, the complement $X\backslash A$ is q-convex.

Thus Theorem 1.1 furnishes closed complex curves in $X\backslash A$ whenever $q \le \dim X - 1$, which is equivalent to $\dim A \ge 1$ (for further examples, see Grauert [38] and Colţoiu [13]).

The following consequence of Theorem 1.1 was proved in [21] in the special case when $X_{\mathrm{sing}} = \emptyset$ and $D = \triangle$.

COROLLARY 1.3
Let X be an irreducible $(n-1)$-complete complex space of dimension $n > 1$, and let D be a bordered Riemann surface. Given a $\mathscr{C}^2$-map $f : \bar{D} \to X$ which is holomorphic in D and satisfies $f(D) \not\subset X_{\mathrm{sing}}$, a positive integer $k \in \mathbb{N}$, and finitely many points $\{z_j\} \subset D$, there is a sequence of proper holomorphic maps $g_\nu : D \to X$ converging to $f|_D$ uniformly on compacts in D such that each g_ν has the same k-jets as f at each of the points z_j. This holds, in particular, if X is a Stein space.

Let X be a complex manifold. The *Kobayashi-Royden pseudonorm* of a tangent vector $v \in T_xX$ is given by

$$\kappa_X(v) = \inf\big\{\lambda > 0 \colon \exists f : \triangle \to X \text{ holomorphic},\ f(0) = x,\ f'(0) = \lambda^{-1}v\big\}.$$

The same quantity is obtained by using only maps that are holomorphic in small neighborhoods of $\bar{\triangle}$ in $\mathbb{C}$. Corollary 1.3 implies the following.

COROLLARY 1.4
If X is an $(n-1)$-complete complex manifold of dimension $n > 1$, then its infinitesimal Kobayashi-Royden pseudometric κ_X is computable in terms of proper *holomorphic discs $f : \triangle \to X$.*

On a quasi-projective algebraic manifold X, the pseudometric κ_X and its integrated form, the Kobayashi pseudodistance, are also computable by algebraic curves (see [15, Corollary 1.2]).

It is natural to inquire which homotopy classes of maps $D \to X$ from a bordered Riemann surface admit a proper holomorphic representative. Hyperbolicity properties of X may impose a major obstruction on the existence of a holomorphic map in a given nontrivial homotopy class (see [50], [51], [22]). The following opposite property is important in Oka-Grauert theory.

A complex manifold X is said to enjoy the *m-dimensional convex approximation property* (CAP_m) if every holomorphic map $U \to X$ from an open set $U \subset \mathbb{C}^m$ can be approximated uniformly on any compact convex set $K \subset U$ by entire maps $\mathbb{C}^m \to X$ (see [26]).

COROLLARY 1.5

Let X be an $(n-1)$-complete complex manifold of dimension $n > 1$. If X satisfies CAP_{n+1}, then for every continuous map $f\colon D \to X$ from a bordered Riemann surface D, there exists a proper holomorphic map $g\colon D \to X$ homotopic to f. If f is holomorphic on a neighborhood of a compact subset $K \subset D$, then g can be chosen to approximate f as close as desired on K. This holds, in particular, if $X = \mathbb{CP}^n \backslash A$, where $n \geq 4$ and $A \subset \mathbb{CP}^n$ is a closed complex submanifold of dimension $d \in \{[(n+1)/2], \ldots, n-2\}$.

Proof

We may assume that $\bar{D} = \{z \in S\colon v(z) \leq 0\}$, where S is an open Riemann surface and $v\colon S \to \mathbb{R}$ is a smooth function with $dv \neq 0$ on $bD = \{v = 0\}$. Choose numbers $c_0 < 0 < c_1$ close to zero so that v has no critical values on $[c_0, c_1]$. Let $D_j = \{z \in S\colon v(z) < c_j\}$ for $j = 0, 1$. We may assume $K \subset D_0$. There is a homotopy of smooth maps $\tau_t\colon D_1 \to D_1$ ($t \in [0, 1]$) such that τ_0 is the identity on D_1, $\tau_1(D_1) = D_0$, and for all $t \in [0, 1]$ we have $\tau_t(D) \subset D$, and τ_t equals the identity map near K. Set $\widetilde{f} = f \circ \tau_1\colon D_1 \to X$. Note that $\widetilde{f}|_D$ is homotopic to f via the homotopy $f \circ \tau_t|_D$ ($t \in [0, 1]$).

By the main result [26, Theorem 1.2], the CAP_{n+1} property of X implies the existence of a holomorphic map $f_1\colon D_1 \to X$ homotopic to $\widetilde{f}\colon D_1 \to X$. Then $f_1|_D$ is homotopic to $\widetilde{f}|_D$ and hence to f. Theorem 1.1, applied to the map $f_1|_{\bar{D}}\colon \bar{D} \to X$, furnishes a proper holomorphic map $g\colon D \to X$ homotopic to $f_1|_D$ and hence to f. In addition, f_1 and g can be chosen to approximate f uniformly on K.

The last statement follows from the aforementioned fact that $\mathbb{CP}^n \backslash A$ is $(n-1)$-complete if A is as in the statement of the corollary (see [65]), and it enjoys CAP_m for all $m \in \mathbb{N}$ provided that $\dim A \leq n-2$ (see [26]). □

By [26] and [25], the property $\mathrm{CAP} = \bigcap_{m=1}^{\infty} \mathrm{CAP}_m$ of a complex manifold X is equivalent to the classical *Oka property* concerning the existence and the homotopy classification of holomorphic maps from Stein manifolds to X. Examples in [40] and [26] show that Corollary 1.5 fails in general if X does not enjoy CAP, and the most that one can expect is to find a proper map $D \to X$ in the given homotopy class which is holomorphic with respect to some complex structure on the smooth 2-surface D. This indeed follows by combining Theorem 1.1 with a very special case of the main result [33, Theorem 1.1, page 616].

COROLLARY 1.6

Let X be an $(n-1)$-complete complex manifold of dimension $n > 1$, and let $\bar{D}$ be a compact, connected, oriented real surface with boundary. For every continuous map

$f\colon D \to X$, there exist a complex structure J on D and a proper J-holomorphic map $g\colon D \to X$ which is homotopic to f.

Another result of independent interest is Theorem 2.1 to the effect that a compact complex curve with $\mathscr{C}^2$-boundary in a complex space admits a basis of open Stein neighborhoods. The following special case is proved in §2.

THEOREM 1.7
Let X be an n-dimensional complex manifold. If D is a relatively compact, smoothly bounded domain in an open Riemann surface S and $f\colon \bar{D} \hookrightarrow X$ is a $\mathscr{C}^2$-embedding that is holomorphic in D, then $f(\bar{D})$ has a basis of open Stein neighborhoods in X which are biholomorphic to open neighborhoods of $\bar{D} \times \{0\}^{n-1}$ in $S \times \mathbb{C}^{n-1}$. In particular, if D is a smoothly bounded planar domain, then $f(\bar{D})$ has a basis of open Stein neighborhoods in X which are biholomorphic to domains in $\mathbb{C}^n$.

Royden showed in [70] that for any holomorphically embedded polydisc $f\colon \triangle^k \hookrightarrow X$ in a complex manifold X and for any $r < 1$, the smaller polydisc $f(r\triangle^k) \subset X$ admits open neighborhoods in X biholomorphic to $\triangle^n$ with $n = \dim X$. We have the analogous result for closed analytic discs, showing that they have no appreciation whatsoever of their surroundings.

COROLLARY 1.8
Let X be an n-dimensional complex manifold. For every $\mathscr{C}^2$-embedding $f\colon \bar{\triangle} \hookrightarrow X$ which is holomorphic in $\triangle$, the image $f(\bar{\triangle})$ has a basis of open neighborhoods in X which are biholomorphic to $\triangle^n$.

These and related results are used to obtain new holomorphic approximation theorems (Corollary 2.7, Theorem 5.1).

Outline of proof of Theorem 1.1
Theorem 1.1 is proved in §6 after developing the necessary tools in §§2–5. We begin by perturbing the initial map $f\colon \bar{D} \to X$ to a new map for which $f(bD) \subset X_{\text{reg}}$ (see Theorem 5.1). The rest of the construction is done in such a way that the image of bD remains in the regular part of X. A proper holomorphic map $g\colon D \to X$ is obtained as a limit $g = \lim_{j\to\infty} f_j|_D$ of a sequence of $\mathscr{C}^2$-maps $f_j\colon \bar{D} \to X$ which are holomorphic in D such that the boundaries $f_j(bD)$ converge to infinity.

Our local method of lifting the boundary $f(bD)$ is similar to the one used (in the special case $D = \triangle$) in earlier articles on the subject (see [16], [19], [20], [27], [28], [35]). Since the Levi form $\mathscr{L}_\rho$ is assumed to have at least two positive eigenvalues at every point of $f(bD)$, we get at least one positive eigenvalue in a direction tangential to the level set of ρ at each point $f(z)$, $z \in bD$; this gives a small analytic disc in

X, tangential to the level set of ρ at $f(z)$, along which ρ increases quadratically. By solving a certain Riemann-Hilbert boundary value problem, we obtain a local holomorphic map whose boundary values on the relevant part of bD are close to the boundaries of these discs, and hence $\rho \circ f$ has increased there. (One positive eigenvalue of $\mathscr{L}_\rho$ does not suffice since the corresponding eigenvector may be transverse to the level set of ρ and cannot be used in the construction.)

To globalize the construction, we develop a new method of patching holomorphic maps by improving a technique from the recent work of Forstnerič [26] on localization of the Oka principle. We embed a given map $f\colon \bar{D} \to X$ into a *spray of maps*, that is, a family of maps $f_t\colon \bar{D} \to X$ depending holomorphically on the parameter t in a Euclidean space and satisfying a certain submersivity property outside of an exceptional subvariety. The local modification method explained above gives a new spray near a part of the boundary bD; by ensuring that the two sprays are sufficiently close to each other on the intersection of their domains $\overline{D_0 \cap D_1}$, we patch them into a new spray over $\overline{D_0 \cup D_1}$ (see Proposition 4.3). This is accomplished by finding a fiberwise biholomorphic transition map between them and decomposing it into a pair of maps over $\bar{D}_0$ (resp., $\bar{D}_1$) which are used to correct the two sprays so as to make them agree over $\overline{D_0 \cap D_1}$.

The main step, namely, a decomposition of the transition map (Theorem 3.2), is achieved by a rapidly convergent iteration. This result generalizes the classical Cartan lemma to nonlinear maps, with $\mathscr{C}^r$-estimates up to the boundary. Unlike in [26, Lemma 2.1], the base domains do not shrink in our present construction — this is not allowed since all action in the construction of proper maps takes place at the boundary.

Our method of gluing sprays is also useful in proving holomorphic approximation theorems (see Theorem 5.1).

One of the difficult problems in earlier works has been to avoid running into a critical point of the given exhaustion function $\rho\colon X \to \mathbb{R}$. For Stein manifolds, this problem was solved by Globevnik [35]. Here we apply an alternative method from [23] and cross each critical level by using a different function constructed especially for this purpose.

We believe that the methods developed in this article are applicable in other problems involving holomorphic maps. With this in mind, many of the new technical tools are obtained in the more general context of strongly pseudoconvex domains in Stein manifolds.

2. Stein neighborhoods of smoothly bounded complex curves

Let $(X, \mathscr{O}_X)$ be a complex space. We denote by $\mathscr{O}(X)$ the algebra of all holomorphic functions on X, endowed with the compact-open topology. A compact subset K of X is said to be $\mathscr{O}(X)$-*convex* if for any point $p \in X \backslash K$, there exists $f \in \mathscr{O}(X)$ with $|f(p)| > \sup_K |f|$. If X is Stein and K is contained in a closed complex subvariety X' of X, then K is $\mathscr{O}(X')$-convex if and only if it is $\mathscr{O}(X)$-convex. (For Stein spaces, we refer to [41] and [47].)

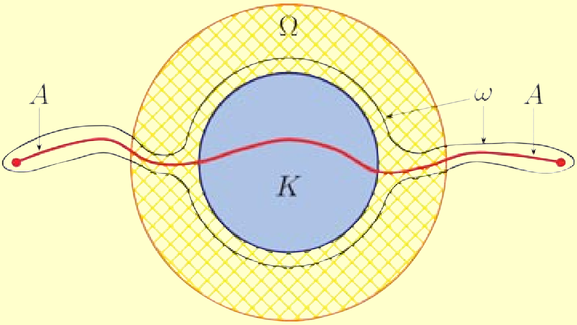

Figure 1. Theorem 2.1

We say that a compact set A in a complex space X is a *complex curve with* $\mathscr{C}^r$*-boundary* bA *in* X if

(i) $A \backslash bA$ is a closed, purely one-dimensional complex subvariety of $X \backslash bA$ without compact irreducible components; and

(ii) every point $p \in bA$ has an open neighborhood $V \subset X$ and a biholomorphic map $\phi\colon V \to V' \subset \Omega \subset \mathbb{C}^N$ onto a closed complex subvariety V' in an open subset $\Omega \Subset \mathbb{C}^N$ such that $\phi(A \cap V)$ is a one-dimensional complex submanifold of Ω with $\mathscr{C}^r$-boundary $\phi(bA \cap V)$.

Note that bA consists of finitely many closed Jordan curves and has no isolated points, but it may contain some singular points of X.

THEOREM 2.1

Let A be a compact complex curve with $\mathscr{C}^2$-boundary in a complex space X. Let K be a compact $\mathscr{O}(\Omega)$-convex set in a Stein open set $\Omega \subset X$. If $bA \cap K = \emptyset$ and $A \cap K$ is $\mathscr{O}(A)$-convex, then $A \cup K$ has a fundamental basis of open Stein neighborhoods ω in X (see Figure 1).

Theorem 2.1 is the main result of this section (see also Theorem 2.6). For $X = \mathbb{C}^n$, this follows from results of Wermer [76] and Stolzenberg [75]. We use only the special case with $K = \emptyset$, but the proof of the general case is not essentially more difficult, and we include it for future applications. The necessity of $\mathscr{O}(A)$-convexity of $K \cap A$ is seen by taking $X = \mathbb{C}^2$, $A = \{(z, 0)\colon |z| \le 3\}$, and $K = \{(z, w)\colon 1 \le |z| \le 2,\ |w| \le 1\}$. Every Stein neighborhood of $A \cup K$ contains the bidisc $\{(z, w)\colon |z| \le 2,\ |w| \le 1\}$.

In this connection, we mention a result of Siu [73, Main Theorem, page 89] to the effect that a closed Stein subspace (without boundary) of any complex space admits an open Stein neighborhood. Extensions to the q-convex case and simplifications of the proof were given by Colţoiu [12] and Demailly [14]. These results do not seem to apply directly to subvarieties with boundaries.

Proof

We adapt [25, proof of Theorem 2.1]. (It is based on the proof of Siu's theorem [73, Main Theorem, page 89] given in [14].) We begin with preliminary results. We have $bA = \bigcup_{j=1}^{m} C_j$, where each C_j is a closed Jordan curve of class $\mathscr{C}^2$ (a diffeomorphic image of the circle $T = \{z \in \mathbb{C} \colon |z| = 1\}$).

LEMMA 2.2

There are a Stein open neighborhood $U_j \subset X$ *of* C_j*, with* $\overline{U}_j \cap K = \emptyset$*, and a holomorphic embedding* $Z = (z, w) \colon U_j \to \mathbb{C}^{1+n_j}$ *for some* $n_j \in \mathbb{N}$ *such that* $Z(U_j)$ *is a closed complex subvariety of the set*

$$U'_j = \bigl\{(z, w) \in \mathbb{C}^{1+n_j} \colon 1 - r_j < |z| < 1 + r_j,\ |w_1| < 1, \ldots,\ |w_{n_j}| < 1\bigr\}$$

for some $0 < r_j < 1$*, and*

$$Z(A \cap U_j) = \bigl\{(z, w) \in U'_j \colon z \in \Gamma_j,\ w = g_j(z)\bigr\},$$

where

$$\Gamma_j = \bigl\{z = re^{i\theta} \in \mathbb{C} \colon 1 - r_j < r \le h_j(\theta)\bigr\},$$

h_j *is a* $\mathscr{C}^2$*-function close to* 1 *(in particular,* $|h_j(\theta) - 1| < r_j$ *for every* $\theta \in \mathbb{R}$*), and* $g_j = (g_{j,1}, \ldots, g_{j,n_j}) \colon \Gamma_j \to \triangle^{n_j}$ *is a* $\mathscr{C}^2$*-map that is holomorphic in the interior of* Γ_j*.*

Proof

We claim that C_j, being a totally real submanifold of class $\mathscr{C}^2$ in X, admits a basis of open Stein neighborhoods in X. This is standard when X is smooth (without singularities), in which case the squared distance to C_j with respect to any smooth Riemannian metric on X is a strongly plurisubharmonic function in a neighborhood of C_j, and its sublevel sets provide a basis of open Stein neighborhoods of C_j. In the general case, when C_j contains some singular points of X we cover C_j by finitely many open sets $U_k \subset X$ ($k = 1, \ldots, m_j$) such that each U_k admits a holomorphic embedding $\phi_k \colon U_k \hookrightarrow \Omega_k \subset \mathbb{C}^{N_k}$ onto a closed complex subvariety $\phi_k(U_k)$ in an open set $\Omega_k \subset \mathbb{C}^{N_k}$. The function $\rho_k(x) = \mathrm{dist}^2(\phi_k(x), \phi_k(C_j \cap U_k)) \ge 0$ ($x \in U_j$) is then strongly plurisubharmonic near the set $\rho_k^{-1}(0) = C_j \cap U_k$. (We are using the Euclidean distance in the above definition of ρ_k.) Patching these functions $\rho_1, \ldots, \rho_{m_j}$ by a smooth partition of unity along C_j in X, we obtain a strongly plurisubharmonic function $\rho \ge 0$ in a neighborhood of C_j which vanishes precisely on C_j, and the sublevel sets $\{\rho < c\}$ for small $c > 0$ provide a Stein neighborhood basis of C_j (see [62]). The details of the patching argument are similar to the nonsingular case and are omitted.

Choose a Stein open neighborhood $U_j \Subset X$ of C_j. By shrinking U_j slightly around C_j, we may assume that U_j embeds holomorphically into a Euclidean space $\mathbb{C}^{1+n_j}$. Denote by $C'_j \subset \mathbb{C}^{1+n_j}$ (resp., by A') the image of C_j (resp., of $A \cap U_j$) under this embedding. We identify the circle T with $T \times \{0\}^{n_j} \subset \mathbb{C}^{1+n_j}$. The complexified tangent bundle to C'_j and the complex normal bundle to C'_j in $\mathbb{C}^{1+n_j}$ are trivial (since every complex vector bundle over a circle is trivial). Using standard techniques for totally real submanifolds (see, e.g., [31]), we find a $\mathscr{C}^2$-diffeomorphism Φ_j from a tube around C'_j in $\mathbb{C}^{1+n_j}$ onto a tube around the circle T such that $\Phi_j(C'_j) = T$ and such that $\bar\partial\, \Phi_j$ and its total first derivative $D^1(\bar\partial\, \Phi_j)$ vanish on C'_j.

By [31, Theorems 1.1, 1.2], we can approximate Φ_j in a tube around C'_j by a biholomorphic map Φ'_j that maps C'_j very close to T and that spreads a collar around C'_j in A' as a graph over an annular domain in the first coordinate axis. Composing the initial embedding $U_j \hookrightarrow \mathbb{C}^{1+n_j}$ with Φ'_j, we obtain (after shrinking U_j around C_j) the situation in the lemma. □

Using the notation in the statement of Lemma 2.2, we set

$$\Lambda_j = \big\{x \in U_j : z(x) \in \Gamma_j\big\} \subset X, \tag{2.1}$$

$$\phi_j(x) = w(x) - g_j\big(z(x)\big) \in \mathbb{C}^{n_j}, \quad x \in \Lambda_j. \tag{2.2}$$

We can extend $|\phi_j|^2$ to a $\mathscr{C}^2$-function on U_j which is positive on $U_j \backslash \Gamma_j$. Choose additional open sets $U_{m+1}, \dots, U_N$ in X whose closures do not intersect any of the sets $U_j \backslash \Lambda_j$ for $j = 1, \dots, m$ such that $A \cup K \subset \bigcup_{j=1}^N U_j$. By choosing these sets sufficiently small, we also get for each $j \in \{m+1, \dots, N\}$ a holomorphic map $\phi_j : U_j \to \mathbb{C}^{n_j}$ whose components generate the ideal sheaf of A at every point of U_j. If $U_j \cap A = \emptyset$ for some j, we take $n_j = 1$ and $\phi_j(x) = 1$. Choose slightly smaller open sets $V_j \Subset U_j$ $(j = 1, \dots, N)$ such that $A \cup K \subset \bigcup_{j=1}^N V_j$. Choose an open set $V \subset X$ with $A \cup K \subset V \Subset \bigcup_{j=1}^N V_j$, and let

$$\Lambda = \bigcup_{j=1}^{m} (\overline{V} \cap \Lambda_j) \cup \bigcup_{j=m+1}^{N} (\overline{V} \cap V_j). \tag{2.3}$$

LEMMA 2.3

There are a family of $\mathscr{C}^2$*-functions* $v_\delta : V \to \mathbb{R}$ $(\delta \in (0,1])$ *and a constant* $M > -\infty$ *such that* $i\partial\bar\partial v_\delta \geq M$ *on* Λ *for all* $\delta \in (0,1)$ *and such that* $v_0(x) = \lim_{\delta\to 0} v_\delta(x)$ *is of class* $\mathscr{C}^2$ *on* $V \backslash A$ *and satisfies* $v_0|_A = -\infty$.

Proof

We adapt [14, proof of Lemma 5]. Let rmax denote a regularized maximum (see [14, page 286]); this function is increasing and convex in all variables (hence it preserves plurisubharmonicity), and it can be chosen as close as desired to the usual maximum.

On every set V_j, we choose a smooth function $\tau_j\colon V_j \to \mathbb{R}$ which tends to $-\infty$ at bV_j. For each $\delta \in [0,1]$, we set

$$v_{\delta,j}(x) = \log\big(\delta + |\phi_j(x)|^2\big) + \tau_j(x), \quad x \in V_j,$$

and $v_\delta(x) = \operatorname{rmax}(\dots, v_{\delta,j}(x), \dots)$, where the regularized maximum is taken over all indices $j \in \{1,\dots,N\}$ for which $x \in V_j$. As $\delta \to 0$, v_δ decreases to v_0 and $\{v_0 = -\infty\} = A$. Since the generators ϕ_j and ϕ_k for the ideal sheaf of A can be expressed in terms of one another on $U_j \cap U_k$, the quotient $|\phi_j|/|\phi_k|$ is bounded on $\overline{V}_j \cap \overline{V}_k$, and hence $(\delta + |\phi_j|^2)/(\delta + |\phi_k|^2)$ is bounded on $\overline{V}_j \cap \overline{V}_k$ uniformly with respect to $\delta \in [0,1]$. Since τ_j tends to $-\infty$ along bV_j, none of the values $v_{\delta,j}(x)$ for x sufficiently near bV_j contributes to the value of $v_\delta(x)$ since the other functions take over in rmax, and this property is uniform with respect to $\delta \in [0,1]$. Since $\log(\delta + |\phi_j(x)|^2)$ is plurisubharmonic on Λ_j if $j \in \{1,\dots,m\}$ (resp., on U_j if $j \in \{m+1,\dots,N\}$), we have $i\partial\overline{\partial}\, v_{\delta,j} \ge i\partial\overline{\partial}\, \tau_j$ on the respective sets. The above argument therefore gives a uniform lower bound for $i\partial\overline{\partial}\, v_\delta$ on the compact set Λ (see (2.3)). However, we cannot control the Levi forms of v_δ from below on the sets $V_j \backslash \Lambda_j$ for $j \in \{1,\dots,m\}$ since ϕ_j fails to be holomorphic there. □

LEMMA 2.4

Let $U \subset X$ be an open set containing $A \cup K$. There exists a neighborhood W of $A \cup K$ with $\overline{W} \subset U$ and a $\mathscr{C}^2$-function $\rho\colon X \to \mathbb{R}$ which is strongly plurisubharmonic on $\overline{W}$ such that $\rho < 0$ on K and $\rho > 0$ on bW.

Proof

Since $A \cap K$ is $\mathscr{O}(A)$-convex, there exists a compact neighborhood $K' \subset U \cap \Omega$ of K such that the set $K' \cap A \subset A \backslash bA$ is also $\mathscr{O}(A)$-convex. Since K is $\mathscr{O}(\Omega)$-convex, there is a smooth strongly plurisubharmonic function $\rho_0\colon \Omega \to \mathbb{R}$ such that $\rho_0 < 0$ on K and $\rho_0 > 1$ on $\Omega \backslash K'$ (see [47, Theorem 5.1.5, page 117]). Set $\Omega_c = \{x \in \Omega\colon \rho_0(x) < c\}$. Fixing a number c with $0 < c < 1/2$, we have $K \subset \Omega_c \subset \Omega_{2c} \subset K'$.

Since the restricted function $\rho_0|_{A\cap\Omega}$ is strongly subharmonic and the set $K' \cap A$ is $\mathscr{O}(A)$-convex, a standard argument (see [25, page 737]) gives another smooth function $\widetilde{\rho}_0\colon X \to \mathbb{R}$ which agrees with ρ_0 in a neighborhood of K' in X such that $\widetilde{\rho}_0|_A$ is strongly subharmonic, $\widetilde{\rho}_0 > c$ on $A \backslash \overline{\Omega}_c$, $\widetilde{\rho}_0 > 2c$ on $A \backslash \overline{\Omega}_{2c}$, and $\widetilde{\rho}_0|_{bA} = c_0 \ge 1$ is constant.

Choose a strongly increasing convex function $h\colon \mathbb{R} \to \mathbb{R}$ satisfying $h(t) \ge t$ for all $t \in \mathbb{R}$, $h(t) = t$ for $t \le c$, and $h(t) > t + 1$ for $t \ge 2c$. The function

$$\rho_1 = h \circ \widetilde{\rho}_0\colon X \to \mathbb{R} \tag{2.4}$$

is then strongly plurisubharmonic on K' and along A, and it satisfies

(i) $\rho_1 = \widetilde{\rho}_0 = \rho_0$ on $\overline{\Omega}_c$,
(ii) $\rho_1 \geq \widetilde{\rho}_0 > c$ on $A \backslash \overline{\Omega}_c$,
(iii) $\rho_1 > \widetilde{\rho}_0 + 1$ on $A \backslash \overline{\Omega}_{2c}$, and
(iv) $\rho_1|_{bA} = c_1 > 2$.

To complete the proof of Lemma 2.4, we need the following result (see [14, Theorem 4]).

LEMMA 2.5

Let A be a compact complex curve with $\mathscr{C}^2$-boundary in a complex space X. For every function $\rho_1 \colon X \to \mathbb{R}$ of class $\mathscr{C}^2$ such that $\rho_1|_A$ is strongly subharmonic, there exists a $\mathscr{C}^2$-function $\rho_2 \colon X \to \mathbb{R}$ which is strongly plurisubharmonic in a neighborhood of A and satisfies $\rho_2|_A = \rho_1|_A$.

Proof

Let $\{U_j \colon j = 1, \dots, N\}$ be the open covering of A chosen at the beginning of the proof of Theorem 2.1. (For the present purpose, we delete those sets that do not intersect A.) For each index $j \in \{1, \dots, m\}$, let $Z = (z, w) \colon U_j \to U'_j \subset \mathbb{C}^{1+n_j}$, Γ_j, Λ_j, and ϕ_j be as above. Denote by $\psi'_j \colon \Gamma_j \times \mathbb{C}^{n_j} \to \mathbb{R}$ the unique function that is independent of the variable $w \in \mathbb{C}^{n_j}$ and satisfies $\rho_1 = \psi'_j \circ Z$ on $A \cap U_j$. We extend ψ'_j to a $\mathscr{C}^2$-function $\psi'_j \colon U'_j \to \mathbb{R}$ which is independent of the w-variable and set

$$\psi_j = \psi'_j \circ Z \colon U_j \to \mathbb{R}. \tag{2.5}$$

Then $\psi_j|_{A \cap U_j} = \rho_1$, and there is an open set $\widetilde{\Gamma}_j \subset \{1 - r_j < |z| < 1 + r_j\}$, with $\Gamma_j \subset \widetilde{\Gamma}_j$, such that ψ_j is subharmonic in the open set

$$\widetilde{U}_j = \left\{x \in U_j \colon z(x) \in \widetilde{\Gamma}_j\right\} \subset X. \tag{2.6}$$

By choosing the remaining sets U_j for $j \in \{m+1, \dots, N\}$ sufficiently small, we also get a holomorphic map $\phi_j \colon U_j \to \mathbb{C}^{n_j}$, whose components generate the ideal sheaf of A at every point of U_j, and a strongly plurisubharmonic function $\psi_j \colon U_j \to \mathbb{R}$ extending $\rho_1|_{A \cap U_j}$.

Choose a smooth partition of unity $\{\theta_j\}$ on a neighborhood of A in X with $\operatorname{supp} \theta_j \subset U_j$ for $j = 1, \dots, N$. Fix an $\epsilon > 0$, and set

$$\rho_2(x) = \sum_{j=1}^{N} \theta_j(x) \left(\psi_j(x) + \epsilon^3 \log(1 + \epsilon^{-4} |\phi_j(x)|^2)\right).$$

For $x \in A$, we have $\rho_2(x) = \sum_j \theta_j(x) \psi_j(x) = \rho_1(x)$. One can easily verify that ρ_2 is strongly plurisubharmonic in a neighborhood of A in X provided that $\epsilon > 0$ is chosen sufficiently small. Indeed, as $\epsilon \to 0$, the function $\epsilon^3 \log(1 + \epsilon^{-4} |\phi_j(x)|^2)$

is of size $O(\epsilon^3)$, its first derivatives are of size $O(\epsilon)$, and its Levi form at points of $A_{\mathrm{reg}} \cap U_j$ in the direction normal to A is of size comparable to ϵ^{-1}, which implies that the Levi form of ρ_2 is positive definite at each point of A provided that $\epsilon > 0$ is chosen sufficiently small (see [14, proof of Theorem 4] for the details). □

With ρ_1 given by (2.4) and ρ_2 furnished by Lemma 2.5, we set

$$\rho = \mathrm{rmax}\{\widetilde{\rho}_0, \rho_2 - 1\}.$$

It is easily verified that ρ is strongly plurisubharmonic on a compact neighborhood $\overline{W} \subset U$ of the set $A \cup \overline{\Omega}_c$, $\rho = \widetilde{\rho}_0 = \rho_0$ on $\overline{\Omega}_c$ (hence $\rho < 0$ on K), $\rho = \rho_2 - 1 > \widetilde{\rho}_0$ in a neighborhood of $A \backslash \Omega_{2c}$, and $\rho|_{bA}$ has a constant value $C > 1$. After shrinking W around $A \cup \overline{\Omega}_c$, we also have $\rho > 0$ on bW. This concludes the proof of Lemma 2.4. □

Completion of the proof of Theorem 2.1

We use the notation established at the beginning of the proof: $U_j \subset X$ is an open Stein neighborhood of a boundary curve $C_j \subset bA$, Λ_j and $\phi_j : U_j \to \mathbb{C}^{n_j}$ are defined by (2.1) (resp., by (2.2)), and $\psi_j : U_j \to \mathbb{R}$ is defined by (2.5).

Let V be an open set containing $A \cup K$, and let $v_\delta : V \to \mathbb{R}$ ($\delta \in [0, 1]$) be a family of functions furnished by Lemma 2.3. Let Λ denote the corresponding set (2.3) on which $i\partial\overline{\partial}\, v_\delta$ is bounded from below uniformly with respect to $\delta \in (0, 1]$. As δ decreases to zero, the functions v_δ decrease monotonically to a function v_0 satisfying $\{v_0 = -\infty\} = A$. By subtracting a constant, we may assume that $v_\delta \le v_1 < 0$ on K for every $\delta \in [0, 1]$.

Given an open set $U \subset X$ containing $A \cup K$, we must find a Stein neighborhood $\omega \subset U$ of $A \cup K$. We may assume that $\overline{U} \subset V$. Let ρ be a function furnished by Lemma 2.4; thus ρ is strongly plurisubharmonic on the closure $\overline{W} \subset U$ of an open set $W \supset A \cup K$, $\rho|_K < 0$, and $\rho|_{bW} > 0$. Let

$$\rho_{\epsilon,\delta} = \rho + \epsilon\, v_\delta : \overline{W} \to \mathbb{R}.$$

Choose $\epsilon > 0$ sufficiently small such that $\rho_{\epsilon,0} > 0$ on bW (such ϵ exists since $\{v_0 = -\infty\} = A$); hence $\rho_{\epsilon,\delta} \ge \rho_{\epsilon,0} > 0$ on bW for every $\delta \in [0, 1]$. Decreasing $\epsilon > 0$ if necessary, we may assume that $\rho_{\epsilon,\delta}$ is strongly plurisubharmonic on $\Lambda \cap \overline{W}$ for every $\delta \in (0, 1]$ (since the positive Levi form of ρ compensates the small negative part of the Levi form of $\epsilon\, v_\delta$). Fix an ϵ with these properties. Now, choose a sufficiently small $\delta > 0$ such that $\rho_{\epsilon,\delta} < 0$ on A. (This is possible since v_δ decreases to v_0, which equals $-\infty$ on A.) Note that $\rho_{\epsilon,\delta} < 0$ on K since both ρ and v_δ are negative on K. By continuity, $\rho_{\epsilon,\delta}$ is strongly plurisubharmonic also on the set $\overline{W} \cap \widetilde{U}_j$ for every $j = 1, \ldots, m$, where $\widetilde{U}_j \subset U_j$ is an open set of the form (2.6).

The function $\psi_j\colon \widetilde{U}_j \to \mathbb{R}$ (see (2.5)) is plurisubharmonic on the open set $\widetilde{U}_j$ (see (2.6)) that contains Λ_j, ψ_j has a constant value c_1 on the curve $C_j \subset bA$, and $\{\psi_j \le c_1\} = \Lambda_j \supset A \cap U_j$. Let $\chi : \mathbb{R} \to \mathbb{R}_+$ be a smooth increasing convex function with $\chi(t) = 0$ for $t \le c_1$ and $\chi(t) > 0$ for $t > c_1$. The plurisubharmonic function $\chi \circ \psi_j\colon \widetilde{U}_j \to \mathbb{R}$ then vanishes on Λ_j and is positive on $\widetilde{U}_j \backslash \Lambda_j$; extending it by zero along A, we obtain a plurisubharmonic function $\psi\colon V \to \mathbb{R}_+$ which vanishes on $\overline{W} \cap \Lambda$ and is positive on each of the sets $\widetilde{U}_j \backslash \Lambda_j$ (where it agrees with $\chi \circ \psi_j$). By choosing χ to grow sufficiently fast on $\{t > c_1\}$, we can ensure that the sublevel set

$$\omega = \Big\{x \in W\colon \psi(x) + \rho_{\epsilon,\delta}(x) < 0\Big\} \Subset W$$

(which contains $A \cup K$) is contained in the set on which $\rho_{\epsilon,\delta}$ is strongly plurisubharmonic. The purpose of adding ψ is to round off the sublevel set sufficiently close to bA, where it exists from $\Lambda \cap \overline{W}$, thereby ensuring that ω remains in the region where the defining function $\psi + \rho_{\epsilon,\delta}$ is strongly plurisubharmonic. Narasimhan's theorem [62, Theorem, page 355] now implies that ω is a Stein domain. This completes the proof of Theorem 2.1. □

The restriction to one-dimensional subvarieties $A \subset X$ was essential only in the proof of Lemma 2.2. For higher-dimensional subvarieties, we have the following partial result.

THEOREM 2.6

Let $h\colon X \to S$ be a holomorphic map of a complex space X to a complex manifold S, and let $D \Subset S$ be a strongly pseudoconvex Stein domain in S. Let $f\colon \bar{D} \to X$ be a $\mathscr{C}^2$-section of h (i.e., $h(f(z)) = z$ for $z \in \bar{D}$) which is holomorphic in D. If $f(bD) \subset X_{\mathrm{reg}}$ and h is a submersion near $f(bD)$, then $A = f(\bar{D})$ has a basis of open Stein neighborhoods in X.

Proof

The only necessary change in the proof is in the construction of the sets Λ_j (2.1) and the functions ϕ_j (2.2), which describe the subvariety $A \subset X$ in a neighborhood of its boundary. When $\dim A = 1$, we can choose ϕ_j globally around the respective boundary curve $C_j \subset bA$ due to the existence of a Stein neighborhood of C_j. When $\dim A > 1$, this is no longer possible, and hence this step must be localized as follows.

Fix a point $p \in bD$, and let $q = f(p) \in bA \subset X_{\mathrm{reg}}$. Since h is a submersion near q, there are local holomorphic coordinates $x = (z, w)$ in an open neighborhood $U \subset X$ of q, and there is an open neighborhood $U' \subset S$ of the point $p = h(q)$ such that $h(x) = h(z, w) = z \in U'$ for $x \in U$, and $f(z) = (z, g(z))$ for $z \in U' \cap \bar{D}$. We take $\Lambda = \{x = (z, w) \in U : z \in U' \cap \bar{D}\}$ and $\phi(x) = \phi(z, w) = w - g(z)$. Covering

bA by finitely many such neighborhoods, the rest of the proof of Theorem 2.1 applies mutatis mutandis. $\square$

COROLLARY 2.7
Let S and X be complex manifolds, and let $D \Subset S$ be a strongly pseudoconvex Stein domain with boundary of class $\mathscr{C}^\ell$. If $2 \le r \le \ell$, then every $\mathscr{C}^r$-map $f : \bar{D} \to X$ which is holomorphic in D is a $\mathscr{C}^r(\bar{D})$-limit of a sequence of maps $f_j : U_j \to X$ which are holomorphic in small open neighborhoods of $\bar{D}$ in S.

For maps from Riemann surfaces, a stronger result is proved in §5.

Proof
When $S = \mathbb{C}^n$, $X = \mathbb{C}^N$, $\ell = 2$, and $r = 0$, this classical result on uniform approximation of holomorphic functions that are continuous up to the boundary follows from the Henkin-Ramírez integral kernel representation of functions in $\mathscr{A}(D)$ (see Henkin [42], Ramírez [66], Kerzman [49], Lieb [55], Henkin and Leiterer [44, page 87]). Another approach that works for $0 \le r \le \ell$, $2 \le \ell$, is via the solution to the $\bar{\partial}$-equation with $\mathscr{C}^r$-estimates (see Range and Siu [68], Lieb and Range [57], Michel and Perotti [60], and [56, Chapter 8, §3, Theorem 3.43]).

Assume now that X is a complex manifold and $2 \le r \le \ell$. By Theorem 2.6, the graph $G_f = \{(z, f(z)) : z \in \bar{D}\}$ admits an open Stein neighborhood Ω in $S \times X$. Choose a proper holomorphic embedding $\psi : \Omega \hookrightarrow \mathbb{C}^N$ and a holomorphic retraction $\pi : W \to \psi(\Omega)$ from an open neighborhood $W \subset \mathbb{C}^N$ of $\psi(\Omega)$ onto $\psi(\Omega)$. Choose a neighborhood $U \subset S$ of $\bar{D}$ and a sequence of holomorphic maps $g_j : U \to \mathbb{C}^N$ such that the sequence $g_j|_{\bar{D}}$ converges in $\mathscr{C}^r(\bar{D})$ to the map $z \to \psi(z, f(z))$ as $j \to +\infty$. Denote by $\mathrm{pr}_X : S \times X \to X$ the projection $(z, x) \to x$. Let $U_j = \{z \in U : g_j(z) \in W\}$. The sequence $f_j = \mathrm{pr}_X \circ \psi^{-1} \circ \pi \circ g_j : U_j \to X$ then satisfies Corollary 2.7. $\square$

Proofs of Theorem 1.7 and Corollary 1.8
Let $D \Subset S$ be a smoothly bounded domain in an open Riemann surface S, and let $f : \bar{D} \hookrightarrow X$ be a $\mathscr{C}^2$-embedding that is holomorphic in D. By Theorem 2.1, the image $f(\bar{D})$ admits an open Stein neighborhood $\Omega \subset X$. Choose a proper holomorphic embedding $\psi : \Omega \hookrightarrow \mathbb{C}^N$, and let $\Sigma = \psi(\Omega) \subset \mathbb{C}^N$. Also, choose a holomorphic retraction $\pi : W \to \Sigma$ from an open neighborhood $W \subset \mathbb{C}^N$ of Σ onto Σ. The embedding $\psi \circ f : \bar{D} \hookrightarrow \Sigma$ extends to a $\mathscr{C}^r$-map F from a neighborhood of $\bar{D}$ in S to Σ; as $r \ge 2$, $\bar{\partial} F$ and its first derivative $D^1(\bar{\partial} F)$ vanish on $\bar{D}$.

Set $A = F(\bar{D}) \subset \Sigma$. Let $\nu = T\Sigma|_A / TA$ denote the complex normal bundle of the embedding $F : \bar{D} \hookrightarrow \Sigma$; this bundle is holomorphic over $\mathrm{Int} A = F(D)$ and is continuous (even of class $\mathscr{C}^1$) up to the boundary. An application of Theorem B for vector bundles that are holomorphic in the interior and continuous up to the boundary (see [46], [53], [68]) gives a direct sum splitting $T\Sigma|_A = TA \oplus \nu$ which is holomorphic over Int A and continuous up to the boundary. (It suffices to follow the proof for vector bundles over open Stein manifolds; see, e.g., [41, page 256].)

Since A is a bordered Riemann surface, the bundle ν is topologically trivial and hence also holomorphically trivial in the sense that it is isomorphic to the product bundle $A \times \mathbb{C}^{n-1}$ ($n = \dim X = \dim \Sigma$) by a continuous complex vector bundle isomorphism that is holomorphic over the interior of A (see [45, Theorem 2], [52]). Hence there exist continuous vector fields $v_1, \dots, v_{n-1}$ tangent to $\nu \subset T\Sigma|_A$ which are holomorphic in the interior of A and generate ν at every point of A. Considering these fields as maps $A \to T\mathbb{C}^N = \mathbb{C}^N \times \mathbb{C}^N$, we can approximate them uniformly on A by vector fields (still denoted $v_1, \dots, v_{n-1}$) that are holomorphic in a neighborhood of A in Σ and tangent to Σ. (The last condition can be fulfilled by composing them with the differential of the retraction $\pi\colon W \to \Omega$.) If the approximations are sufficiently close on A, then the new vector fields are also linearly independent at each point of A and transverse to TA. The flow θ_j^t of v_j is defined and holomorphic for sufficiently small values of $t \in \mathbb{C}$ beginning at any point near A. The map

$$\widetilde{F}(z, t_1, \dots, t_{n-1}) = \theta_1^{t_1} \circ \cdots \circ \theta_{n-1}^{t_{n-1}} \circ F(z)$$

is a diffeomorphism from an open neighborhood of $\bar{D} \times \{0\}^{n-1}$ in $S \times \mathbb{C}^{n-1}$ onto an open neighborhood of $A = F(\bar{D})$ in $\Sigma \subset \mathbb{C}^N$. $\widetilde{F}$ is holomorphic in the variables $t = (t_1, \dots, t_{n-1})$ and satisfies $\frac{\partial \widetilde{F}}{\partial \bar{z}}(z, t) = 0$ for $z \in \bar{D}$.

Choose a strongly subharmonic $\mathscr{C}^2$-function $\rho\colon S \to \mathbb{R}$ such that $D = \{z \in S\colon \rho(z) < 0\}$ and $d\rho(z) \neq 0$ for every $z \in bD = \{\rho = 0\}$. For $\epsilon \geq 0$ (small and variable) and $M > 0$ (large and fixed), the set

$$O_\epsilon = \left\{(z, t) \in S \times \mathbb{C}^{n-1}\colon \rho(z) + M|t|^2 < \epsilon\right\}$$

is strongly pseudoconvex with $\mathscr{C}^2$-boundary and is contained in the domain of $\widetilde{F}$. (The latter condition is achieved by choosing $M > 0$ sufficiently large.) Note that $\bar{D} \times \{0\}^{n-1} \subset O_\epsilon$ for $\epsilon > 0$. The properties of $\widetilde{F}$ described above imply that $\|\bar{\partial}\widetilde{F}\|_{L^\infty(O_\epsilon)} = o(\epsilon)$ as $\epsilon \to 0$. There are constants $C > 0$ and $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0)$, the equation $\bar{\partial}U = \bar{\partial}\widetilde{F}$ has a solution $U = U_\epsilon \in \mathscr{C}^2(O_\epsilon)$ satisfying a uniform estimate

$$\|U_\epsilon\|_{L^\infty(O_\epsilon)} \leq C\|\bar{\partial}\widetilde{F}\|_{L^\infty(O_\epsilon)} = o(\epsilon) \tag{2.7}$$

(see [43], [56], [68], and the discussion in §3). The map

$$G_\epsilon = \pi \circ (\widetilde{F} - U_\epsilon)\colon O_\epsilon \to \Sigma \subset \mathbb{C}^N$$

is then holomorphic, and it is homotopic to $\widetilde{F}|_{O_\epsilon}$ through the homotopy $G_{\epsilon,s} = \pi \circ (\widetilde{F} - sU_\epsilon) \in \Sigma$ ($s \in [0,1]$) satisfying $\|G_{\epsilon,s} - \widetilde{F}\|_{L^\infty(O_\epsilon)} = o(\epsilon)$ as $\epsilon \to 0$, uniformly in $s \in [0,1]$. Choosing $\epsilon > 0$ sufficiently small, we conclude that $G_{\epsilon,s}(z,t) \in \Sigma \backslash \widetilde{F}(\bar{O}_0)$ for each $(z,t) \in bO_{\epsilon/2}$ and $s \in [0,1]$. It follows that for each point $x \in \widetilde{F}(\bar{O}_0)$, the number of solutions $(z,t) \in O_{\epsilon/2}$ of the equation $G_{\epsilon,s}(z,t) = x$, counted with algebraic multiplicities, does not depend on $s \in [0,1]$, and hence it equals one (its value at $s = 0$). Taking $s = 1$, we see that the set $G_\epsilon(O_{\epsilon/2})$ contains $\widetilde{F}(\bar{O}_0) \supset A$.

From (2.7) and the interior elliptic regularity estimates (see [31, Lemma 3.2]), we also see that $\|dU_\epsilon\|_{L^\infty(O_{\epsilon/2})} = o(1)$ as $\epsilon \to 0$, and hence G_ϵ is an injective immersion on $O_{\epsilon/2}$ for every sufficiently small $\epsilon > 0$ (since it is a $\mathscr{C}^1$-small perturbation of $\widetilde{F}$). For such values of ϵ, the set $U_\epsilon := \psi^{-1}(G_\epsilon(O_{\epsilon/2})) \subset X$ is an open Stein neighborhood of $f(\bar{D})$, and U_ϵ is biholomorphic (via $\psi^{-1} \circ G_\epsilon$) to the domain $O_{\epsilon/2} \subset S \times \mathbb{C}^{n-1}$.

Since X can be replaced by an arbitrary open neighborhood of $f(\bar{D})$ in the above construction, this concludes the proof of Theorem 1.7. □

The same proof gives Corollary 1.8. □

3. A Cartan-type lemma with estimates up to the boundary

In this section, we prove one of our main tools, Theorem 3.2.

Definition 3.1

A pair of relatively compact open subsets $D_0, D_1 \Subset S$ in a complex manifold S is said to be a *Cartan pair* of class $\mathscr{C}^\ell$ ($\ell \geq 2$) if

(i) the sets D_0, D_1, $D = D_0 \cup D_1$ and $D_{0,1} = D_0 \cap D_1$ are Stein domains with strongly pseudoconvex boundaries of class $\mathbb{C}^\ell$, and

(ii) $\overline{D_0 \backslash D_1} \cap \overline{D_1 \backslash D_0} = \emptyset$ (the separation property).

Replacing S by a suitably chosen neighborhood of $\overline{D_0 \cup D_1}$, we can assume that S is a Stein manifold.

Let P be a bounded open set in $\mathbb{C}^n$. We denote the variable in S by z and the variable in $\mathbb{C}^n$ by $t = (t_1, \ldots, t_n)$. For each pair of integers $r, s \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, we denote by $\mathscr{C}^{r,s}(\bar{D} \times P)$ the space of all functions $f \colon \bar{D} \times P \to \mathbb{C}$ with bounded partial derivatives up to order r in the z-variable and up to order s in the t-variable, endowed with the norm

$$\|f\|_{\mathscr{C}^{r,s}(D \times P)} = \sup\Big\{|D_z^\mu D_t^\nu f(z,t)| \colon z \in \bar{D},\ t \in P,\ |\mu| \leq r,\ |\nu| \leq s\Big\} < +\infty.$$

Here D_t^ν denotes the partial derivative of order $\nu \in \mathbb{Z}^{2n}$ with respect to the real and imaginary parts of the components t_j of $t \in \mathbb{C}^n$. The same definition applies to D_z^μ when $S = \mathbb{C}^m$; in general, we cover $\bar{D}$ with a finite system of local holomorphic charts $U_j \Subset V_j \subset S$, with biholomorphic maps $\phi_j \colon V_j \to V_j' \subset \mathbb{C}^m$, and take at each point $z \in \bar{D}$ the maximum of the above norms calculated in the ϕ_j-coordinates with respect to those charts (V_j, ϕ_j) for which $z \in U_j$. Alternatively, we can measure the z-derivatives with respect to a smooth Hermitian metric on S; the two choices yield equivalent norms on $\mathscr{C}^{r,s}(\bar{D} \times P)$. Set

$$\mathscr{A}^{r,s}(D \times P) = \mathscr{O}(D \times P) \cap \mathscr{C}^{r,s}(\bar{D} \times P), \quad r, s \in \mathbb{Z}_+.$$

For $t = (t_1, \dots, t_n) \in \mathbb{C}^n$, we write $|t| = \left(\sum |t_j|^2\right)^{1/2}$. For a map $f = (f_1, \dots, f_n) \colon \bar{D} \times P \to \mathbb{C}^n$ with components $f_j \in \mathscr{C}^{r,s}(\bar{D} \times P)$, we set

$$\|f\|_{\mathscr{C}^{r,s}(D \times P)} = \left(\sum_{j=1}^{n} \|f_j\|^2_{\mathscr{C}^{r,s}(D \times P)}\right)^{1/2}.$$

Let $\mathbb{B}(t; \delta) \subset \mathbb{C}^n$ denote the ball of radius $\delta > 0$ centered at $t \in \mathbb{C}^n$. For any subset $P \subset \mathbb{C}^n$ and $\delta > 0$, we set

$$P_{-\delta} = \left\{t \in P : \mathbb{B}(t; \delta) \subset P\right\}.$$

THEOREM 3.2 (Generalized Cartan lemma)
Let (D_0, D_1) be a Cartan pair of class $\mathscr{C}^\ell$ ($\ell \geq 2$), and let P be a bounded open set in $\mathbb{C}^n$ containing the origin. Set $D = D_0 \cup D_1$ and $D_{0,1} = D_0 \cap D_1$. Given $\delta^* > 0$ and $r \in \{0, 1, \dots, \ell\}$, there exist numbers $\epsilon^* > 0$ and $M_{r,s} \geq 1$ ($s = 0, 1, 2, \dots$) satisfying the following. For every map $\gamma \colon \bar{D}_{0,1} \times P \to \mathbb{C}^n$ of class $\mathscr{A}^{r,0}(D_{0,1} \times P)^n$ satisfying

$$\gamma(z, t) = t + c(z, t), \quad \|c\|_{\mathscr{C}^{r,0}(D_{0,1} \times P)} < \epsilon^*,$$

there exist maps $\alpha \colon \bar{D}_0 \times P_{-\delta^*} \to \mathbb{C}^n$, $\beta \colon \bar{D}_1 \times P_{-\delta^*} \to \mathbb{C}^n$ of the form

$$\alpha(z, t) = t + a(z, t), \qquad \beta(z, t) = t + b(z, t),$$

with $a \in \mathscr{A}^{r,s}(D_0 \times P_{-\delta^*})^n$ and $b \in \mathscr{A}^{r,s}(D_1 \times P_{-\delta^*})^n$ for all $s \in \mathbb{Z}_+$, which are fiberwise injective holomorphic and satisfy

$$\gamma\big(z, \alpha(z, t)\big) = \beta(z, t), \quad z \in \bar{D}_{0,1}, \ t \in P_{-\delta^*}, \tag{3.1}$$

and also the estimates

$$\|a\|_{\mathscr{C}^{r,s}(D_0 \times P_{-\delta^*})} \leq M_{r,s} \cdot \|c\|_{\mathscr{C}^{r,0}(D_{0,1} \times P)},$$

$$\|b\|_{\mathscr{C}^{r,s}(D_1 \times P_{-\delta^*})} \leq M_{r,s} \cdot \|c\|_{\mathscr{C}^{r,0}(D_{0,1} \times P)}.$$

If $\gamma(z, t) = t + c(z, t)$ is tangent to the map $\gamma_0(z, t) = t$ to order $m \in \mathbb{N}$ at $t = 0$ (i.e., the function $c(\cdot\,, t)$ vanishes to order m at $t = 0$), then α and β can be chosen to satisfy the same property.

Remark 3.3
The relation (3.1) is equivalent to

$$\gamma_z = \beta_z \circ \alpha_z^{-1}, \quad z \in \bar{D}_{0,1}.$$

The classical *Cartan lemma* (see [41, Theorem 7, page 199]) corresponds to the special case when $\alpha_z = \alpha(z, \cdot)$, β_z, and γ_z are linear automorphisms of $\mathbb{C}^n$ depending holomorphically on the point z in the respective base domain. A version of the Cartan lemma without shrinking the base domains was proved by Douady [18] and was proved for matrix-valued functions of class $\mathscr{A}^\infty$ by Sebbar [72, Theorem 1.4]. Berndtsson and Rosay [6] proved a splitting lemma over the disc $\triangle$ for bounded holomorphic maps into $\mathrm{GL}_n(\mathbb{C})$. A key difference between all these results and Theorem 3.2 is that we do not restrict ourselves to fiberwise linear maps. A result similar to Theorem 3.2, but less precise as it requires shrinking of the base domains, is [26, Lemma 2.1], which follows from [23, Theorem 4.1]. That lemma does not suffice for the application in this article, where it is essential that no shrinking be allowed in the base domain.

Theorem 3.2 is proved by a rapidly convergent iteration similar to the one in [23, proof of Theorem 4.1], but with estimates of derivatives. At an inductive step, we split the map $c(z,t) = \gamma(z,t) - t$ into a difference $c = b - a$, where the maps $a\colon \bar{D}_0 \times P \to \mathbb{C}^n$ and $b\colon \bar{D}_1 \times P \to \mathbb{C}^n$ are of class $\mathscr{A}^{r,0}$, with estimates of their $\mathscr{C}^{r,0}$-norms in terms of the $\mathscr{C}^{r,0}$-norm of c (see Lemma 3.4). Set

$$\alpha_z(t) = \alpha(z,t) := t + a(z,t), \qquad \beta_z(t) = \beta(z,t) := t + b(z,t).$$

We then show that for $z \in \bar{D}_{0,1}$ and t in a smaller set $P_{-\delta} \subset \mathbb{C}^n$, with ϵ sufficiently small compared to δ, there exists a map $\widetilde{\gamma}\colon \bar{D}_{0,1} \times P_{-\delta} \to \mathbb{C}^n$ of the form $\widetilde{\gamma}(z,t) = t + \widetilde{c}(z,t)$ satisfying

$$\gamma_z \circ \alpha_z = \beta_z \circ \widetilde{\gamma}_z, \quad z \in \bar{D}_{0,1},$$

and a quadratic estimate

$$\widetilde{\epsilon} = \|\widetilde{c}\|_{\mathscr{C}^{r,0}(D_{0,1} \times P_{-\delta})} \le \mathrm{const} \cdot \frac{\|c\|^2_{\mathscr{C}^{r,0}(D_{0,1} \times P)}}{\delta}$$

(see Lemma 3.5). If $\epsilon = \|c\|_{\mathscr{C}^{r,0}(D_{0,1} \times P)}$ is sufficiently small compared to δ, then $\widetilde{\epsilon}$ is much smaller than ϵ. Choosing a sequence of δ 's with the sum $\delta^*/2$ and assuming that the initial map c is sufficiently small, the sequences of compositions of the maps α_z (resp., β_z), obtained in the individual steps, converge on $P_{-\delta^*/2}$ to limit maps α (resp., β) satisfying $\gamma_z \circ \alpha_z = \beta_z$ for $z \in \bar{D}_{0,1}$. After another shrinking of the fiber by $\delta^*/2$, we obtain injective holomorphic maps on $P_{-\delta^*}$ satisfying the estimates in Theorem 3.2.

We begin by recalling the relevant results on the solvability of the $\bar\partial$-equation. Let D be a relatively compact strongly pseudoconvex domain with boundary of class $\mathscr{C}^\ell$ ($\ell \geq 2$) in a Stein manifold S. Let $\mathscr{C}^r_{0,1}(\bar D)$ denote the space of (0, 1)-forms with $\mathscr{C}^r$-coefficients on $\bar D$, and let $\mathscr{Z}^r_{0,1}(\bar D) = \{f \in \mathscr{C}^r_{0,1}(\bar D) \colon \bar\partial f = 0\}$. According to Range and Siu [68] and Lieb and Range [57, Theorem 1] (see also [60, Theorem 1′]), there exists a linear operator $T \colon \mathscr{C}^0_{0,1}(D) \to \mathscr{C}^0(D)$ satisfying the following properties:

(i) if $f \in \mathscr{C}^0_{0,1}(\bar D) \cap \mathscr{C}^1_{0,1}(D)$ and $\bar\partial f = 0$, then $\bar\partial(Tf) = f$;

(ii) if $f \in \mathscr{C}^0_{0,1}(\bar D) \cap \mathscr{C}^r_{0,1}(D)$ $(1 \leq r \leq \ell)$, then for each $l = 0, 1, \ldots, r$,

$$\|Tf\|_{\mathscr{C}^{l,1/2}(\bar D)} \leq C_l \|f\|_{\mathscr{C}^l_{0,1}(\bar D)}. \tag{3.2}$$

The results in [57] are stated only for the case $bD \in \mathscr{C}^\infty$, but a more careful analysis shows that one needs only $\mathscr{C}^\ell$-boundary in order to get estimates up to order ℓ; this is implicitly contained in the article by Michel and Perotti [60] (the special case of domains without corners). The case of domains in Stein manifolds easily reduces to the Euclidean case by standard techniques (holomorphic embeddings and retractions). Lieb and Range showed that for strongly pseudoconvex domains with smooth boundaries in $\mathbb{C}^n$, the estimates (3.2) also hold for the Kohn solution operator $T = \bar\partial^* N$ (see [59], [58, Corollary 2]). Here $\bar\partial^*$ is the formal adjoint of $\bar\partial$ on (0, 1)-forms (under a suitable choice of a Hermitian metric on S), and N is the corresponding Neumann operator on (0, 1)-forms on D (the inverse of the complex Laplacian $\Box = \bar\partial\,\bar\partial^* + \bar\partial^*\bar\partial$ acting on (0, 1)-forms; see also [56, Chapter 8, §3, Theorem 3.43]; for Sobolev estimates, see [11, Theorem 5.2.6, page 103]).

LEMMA 3.4

Let $D = D_0 \cup D_1 \Subset S$, $D_{0,1} = D_0 \cap D_1$, and $P \subset \mathbb{C}^n$ be as in Theorem 3.2. For every $r \in \{0, 1, \ldots, \ell\}$, there are a constant $C_r \geq 1$, independent of P, and linear operators

$$A \colon \mathscr{A}^{r,0}(D_{0,1} \times P)^n \longrightarrow \mathscr{A}^{r,0}(D_0 \times P)^n,$$
$$B \colon \mathscr{A}^{r,0}(D_{0,1} \times P)^n \longrightarrow \mathscr{A}^{r,0}(D_1 \times P)^n$$

satisfying

$$c = Bc|_{\bar D_{0,1} \times P} - Ac|_{\bar D_{0,1} \times P}, \quad c \in \mathscr{A}^{r,0}(D_{0,1} \times P)^n,$$

and the estimates

$$\|Ac\|_{\mathscr{C}^{r,0}(D_0\times P)} \le C_r\cdot \|c\|_{\mathscr{C}^{r,0}(D_{0,1}\times P)},$$
$$\|Bc\|_{\mathscr{C}^{r,0}(D_1\times P)} \le C_r\cdot \|c\|_{\mathscr{C}^{r,0}(D_{0,1}\times P)}.$$

If c vanishes to order $m\in\mathbb{N}$ at $t=0$, then so do Ac and Bc.

Proof

The separation condition (ii) in the definition of a Cartan pair implies that there exists a smooth function χ on S with values in $[0,1]$ such that $\chi=0$ in an open neighborhood of $\overline{D_0\backslash D_1}$ and $\chi=1$ in an open neighborhood of $\overline{D_1\backslash D_0}$. Note that $\chi(z)c(z,t)$ extends to a function in $\mathscr{C}^{r,0}(\bar D_0\times P)$ which vanishes on $\overline{D_0\backslash D_1}\times P$, and $(\chi(z)-1)c(z,t)$ extends to a function in $\mathscr{C}^{r,0}(\bar D_1\times P)$ which vanishes on $\overline{D_1\backslash D_0}\times P$. Furthermore, $\bar\partial(\chi c)=\bar\partial((\chi-1)c)=c\bar\partial\chi$ is a $(0,1)$-form on $\bar D$ with $\mathscr{C}^r$-coefficients and with support in $\bar D_{0,1}\times P$, depending holomorphically on $t\in P$.

Let T denote a linear solution operator to the $\bar\partial$-equation satisfying (3.2). For any $c\in\mathscr{A}^{r,0}(D_{0,1}\times P)$ and $t\in P$, we set

$$(Ac)(z,t)=\big(\chi(z)-1\big)c(z,t)-T\big(c(\cdot,t)\bar\partial\chi\big)(z),\quad z\in\bar D_0.$$
$$(Bc)(z,t)=\chi(z)c(z,t)-T\big(c(\cdot,t)\bar\partial\chi\big)(z),\quad z\in\bar D_1.$$

Then $Ac-Bc=c$ on $\bar D_{0,1}\times P$, $\bar\partial_z(Ac)=0$, and $\bar\partial_z(Bc)=0$ on their respective domains. The bounded linear operator T commutes with the derivative $\bar\partial_t$ on the parameter t. Since $\bar\partial_t(c(z,t)\bar\partial\chi(z))=0$, we get $\bar\partial_t(Ac)=0$ and $\bar\partial_t(Bc)=0$. The estimates follow from boundedness of T (see (3.2)). □

LEMMA 3.5

Let $D=D_0\cup D_1\Subset S$, $D_{0,1}=D_0\cap D_1$, and $P\subset\mathbb{C}^n$ be as in Theorem 3.2. Given $c\in\mathscr{A}^{r,0}(D_{0,1}\times P)^n$, let $a=Ac$ and $b=Bc$ be as in Lemma 3.4. Let $\alpha\colon\bar D_0\times P\to\mathbb{C}^n$, $\beta\colon\bar D_1\times P\to\mathbb{C}^n$, and $\gamma\colon\bar D_{0,1}\times P\to\mathbb{C}^n$ be given by

$$\alpha(z,t)=t+a(z,t),\qquad \beta(z,t)=t+b(z,t),\qquad \gamma(z,t)=t+c(z,t).$$

Let $C_r\ge 1$ be the constant in Lemma 3.4. There is a constant $K_r>0$ with the following property. If $4\sqrt{n}C_r\|c\|_{\mathscr{C}^{r,0}(D_{0,1}\times P)}<\delta$, then there is a map $\widetilde\gamma\colon\bar D_{0,1}\times P_{-\delta}\to\mathbb{C}^n$ of the form $\widetilde\gamma(z,t)=t+\widetilde c(z,t)$, with $\widetilde c\in\mathscr{A}^{r,0}(D_{0,1}\times P_{-\delta})^n$, satisfying the identity

$$\gamma_z\circ\alpha_z=\beta_z\circ\widetilde\gamma_z,\quad z\in\bar D_{0,1},$$

and the estimate

$$\|\widetilde c\|_{\mathscr{C}^{r,0}(D_{0,1}\times P_{-\delta})}\le K_r\cdot\frac{\|c\|^2_{\mathscr{C}^{r,0}(D_{0,1}\times P)}}{\delta}.$$

If the functions a, b, and c vanish to order $m \in \mathbb{N}$ at $t = 0$, then so does $\widetilde{c}$.

Proof

We begin by estimating the composition $\gamma_z \circ \alpha_z$. Since the same estimate is used for other compositions as well, we formulate the result as an independent lemma.

LEMMA 3.6

Let D be a domain with $\mathscr{C}^1$-boundary in a complex manifold S, let P be an open set in $\mathbb{C}^n$, and let $0 < \delta < 1$. Given maps $\alpha_j(z,t) = t + a_j(z,t)$ $(j = 0, 1)$ with $a_0 \in \mathscr{A}^{r,0}(D \times P)^n$, $a_1 \in \mathscr{A}^{r,0}(D \times P_{-\delta})^n$, and $\|a_1\|_{\mathscr{C}^{r,0}(D\times P_{-\delta})} < \delta/2$, we have for all $(z,t) \in \bar{D} \times P_{-\delta}$,

$$\alpha_0\big(z, \alpha_1(z,t)\big) = t + a_0(z,t) + a_1(z,t) + e(z,t),$$

where

$$\|e\|_{\mathscr{C}^{r,0}(D\times P_{-\delta})} \le \frac{L_r}{\delta} \cdot \|a_0\|_{\mathscr{C}^{r,0}(D\times P)} \cdot \|a_1\|_{\mathscr{C}^{r,0}(D\times P_{-\delta})}$$

for some constant $L_r > 0$ depending only on r and n.

Proof

We have

$$\begin{aligned}\alpha_0\big(z, \alpha_1(z,t)\big) &= \alpha_1(z,t) + a_0\big(z, \alpha_1(z,t)\big)\\ &= t + a_1(z,t) + a_0\big(z, t + a_1(z,t)\big)\\ &= t + a_0(z,t) + a_1(z,t) + e(z,t),\end{aligned}$$

where the error term equals

$$e(z,t) = a_0\big(z, t + a_1(z,t)\big) - a_0(z,t).$$

Fix a point $(z,t) \in \bar{D} \times P_{-\delta}$. Since $|a_1(z,t)| < \delta/2$, the line segment $\lambda \subset \mathbb{C}^n$ with the endpoints t and $\alpha_1(z,t) = t + a_1(z,t)$ is contained in $P_{-\delta/2}$. Using the Cauchy estimates for the partial derivative $\partial_t a_0$, we obtain

$$\begin{aligned}|e(z,t)| &= \left| \int_0^1 (\partial_t a_0)\big(z, t + \tau a_1(z,t)\big) \cdot a_1(z,t)\, d\tau \right|\\ &\le \sup_{t' \in \lambda} \|\partial_t a_0(z,t')\| \cdot |a_1(z,t)|\\ &\le \frac{2\sqrt{n}}{\delta} \cdot \|a_0\|_{\mathscr{C}^{0,0}(D\times P)} \cdot \|a_1\|_{\mathscr{C}^{0,0}(D\times P_{-\delta})},\end{aligned}$$

which is the required estimate for $r = 0$. We proceed to estimate the partial differential of $e(z,t)$:

$$\begin{aligned}\partial_z e(z,t) = (\partial_z a_0)\big(z, t + a_1(z,t)\big) - (\partial_z a_0)(z,t)\\ + (\partial_t a_0)\big(z, t + a_1(z,t)\big)\cdot(\partial_z a_1)(z,t).\end{aligned}$$

The difference in the first line equals

$$\int_0^1 \partial_t(\partial_z a_0)\big(z, t + \tau a_1(z,t)\big)\cdot a_1(z,t)\,d\tau,$$

which can be estimated exactly as above (using the Cauchy estimates for $\partial_t\partial_z a_0$) by

$$\frac{\text{const}}{\delta}\cdot \|a_0\|_{\mathscr{C}^{1,0}(D\times P)}\cdot \|a_1\|_{\mathscr{C}^{0,0}(D\times P_{-\delta})}.$$

Applying the Cauchy estimate for $\partial_t a_0$, we estimate the remaining term in the expression for $e(z,t)$ by

$$\frac{\text{const}}{\delta}\cdot \|a_0\|_{\mathscr{C}^{0,0}(D\times P)}\cdot \|a_1\|_{\mathscr{C}^{1,0}(D\times P_{-\delta})}.$$

This proves the estimate in Lemma 3.6 for $r = 1$.

We proceed in a similar way to estimate the higher-order derivatives of e. In the expression for $\partial_z^k e(z,t)$, we have a main term

$$(\partial_z^k a_0)\big(z, t + a_1(z,t)\big) - (\partial_z^k a_0)(z,t) = \int_0^1 \partial_t(\partial_z^k a_0)\big(z, t + \tau a_1(z,t)\big)\cdot a_1(z,t)\,d\tau,$$

which is estimated by $\text{const}\cdot \delta^{-1}\|a_0\|_{\mathscr{C}^{k,0}(D\times P)}\cdot \|a_1\|_{\mathscr{C}^{0,0}(D\times P_{-\delta})}$. The remaining terms in $e(z,t)$ are products of partial derivatives of order at most k of a_0 (with respect to both z and t variables) with partial derivatives of a_1 of order at most k with respect to the z-variable. Each t-derivative of a_0 can be removed by using the Cauchy estimates, contributing another δ in the denominator. The chain rule shows that each term containing l derivatives of a_0 on the t-variable is multiplied by l factors involving a_1 and its z-derivatives; this gives an estimate $\text{const}\cdot \delta^{-l}\,\|a_0\|_{\mathscr{C}^{k,0}(D\times P)}\cdot \|a_1\|^l_{\mathscr{C}^{k,0}(D\times P_{-\delta})}$. Since we have assumed that $\|a_1\|_{\mathscr{C}^{r,0}(D\times P)} < \delta/2$, this is less than

$$\frac{\text{const}}{\delta}\cdot \|a_0\|_{\mathscr{C}^{k,0}(D\times P)}\cdot \|a_1\|_{\mathscr{C}^{k,0}(D\times P_{-\delta})},$$

and the lemma is proved. □

Now, let α, β, and γ be as in Lemma 3.5. Set $\epsilon = \|c\|_{\mathscr{C}^{r,0}(D_{0,1}\times P)}$; then $\|a\|_{\mathscr{C}^{r,0}(D_0\times P)} \le C_r\epsilon$ and $\|b\|_{\mathscr{C}^{r,0}(D_1\times P)} \le C_r\epsilon$ by Lemma 3.4. Since we have assumed that

$4\sqrt{n}C_r\epsilon < \delta$, Lemma 3.6 with $\alpha_0 = \gamma$ and $\alpha_1 = \alpha$ gives, for $z \in \bar{D}_{0,1}$ and $t \in P_{-\delta}$,

$$\gamma\big(z, \alpha(z,t)\big) = t + c(z,t) + a(z,t) + e(z,t) = \beta(z,t) + e(z,t) \in P_{-\delta/2},$$

where

$$\|e\|_{\mathscr{C}^{r,0}(D_{0,1}\times P_{-\delta})} \le \frac{L_r}{\delta}\cdot \|c\|_{\mathscr{C}^{r,0}(D_{0,1}\times P)}\cdot \|a\|_{\mathscr{C}^{r,0}(D_{0,1}\times P_{-\delta})} \le \frac{L_r C_r \epsilon^2}{\delta}.$$

It remains to find a map $\widetilde{\gamma}(z,t) = t + \widetilde{c}(z,t)$ on $\bar{D}_{0,1} \times P_{-\delta}$ satisfying

$$\beta(z,t) + e(z,t) = \beta\big(z, t + \widetilde{c}(z,t)\big) = t + \widetilde{c}(z,t) + b\big(z, t + \widetilde{c}(z,t)\big)$$

and an estimate

$$\|\widetilde{c}\|_{\mathscr{C}^{r,0}(D_{0,1}\times P_{-\delta})} \le \mathrm{const}\cdot \epsilon^2\delta^{-1}.$$

For the existence of $\widetilde{\gamma}$, it suffices to see that the map β_z is injective on $P_{-\delta/4}$ and $\beta_z(P_{-\delta/4}) \supset P_{-\delta/2}$ for every $z \in \bar{D}_{0,1}$; since $\gamma_z \circ \alpha_z \in P_{-\delta/2}$, we can then take $\widetilde{\gamma}_z = \beta_z^{-1} \circ \gamma_z \circ \alpha_z$. To see the injectivity of β_z, note that for $t, t' \in P_{-\delta/4}$, $t \ne t'$, we have

$$|\beta_z(t) - \beta_z(t')| \ge |t - t'| - |b_z(t) - b_z(t')| \ge |t - t'|\Big(1 - \frac{4\sqrt{n}C_0\epsilon}{\delta}\Big) > 0.$$

(We applied the Cauchy estimate to $\partial_t b_z$.) The inclusion $P_{-\delta/2} \subset \beta_z(P_{-\delta/4})$ follows from the estimate $\|b\|_{\mathscr{C}^{r,0}(D_1\times P)} \le C_r\epsilon \le \delta/(4\sqrt{n})$ by Rouché's theorem.

In order to estimate $\widetilde{c}$, we rewrite its defining equation in the form

$$\begin{aligned}\widetilde{c}(z,t) &= b(z,t) - b\big(z, t + \widetilde{c}(z,t)\big) + e(z,t)\\ &= -\int_0^1 (\partial_t b)\big(z, t + \tau\widetilde{c}(z,t)\big)\cdot \widetilde{c}(z,t)\, d\tau + e(z,t).\end{aligned}$$

Since the path of integration lies in $P_{-\delta/2}$, the Cauchy estimates for $\partial_t b$ give

$$|\widetilde{c}(z,t)| \le \frac{2\sqrt{n}C_0\epsilon}{\delta}\cdot |\widetilde{c}(z,t)| + |e(z,t)| \le \frac{1}{2}\,|\widetilde{c}(z,t)| + |e(z,t)|$$

and hence $|\widetilde{c}(z,t)| \le 2|e(z,t)| \le \mathrm{const}\cdot \epsilon^2\delta^{-1}$. We proceed inductively to estimate the derivatives $\partial_z^k\widetilde{c}$ for $k \le r$ by differentiating the implicit equation for $\widetilde{c}$. The top-order differential $|\partial_z^k\widetilde{c}|$ appearing on the right-hand side is multiplied by a constant less than 1 arising from an estimate on b (just as was done above); subsuming this term by the left-hand side, we obtain the estimates of $|\partial_z^k\widetilde{c}|$ for all $k \le r$. Although we obtain a term δ^r in the denominator, we can cancel $r-1$ powers of δ by appropriate terms of size $O(\epsilon)$, just as we did at the end of proof of Lemma 3.6 to get $\|\widetilde{c}\|_{\mathscr{C}^{r,0}(D_{0,1}\times P_{-\delta})} = O(\epsilon^2\delta^{-1})$. □

Proof of Theorem 3.2

We write $(\gamma\alpha)(z,t) = \gamma(z, \alpha(z,t))$, and similarly for the fiberwise composition of several maps. Let

$$\gamma(z,t) = \gamma_0(z,t) = t + c_0(z,t), \quad \epsilon_0 = \|c_0\|_{\mathscr{C}^{r,0}(D_{0,1}\times P)},$$

and let $\delta^* > 0$ be as in Theorem 3.2. We first describe the inductive procedure and subsequently show convergence, provided that $\epsilon_0 > 0$ is sufficiently small. Let $P_0 = P$ and $P_* = P_{-\delta^*/2}$. For every $k \in \mathbb{Z}_+$, set

$$\delta_k = 2^{-k-2}\delta^*, \quad P_{k+1} = (P_k)_{-\delta_k}.$$

Then $\sum_{k=0}^{\infty} \delta_k = \delta^*/2$, and $\bigcap_{k=0}^{\infty} P_k = \bar{P}_*$. Let $C_r \geq 1$, $K_r \geq 1$, and $L_r \geq 1$ be the constants in Lemmas 3.4, 3.5, and 3.6, respectively. We inductively construct sequences of maps

$$\begin{aligned} \alpha_k(z,t) &= t + a_k(z,t), \quad a_k \in \mathscr{A}^{r,0}(D_0 \times P_k)^n, \\ \beta_k(z,t) &= t + b_k(z,t), \quad b_k \in \mathscr{A}^{r,0}(D_1 \times P_k)^n, \\ \gamma_k(z,t) &= t + c_k(z,t), \quad c_k \in \mathscr{A}^{r,0}(D_{0,1} \times P_k)^n, \end{aligned}$$

such that, setting $\epsilon_k = \|c_k\|_{\mathscr{C}^{r,0}(D_{0,1}\times P_k)}$, the following hold for all $k \in \mathbb{Z}_+$:

(1_k) $\|a_k\|_{\mathscr{C}^{r,0}(D_0\times P_k)} \leq C_r\epsilon_k$, $\|b_k\|_{\mathscr{C}^{r,0}(D_1\times P_k)} \leq C_r\epsilon_k$;

(2_k) $4\sqrt{n}C_r\epsilon_k < \delta_k = 2^{-k-2}\delta^*$;

(3_k) $\gamma_k\alpha_k = \beta_k\gamma_{k+1}$ on $\bar{D}_{0,1} \times P_{k+1}$;

(4_k) $\epsilon_{k+1} = \|c_{k+1}\|_{\mathscr{C}^{r,0}(D_{0,1}\times P_{k+1})} \leq K_r\epsilon_k^2\delta_k^{-1} = (4K_r\delta^{*-1})2^k\epsilon_k^2$.

These conditions imply, for every $k \in \mathbb{Z}_+$,

$$\gamma_0(\alpha_0\alpha_1\cdots\alpha_k) = (\beta_0\beta_1\cdots\beta_k)\gamma_{k+1} \quad \text{on } \bar{D}_{0,1} \times P_{k+1}. \tag{3.3}$$

Assuming that $\epsilon_0 = \|c_0\|_{\mathscr{C}^{r,0}(D_{0,1}\times P)} > 0$ is sufficiently small, we prove that as $k \to +\infty$, the sequence of maps

$$\widetilde{\alpha}_k = \alpha_0\alpha_1\cdots\alpha_k \colon \bar{D}_0 \times P_k \to \mathbb{C}^n \tag{3.4}$$

converges to a map $\alpha\colon \bar{D}_0 \times P_* \to \mathbb{C}^n$, the sequence

$$\widetilde{\beta}_k = \beta_0\beta_1\cdots\beta_k \colon \bar{D}_1 \times P_k \to \mathbb{C}^n \tag{3.5}$$

converges to a map $\beta\colon \bar{D}_1 \times P_* \to \mathbb{C}^n$, and the sequence γ_k converges on $\bar{D}_{0,1} \times P_*$ to the map $(z,t) \to t$. (All convergences are in the $\mathscr{C}^{r,0}$-norms on the respective domains.) In the limit, we obtain a desired splitting

$$\gamma\alpha = \beta \quad \text{on } \bar{D}_{0,1} \times P_*.$$

We begin at $k = 0$ with the given map $\gamma_0(z,t) = t + c_0(z,t)$ on $\bar{D}_{0,1} \times P_0$. Lemma 3.4, applied to c_0, gives maps a_0 and b_0 satisfying (1_0). If (2_0) holds (which is the case if $\epsilon_0 = \|c_0\|_{\mathscr{C}^{r,0}(D_{0,1}\times P_0)} > 0$ is sufficiently small), then Lemma 3.5 furnishes a map $\gamma_1 \colon \bar{D}_{0,1} \times P_1 \to \mathbb{C}^n$ satisfying (3_0) and (4_0).

Assume inductively that for some $k \in \mathbb{N}$, we already have maps satisfying (1_j)–(4_j) for $j = 0, \dots, k-1$, and consequently, (3.3) holds with k replaced by $k-1$. Lemma 3.4, applied to $c_k(z,t) = \gamma_k(z,t) - t$ on $\bar{D}_{0,1} \times P_k$, gives maps a_k and b_k satisfying (1_k). If (2_k) holds (and we show that it does if ϵ_0 is sufficiently small), then Lemma 3.5, applied with $\alpha = \alpha_k$, $\beta = \beta_k$, $\gamma = \gamma_k$, furnishes a map $\widetilde{\gamma} = \gamma_{k+1} \colon \bar{D}_{0,1} \times P_{k+1} \to \mathbb{C}^n$ satisfying (3_k) and (4_k). This completes the inductive step.

To make the induction work, we must ensure that the sequence $\epsilon_k = \|c_k\|_{\mathscr{C}^{r,0}(D_{0,1}\times P_k)}$ satisfies (2_k) for every $k = 0, 1, 2, \dots$. To control this process, we set $N = \max\{4K_r/\delta^*, 1\}$ and define a sequence $\sigma_k > 0$ by

$$\sigma_0 = \epsilon_0, \qquad \sigma_{k+1} = 2^k N \sigma_k^2, \quad k = 0, 1, 2, \dots. \tag{3.6}$$

Any sequence $\epsilon_k \geq 0$ beginning with $\epsilon_0 = \sigma_0$ and satisfying (4_k) for all $k \in \mathbb{Z}_+$ clearly satisfies $\epsilon_k \leq \sigma_k$. If we can ensure (by choosing $\epsilon_0 > 0$ sufficiently small) that

$$\sigma_k < \frac{\delta^*}{2^{k+4}\sqrt{n}C_r}, \quad k \in \mathbb{Z}, \tag{3.7}$$

then $4\sqrt{n}C_r\epsilon_k \leq 4\sqrt{n}C_r\sigma_k < 2^{-k-2}\delta^* = \delta_k$, and hence (2_k) holds.

We look for a solution in the form $\sigma_k = 2^{\mu_k} N^{\nu_k} {\epsilon_0}^{\tau_k}$. From (3.6), we get

$$\mu_{k+1} = 2\mu_k + k, \quad \mu_0 = 0;$$
$$\nu_{k+1} = 2\nu_k + 1, \quad \nu_0 = 0;$$
$$\tau_{k+1} = 2\tau_k, \quad \tau_0 = 1.$$

Solutions are

$$\mu_k = 2^k \sum_{l=1}^{k} l2^{-l} < 2^{k+1}, \qquad \nu_k = 2^k - 1, \qquad \tau_k = 2^k.$$

Therefore

$$\sigma_k < 2^{2^{k+1}} N^{2^k} {\epsilon_0}^{2^k} = (4N\epsilon_0)^{2^k}, \quad k \in \mathbb{N}. \tag{3.8}$$

If $\epsilon_0 = \|c_0\|_{\mathscr{C}^{r,0}(D_{0,1}\times P_0)} > 0$ is sufficiently small, then this sequence converges to zero very rapidly and satisfies (3.7) (see [23, Lemma 4.8, page 166] for more details). For such ϵ_0, we have

$$\|c_k\|_{\mathscr{C}^{r,0}(D_{0,1}\times P_k)} = \epsilon_k \le \sigma_k \le (4N\epsilon_0)^{2^k} \to 0,$$

and hence $\gamma_k(z,t) \to t$ in $\mathscr{C}^{r,0}(\bar{D}_{0,1} \times P_*)$ as $k \to \infty$.

To complete the proof of Theorem 3.2, we must show that the sequences (3.4) and (3.5) also converge in $\mathscr{C}^{r,0}(\bar{D}_0 \times P_*)$ (resp., $\mathscr{C}^{r,0}(\bar{D}_1 \times P_*)$), provided that $\epsilon_0 > 0$ is sufficiently small. Write

$$\widetilde{\alpha}_k(z,t) = t + \widetilde{a}_k(z,t), \qquad \widetilde{\beta}_k(z,t) = t + \widetilde{b}_k(z,t).$$

By Lemma 3.6, we have $\widetilde{a}_{k+1} = \widetilde{a}_k + a_{k+1} + e_{k+1}$, where

$$\|e_{k+1}\|_{\mathscr{C}^{r,0}(D_0\times P_{k+1})} \le \frac{L_r}{\delta_k}\|\widetilde{a}_k\|_{\mathscr{C}^{r,0}(D_0\times P_k)}\|a_{k+1}\|_{\mathscr{C}^{r,0}(D_0\times P_{k+1})}.$$

Assuming a priori that $\|\widetilde{a}_k\|_{\mathscr{C}^{r,0}(D_0\times P_k)} \le 1$ for all $k \in \mathbb{Z}_+$, we get the following estimates for the $\mathscr{C}^{r,0}(D_0 \times P_{k+1})$-norms:

$$\|\widetilde{a}_{k+1} - \widetilde{a}_k\| \le \|a_{k+1}\| + \|e_{k+1}\| \le C_r\Big(1 + \frac{L_r}{\delta_*}2^{k+1}\Big)\epsilon_{k+1} \le R2^{k+1}\epsilon_{k+1}$$

with $R = C_r(1 + L_r/\delta_*)$. Note that $\widetilde{a}_0 = a_0$ and $\|a_0\| \le C_r\epsilon_0$. Hence

$$\|\widetilde{a}_0\|_{\mathscr{C}^{r,0}(D_0\times P_0)} + \sum_{k=0}^{\infty}\|\widetilde{a}_{k+1} - \widetilde{a}_k\|_{\mathscr{C}^{r,0}(D_0\times P_{k+1})} \le C_r\epsilon_0 + R\sum_{k=1}^{\infty}2^k\epsilon_k.$$

Since $\epsilon_k \le \sigma_k \le (4N\epsilon_0)^{2^k}$ for $k \in \mathbb{N}$ (see (3.8)), we see that $R\sum_{k=1}^{\infty}2^k\epsilon_k < \epsilon_0$ if $\epsilon_0 > 0$ is sufficiently small (see [23, Lemma 4.8, page 166] for the details). This justifies the assumption $\|\widetilde{a}_k\|_{\mathscr{C}^{r,0}(D_0\times P_k)} \le 1$ and implies that the sequence $\widetilde{a}_k = \widetilde{a}_0 + \sum_{j=1}^{k}(\widetilde{a}_j - \widetilde{a}_{j-1})$ converges on $\bar{D}_0 \times P_*$ to a limit $a = \lim_{k\to\infty}\widetilde{a}_k$ satisfying $\|a\|_{\mathscr{C}^{r,0}(D_0\times P_*)} \le (C_0 + 1)\epsilon_0$. Hence the estimate in Theorem 3.2 holds for $s = 0$ with the constant $M_{r,0} = C_0 + 1$.

The same proof shows convergence of the sequence $\widetilde{b}_k \to b$ on $\bar{D}_1 \times P_*$ and the estimate $\|b\|_{\mathscr{C}^{r,0}(D_1\times P_*)} \le (C_0 + 1)\epsilon_0$.

By shrinking the fiber domain $P_* = P_{-\delta^*/2}$ by an extra $\delta^*/2$ and applying the Cauchy estimates to the maps $a(z,\cdot)$ and $b(z,\cdot)$, we also obtain the estimates in the $\mathscr{C}^{r,s}$-norms in Theorem 3.2. In addition, if ϵ_0 is sufficiently small, then the maps $\alpha(z,\cdot)\colon P_{-\delta^*} \to \mathbb{C}^n$ and $\beta(z,\cdot)\colon P_{-\delta^*} \to \mathbb{C}^n$ are injective holomorphic for each z in their respective domain $\bar{D}_0$ (resp., $\bar{D}_1$).

This completes the proof of Theorem 3.2. $\square$

Remark 3.7

Theorem 3.2 holds whenever D_0, D_1, $D_{0,1} = D_0 \cap D_1$, $D = D_0 \cup D_1$ are relatively compact domains with $\mathscr{C}^1$-boundaries satisfying the separation condition $\overline{D_0 \backslash D_1} \cap \overline{D_1 \backslash D_0} = \emptyset$, and there exists a linear operator $T : \mathscr{L}^r_{0,1}(\bar{D}) \to \mathscr{C}^r(\bar{D})$ satisfying

$$\bar{\partial}(Tf) = f, \qquad \|Tf\|_{\mathscr{C}^r(\bar{D})} \le C_r \|f\|_{\mathscr{C}^r_{0,1}(\bar{D})}.$$

Strong pseudoconvexity of $D_{0,1}$ is not needed here, but it is used in the gluing of sprays (see Proposition 4.3). The proof of Theorem 3.2 carries over to the *parametric case* when γ depends smoothly on real parameters $s = (s_1, \dots, s_m) \in [0,1]^m \subset \mathbb{R}^m$. Indeed, the proof of Lemma 3.4 remains valid in the parametric case, and the estimates controlling the iteration process are uniform with respect to a finite number of s-derivatives. This gives a family of splittings $\gamma^s_z = \beta^s_z \circ (\alpha^s_z)^{-1}$ for $z \in \bar{D}_{0,1}$ with $\mathscr{C}^k$-dependence on the parameter $s \in [0,1]^m$ for a given $k \in \mathbb{N}$.

4. Gluing sprays on Cartan pairs

In this section, X is an irreducible complex space, and $h : X \to S$ is a holomorphic map to a complex manifold S. Its *branching locus* $\mathrm{br}(h)$ is the union of X_{sing} and the set of all those points in X_{reg} at which h fails to be a submersion; thus $\mathrm{br}(h)$ is an analytic subset of X, $X' = X \backslash \mathrm{br}(h)$ is a connected complex manifold, and $h|_{X'} : X' \to S$ is a holomorphic submersion. For each $x \in X'$, we set $VT_x X = \ker dh_x$, the *vertical tangent space of* X.

A *section* of $h : X \to S$ over a subset $D \subset S$ is a map $f : D \to X$ satisfying $h(f(z)) = z$ for all $z \in D$. Let $D \Subset S$ be a smoothly bounded domain, and let $r \in \mathbb{Z}_+$. A section $f : \bar{D} \to X$ is of class $\mathscr{A}^r(D)$ if it is holomorphic in D and r times continuously differentiable on $\bar{D}$. (At points of $f(bD) \cap X_{\mathrm{sing}}$, we use local holomorphic embeddings of X into a Euclidean space.)

Definition 4.1

An *h-spray of class $\mathscr{A}^r(D)$ with the exceptional set $\sigma = \sigma(f) \subset \bar{D}$ of order $k \ge 0$* is a map $f : \bar{D} \times P \to X$, where P (the *parameter set* of f) is an open subset of a Euclidean space $\mathbb{C}^n$ containing the origin, such that the following hold:

(i) f is holomorphic on $D \times P$ and of class $\mathscr{C}^r$ on $\bar{D} \times P$;

(ii) $h(f(z,t)) = z$ for all $z \in \bar{D}$ and $t \in P$;

(iii) the maps $f(\cdot\,, 0)$ and $f(\cdot\,, t)$ agree on σ up to order k for $t \in P$; and

(iv) for every $z \in \bar{D} \backslash \sigma$ and $t \in P$, we have $f(z,t) \notin \mathrm{br}(h)$, and the map

$$\partial_t f(z,t) : T_t\mathbb{C}^n = \mathbb{C}^n \to VT_{f(z,t)}X$$

is surjective (the *domination condition*).

For a product fibration $h\colon X = S \times Y \to S$, $h(z, y) = z$, we can identify an h-spray $\bar{D} \times P \to S \times Y$ with a *spray of maps* $\bar{D} \times P \to Y$ by composing with the projection $S \times Y \to Y$, $(z, y) \to y$. In this case, (ii) is redundant, and the domination condition (iv) is replaced by the following:

(iv′) if $z \in \bar{D} \backslash \sigma$ and $t \in P$, then $f(z, t) \in Y_{\mathrm{reg}}$, and $\partial_t f(z, t)\colon T_t\mathbb{C}^n \to T_{f(z,t)}Y$ is surjective.

Condition (ii) means that $f_t = f(\cdot\,, t)\colon \bar{D} \to X$ is a section of h of class $\mathscr{A}^r(D)$ for every $t \in P$, and by (i), these sections depend holomorphically on the parameter t. We call f_0 the *core* (or *central*) *section* of the spray. Conditions (iii) and (iv) imply that the exceptional set $\sigma(f)$ is locally defined by functions of class $\mathscr{A}^r(D)$.

Unlike the sprays used in Oka-Grauert theory, which are defined for all values $t \in \mathbb{C}^n$ but are dominant only at the core section f_0, *our sprays are local with respect to* t and dominant at every point (z, t) with $z \notin \sigma$. In applications, the parameter domain P is allowed to shrink.

LEMMA 4.2 (Existence of sprays)

Let $h\colon X \to S$ be a holomorphic map of a complex space X to a complex manifold S. Let $r \geq 2$ and $k \geq 0$ be integers. Let D be a relatively compact domain with strongly pseudoconvex boundary of class $\mathscr{C}^2$ in a Stein manifold S, and let $\sigma \subset \bar{D}$ be the common zero set of finitely many functions in $\mathscr{A}^r(D)$. Given a section $f_0\colon \bar{D} \to X$ of class $\mathscr{A}^r(D)$ such that the set $\{z \in \bar{D}\colon f(z) \in \mathrm{br}(h)\}$ *does not intersect bD and is contained in σ, there exists an h-spray $f\colon \bar{D} \times P \to X$ of class $\mathscr{A}^r(D)$ with the core section f_0 and with the exceptional set σ of order k.*

Proof

By Theorem 2.6, there exists a Stein open set $\Omega \subset X$ containing $f_0(\bar{D})$. (This is the only place in the proof where the assumption $r \geq 2$ is used.) According to [24, Proposition 2.2] (for manifolds, see [32, Lemma 5.3]), there exist an integer $n \in \mathbb{N}$, an open set $V \subset \Omega \times \mathbb{C}^n$ containing $\Omega \times \{0\}$, and a holomorphic *spray* map $s\colon V \to \Omega$ satisfying the following:

(a) $s(x, 0) = x$ for $x \in \Omega$;

(b) $h(s(x, t)) = h(x)$ for $(x, t) \in V$;

(c) $s(x, t) = x$ when $(x, t) \in V$ and $x \in \mathrm{br}(h)$; and

(d) for each $(x, t) \in V$ with $x \in \Omega \backslash \mathrm{br}(h)$, we have $s(x, t) \in X \backslash \mathrm{br}(h)$, and the partial differential $\partial_t s(x, t)|_{t=0}\colon T_0\mathbb{C}^n \to VT_xX = \ker dh_x$ is surjective.

A map s with these properties is obtained by composing small complex time flows of certain holomorphic vector fields on Ω which vanish on $\mathrm{br}(h) \cap \Omega$ and are tangential to the fibers of h.

By the hypothesis, we have $\sigma = \{z \in \bar{D}\colon g_1(z) = 0, \ldots, g_m(z) = 0\}$, where $g_1, \ldots, g_m \in \mathscr{A}^r(D)$. We can assume that $\sup_{z \in \bar{D}} |g_j(z)| < 1$ for $j = 1, \ldots, m$.

Denote the coordinates on $(\mathbb{C}^n)^m = \mathbb{C}^{nm}$ by $t = (t_1, \ldots, t_m)$, where $t_j = (t_{j,1}, \ldots, t_{j,n}) \in \mathbb{C}^n$ for $j = 1, \ldots, m$. Let $l \in \mathbb{N}$. The map $\phi_l \colon \bar{D} \times (\mathbb{C}^n)^m \to \mathbb{C}^n$, defined by

$$\phi_l(z, t_1, \ldots, t_m) = \sum_{j=1}^{m} g_j(z)^{k+l}\, t_j,$$

is a linear submersion $\mathbb{C}^{nm} \to \mathbb{C}^n$ over each point $z \in \bar{D}\backslash\sigma$, and it vanishes to order $k + l$ on σ. Let $P \subset \mathbb{C}^{nm}$ be a bounded open set containing the origin. By choosing the integer l sufficiently large, we can ensure that the map

$$f(z, t) = s\big(f_0(z), \phi_l(z, t)\big) \in X$$

is a spray $\bar{D} \times P \to X$ with the core section f_0 and with the exceptional set σ of order k. All conditions except Definition 4.1(iv) are evident. To get (iv), let Σ denote the set of all points $(x, t) \in V$ such that either $x \in \mathrm{br}(h)$, or $x \notin \mathrm{br}(h)$ and the maps $\partial_t s(x, t) \colon T_t\mathbb{C}^n \to VT_{s(x,t)}X$ fail to be surjective. Then Σ is a closed analytic subset of V satisfying $\Sigma \cap (\Omega \times \{0\}) = \mathrm{br}(h) \times \{0\}$ according to property (d) of s. Analyticity of Σ is clear except perhaps near the points $(x_0, t_0) \in V$ with $x_0 \in \mathrm{br}(h)$. To see the analyticity near such points, we choose a holomorphic embedding $\psi \colon U \to \widetilde{U} \subset \mathbb{C}^N$ of a small open neighborhood $U \subset X$ of x_0 onto a local complex subvariety $\widetilde{U} = \psi(U) \subset \mathbb{C}^N$ with $\psi(x_0) = 0$. Note that $s(x_0, t_0) = x_0$. There is a holomorphic map $\tilde{s}$ from a neighborhood of $(0, t_0) \in \mathbb{C}^N \times \mathbb{C}^n$ to $\mathbb{C}^N$ such that $\tilde{s}(0, t_0) = 0$ and $\tilde{s}(\psi(x), t) = \psi(s(x, t))$; that is, $\tilde{s}$ is a local holomorphic extension of s if U is identified with its image $\widetilde{U} \subset \mathbb{C}^N$. Locally near the point (x_0, t_0), Σ corresponds to the set of points $(w, t) \in \mathbb{C}^N \times \mathbb{C}^n$ near $(0, t_0)$ such that $w \in \widetilde{U}$ and the partial differential $\partial_t \tilde{s}(w, t)$ has rank less than $\dim VT(X\backslash\mathrm{br}(h))$; the latter dimension is constant since X is assumed irreducible. Clearly, the latter set is analytic. The contact between Σ and $\Omega \times \{0\}$ is necessarily of finite order along their intersection $\mathrm{br}(h) \times \{0\}$. By choosing $l \in \mathbb{Z}_+$ large enough, we ensure that $\phi_l(z, t) \in V\backslash\Sigma$ for every $z \in \bar{D}\backslash\sigma$ and $t \in P$. For such choices, f also satisfies property (iv). □

The following proposition provides the main tool for gluing holomorphic sections on Cartan pairs by preserving their boundary regularity.

PROPOSITION 4.3 (Gluing sprays)
Let $h \colon X \to S$ be a holomorphic map from a complex space X onto a Stein manifold S. Let (D_0, D_1) be a Cartan pair of class $\mathscr{C}^\ell$ ($\ell \geq 2$) in S (see Definition 3.1), and let $D = D_0 \cup D_1$, $D_{0,1} = D_0 \cap D_1$. Given integers $r \in \{0, 1, \ldots, \ell\}$, $k \in \mathbb{Z}_+$, and an h-spray $f \colon \bar{D}_0 \times P_0 \to X$ of class $\mathscr{A}^r(D_0)$ with the exceptional set $\sigma(f)$ of order k

and satisfying $\sigma(f) \cap \bar{D}_{0,1} = \emptyset$*, there is an open set* $P \Subset P_0$ *containing* $0 \in \mathbb{C}^n$ *such that the following hold.*

For every h-spray $f' : \bar{D}_1 \times P_0 \to X$ *of class* $\mathscr{A}^r(D_1)$ *with the exceptional set* $\sigma(f')$ *of order k, with* $\sigma(f') \cap \bar{D}_{0,1} = \emptyset$*, such that* f' *is sufficiently* $\mathscr{C}^r$ *close to* f *on* $\bar{D}_{0,1} \times P_0$*, there exists an h-spray* $g : \bar{D} \times P \to X$ *of class* $\mathscr{A}^r(D)$ *with the exceptional set* $\sigma(g) = \sigma(f) \cup \sigma(f')$ *of order k whose restriction* $g : \bar{D}_0 \times P \to X$ *is as close as desired to* $f : \bar{D}_0 \times P \to X$ *in the* $\mathscr{C}^r$*-topology. The core section* $g_0 = g(\cdot\,, 0)$ *is homotopic to* f_0 *on* $\bar{D}_0$*, and* g_0 *is homotopic to* f'_0 *on* $\bar{D}_1$*. In addition,* g_0 *agrees with* f_0 *up to order k on* $\sigma(f)$*, and* g_0 *agrees with* f'_0 *up to order k on* $\sigma(f')$*.*

If f *and* f' *agree to order* $m \in \mathbb{N}$ *along* $\bar{D}_{0,1} \times \{0\}$*, then g can be chosen to agree with* f *to order m along* $\bar{D}_0 \times \{0\}$ *and to agree with* f' *to order m along* $\bar{D}_1 \times \{0\}$*.*

Proof

First, we find a holomorphic transition map between the two sprays (see Lemma 4.4); decomposing this map by Theorem 3.2, we can adjust the two sprays to match them over $\bar{D}_{0,1}$. The first step is accomplished by the following lemma applied on the strongly pseudoconvex domain $D_{0,1}$.

LEMMA 4.4

Let $D \Subset S$ *be a strongly pseudoconvex domain with* $\mathscr{C}^\ell$*-boundary* ($\ell \geq 2$) *in a Stein manifold S, let* P_0 *be a domain in* $\mathbb{C}^n$ *containing the origin, and let* $f : \bar{D} \times P_0 \to X$ *be a spray of class* $\mathscr{A}^r(D)$ ($0 \leq r \leq \ell$) *with trivial exceptional set. Choose* $\epsilon^* > 0$*. There exists an open set* $P_1 \subset \mathbb{C}^n$*, with* $0 \in P_1 \Subset P_0$*, satisfying the following. For every spray* $f' : \bar{D} \times P_0 \to X$ *of class* $\mathscr{A}^r(D)$ *which approximates* f *sufficiently closely in the* $\mathscr{C}^r$ *-topology, there exists a map* $\gamma : \bar{D} \times P_1 \to \mathbb{C}^n$ *of class* $\mathscr{A}^{r,0}(D \times P_1)$ *satisfying*

$$\gamma(z,t) = t + c(z,t), \quad \|c\|_{\mathscr{C}^{r,0}(D\times P_1)} < \epsilon^*, \tag{4.1}$$

$$f(z,t) = f'\bigl(z, \gamma(z,t)\bigr), \quad (z,t) \in \bar{D} \times P_1. \tag{4.2}$$

If f *and* f' *agree to order m along* $\bar{D} \times \{0\}$*, then we can choose* γ *of the form* $\gamma(z,t) = t + \sum_{|J|=m} \widetilde{c}_J(z,t) t^J$ *with* $\widetilde{c}_J \in \mathscr{A}^{r,0}(D \times P_1)^n$*.*

Assuming Lemma 4.4 for the moment, we conclude the proof of Proposition 4.3 as follows. Let γ and P_1 be as in the conclusion of Lemma 4.4. (We emphasize that this lemma is applied on the set $D_{0,1}$.) Choose an open set $P \subset \mathbb{C}^n$ with $0 \in P \Subset P_1$. For $\epsilon^* > 0$ chosen sufficiently small, Theorem 3.2 applied to γ gives a decomposition

$$\gamma\bigl(z, \alpha(z,t)\bigr) = \beta(z,t), \quad (z,t) \in \bar{D}_{0,1} \times P, \tag{4.3}$$

where $\alpha\colon \bar{D}_0 \times P \to P_1 \subset \mathbb{C}^n$ and $\beta\colon \bar{D}_1 \times P \to P_1 \subset \mathbb{C}^n$ are maps of class $\mathscr{A}^{r,0}$. Replacing t by $\alpha(z,t)$ in (4.2) gives

$$f\big(z, \alpha(z,t)\big) = f'\big(z, \beta(z,t)\big), \quad (z,t) \in \bar{D}_{0,1} \times P. \tag{4.4}$$

Hence the two sides define a map $g\colon \bar{D} \times P \to X$ of class $\mathscr{C}^r(\bar{D} \times P)$ which is holomorphic in $D \times P$. Since the maps α and β are injective holomorphic on the fibers $\{z\} \times P$, g is a spray with the exceptional set $\sigma(g) = \sigma(f) \cup \sigma(f')$.

The estimates on α and β in Theorem 3.2 show that their distances from the identity map are controlled by the number ϵ^* and hence (in view of Lemma 4.4) by the $\mathscr{C}^r$-distance of f' to f on $\bar{D}_{0,1} \times P_0$. Hence the new spray g approximates f in $\mathscr{C}^r(\bar{D}_0 \times P)$. On the other hand, we do not get any obvious control on the $\mathscr{C}^r$-distance between f' and g on $\bar{D}_1 \times P$, the problem being that the $\mathscr{C}^r$-norm of f' is not a priori bounded, and precomposing f' by a map β (even if it is close to the identity map) can still cause a big change. However, in our application in §6, we need only control the range (location) of g, and this is ensured by the construction.

Finally, if f and f' agree to order m along $\bar{D}_{0,1} \times \{0\}$, then by Lemma 4.4, we can choose γ of the form $\gamma(z,t) = t + \sum_{|J|=m} \widetilde{c}_J(z,t)t^J$ with $\widetilde{c}_J \in \mathscr{A}^{r,0}(D_{0,1} \times P_1)^n$ for each multi-index J. Theorem 3.2 then gives a decomposition (4.3), where $\alpha(z,t) = t + \sum_{|J|=m} \widetilde{a}_J(z,t)t^J$ and $\beta(z,t) = t + \sum_{|J|=m} \widetilde{b}_J(z,t)t^J$, thereby ensuring that the spray g (4.4) agrees with f (resp., f') to order m at $t = 0$. This proves Proposition 4.3, granted that Lemma 4.4 holds. □

Proof of Lemma 4.4

Let E denote the subbundle of $\bar{D} \times \mathbb{C}^n$ with fibers

$$E_z = \ker\big(\partial_t f(z,t)|_{t=0}\colon \mathbb{C}^n \to VT_{f(z,0)}X\big), \quad z \in \bar{D}.$$

This subbundle is holomorphic over D and of class $\mathscr{C}^r$ on $\bar{D}$. We claim that E is complemented; that is, there exists a complex vector subbundle $G \subset \bar{D} \times \mathbb{C}^n$ which is continuous on $\bar{D}$ and holomorphic over D such that $\bar{D} \times \mathbb{C}^n = E \oplus G$. For holomorphic vector bundles on open Stein manifolds, this follows from Cartan's Theorem B [41, page 256]; the same proof applies in the category of holomorphic vector bundles with continuous boundary values over a strongly pseudoconvex domain by using the corresponding versions of Theorem B due to Leiterer [53] and Heunemann [46]. Finally we use a result of Heunemann [45] to approximate G uniformly on $\bar{D}$ by a holomorphic vector subbundle (still denoted G) of $U \times \mathbb{C}^n$ over an open neighborhood $U \supset \bar{D}$; a simple proof of this result can be found in the appendix to this article.

For each fixed $z \in U$, we write $\mathbb{C}^n \ni t = t'_z \oplus t''_z$ with $t'_z \in E_z$ and $t''_z \in G_z$. The partial differential $\partial_t|_{t=0} f(\cdot\,, t)$ gives an isomorphism $G|_{\bar{D}} \to VT_{f_0(\bar{D})}X$, and it vanishes on E. The implicit function theorem now gives an open neighborhood

$P_1 \Subset P_0$ of $0 \in \mathbb{C}^n$ such that for each spray $f' \colon \bar{D} \times P_0 \to X$ which is sufficiently $\mathscr{C}^r$ close to f on $\bar{D} \times P_0$, there is a unique map

$$\widetilde{\gamma}(z, t'_z \oplus t''_z) = t'_z \oplus \bigl(t''_z + \widetilde{c}(z,t)\bigr) \in E_z \oplus G_z = \mathbb{C}^n$$

of class $\mathscr{A}^{r,0}(D \times P_1)$ solving $f(z, \widetilde{\gamma}(z,t)) = f'(z,t)$, and $\|\widetilde{c}\|_{\mathscr{A}^{r,0}(D_{0,1} \times P_1)}$ is controlled by the $\mathscr{C}^r$-distance between f and f' on $\bar{D} \times P_0$. After shrinking P_1, the fiberwise inverse $\gamma(z,t) = t' \oplus (t''_z + c''(z,t))$ of γ then satisfies (4.2), and $\|c''\|_{\mathscr{A}^{r,0}(D_{0,1} \times P_1)}$ is controlled by the $\mathscr{C}^r$-distance between f and f' on $\bar{D} \times P_0$. □

Remark 4.5

The additions to Theorem 3.2, explained in Remark 3.7, yield the corresponding additions to Proposition 4.3. First of all, one can relax the definition of a spray by omitting the condition regarding the exceptional set. The only essential condition needed in Proposition 4.3 is that the spray f is *dominating on* $\bar{D}_{0,1}$, in the sense that its t-differential is surjective on this set at $t = 0$. (This notion of domination agrees with the one introduced by Gromov [40].) Approximating such spray f sufficiently closely in the $\mathscr{C}^r$-topology on $\bar{D}_0 \times P$ (for some open neighborhood $P \subset \mathbb{C}^n$ of the origin) by another spray f', we can glue f and f' into a new spray g over $\bar{D}_0 \cup \bar{D}_1$ which is dominating over $\bar{D}_{0,1}$. The *exceptional set* condition in Definition 4.1 is needed only when one wishes to interpolate a given spray on a subvariety of $\bar{D}_0$. The parametric version of Theorem 3.2 (see Remark 3.7) also gives the corresponding parametric version of Proposition 4.3, in which the two h-sprays f and f' depend smoothly on a real parameter $s \in [0,1]^m \subset \mathbb{R}^m$. The remaining ingredients of the proof (such as Lemma 4.4) carry over to the parametric case without difficulties.

5. Approximation of holomorphic maps to complex spaces

In this section, we prove the following approximation theorem for maps of bordered Riemann surfaces to arbitrary complex spaces. This result is used in the proof of Theorem 1.1 to replace the initial map by another one that maps the boundary into the regular part of the space.

THEOREM 5.1

Let D be a connected, relatively compact, smoothly bounded domain in an open Riemann surface S, let X be a complex space, and let $f \colon \bar{D} \to X$ be a map of class $\mathscr{C}^r$ $(r \geq 2)$ which is holomorphic in D. Given finitely many points $z_1, \dots, z_l \in D$ and an integer $k \in \mathbb{N}$, there is a sequence of holomorphic maps $f_\nu \colon U_\nu \to X$ in open sets $U_\nu \subset S$ containing $\bar{D}$ such that f_ν agrees with f to order k at z_j for $j = 1, \dots, l$ and $\nu \in \mathbb{N}$, and the sequence f_ν converges to f in $\mathscr{C}^r(\bar{D})$ as $\nu \to +\infty$. If $f(D) \not\subset X_{\mathrm{sing}}$, we can also ensure that $f_\nu(bD) \subset X_{\mathrm{reg}}$ for each $\nu \in \mathbb{N}$.

Proof

We proceed by induction on $n = \dim X$. The result trivially holds for $n = 0$. Assume that it holds for all complex spaces of dimension less than n for some $n > 0$, and let $\dim X = n$. If $f(D) \subset X_{\mathrm{sing}}$, then the conclusion holds by applying the inductive hypothesis with the complex space X_{sing}. Suppose now that $f(D) \not\subset X_{\mathrm{sing}}$. The set

$$\sigma = \left\{z \in \bar{D}\colon\ f(z) \in X_{\mathrm{sing}}\right\} \tag{5.1}$$

is compact, $\sigma \cap D$ is discrete, and $\sigma \cap bD$ has empty relative interior in bD. Indeed, as X_{sing} is an analytic subset of X, and hence complete pluripolar, the existence of a nonempty arc in bD which f maps to X_{sing} implies $f(\bar{D}) \subset X_{\mathrm{sing}}$, in contradiction to our assumption.

Set $K = \{z_1, \dots, z_l\}$. Let $bD = \bigcup_{j=1}^m C_j$, where each C_j is a closed Jordan curve. For each $j = 1, \dots, m$, we choose a point $p_j \in C_j \backslash \sigma$ and an open set $U_j \subset S$ such that $p_j \in U_j$ and $\overline{U}_j$ does not intersect $\sigma \cup K$. We choose the sets U_j so small that $f(\bar{D} \cap \overline{U}_j)$ is contained in a local chart of X_{reg}.

LEMMA 5.2

The map f can be approximated in $\mathscr{C}^r(\bar{D}, X)$ by maps $f'\colon \bar{D}' \to X$ of class $\mathscr{A}^r(D', X)$, where $D' \subset S$ is a smoothly bounded domain (depending on f') satisfying $D \cup \{p_j\}_{j=1}^m \subset D' \subset D \cup \left(\bigcup_{j=1}^m U_j\right)$. In addition, we can choose f' such that it agrees with f to order k at z_j for $j \in \{1, \dots, l\}$.

Proof

By Theorem 2.1, the graph of f over $\bar{D}$ has an open Stein neighborhood in $S \times X$. It follows that the set σ (see (5.1)) is the common zero set of finitely many functions in $\mathscr{A}^r(D)$. By Lemma 4.2, there is a spray $\widetilde{f}\colon \bar{D} \times P \to X$ ($P \subset \mathbb{C}^N$) of class $\mathscr{A}^r(D)$, with the core map $\widetilde{f}(\cdot\,, 0) = f$ and the exceptional set $\tilde{\sigma} = \sigma \cup K$ of order k.

After shrinking the parameter set $P \subset \mathbb{C}^N$ of $\widetilde{f}$ around $0 \in \mathbb{C}^N$, we may assume that $\widetilde{f}$ maps the set $E_j = (\overline{U}_j \cap \bar{D}) \times \bar{P}$ into a local chart $\Omega \subset X_{\mathrm{reg}}$ for each $j = 1, \dots, m$. Hence we can approximate the restriction of $\widetilde{f}$ to E_j as close as desired in the $\mathscr{C}^r$-sense by a spray $\widetilde{g}_j : \overline{V}_j \times P \to X_{\mathrm{reg}}$, where V_j is an open set in S (depending on $\widetilde{g}_j$) satisfying $U_j \cap \bar{D} \subset V_j \subset U_j$.

If the approximations are sufficiently close, Lemma 4.4 furnishes a transition map γ_j between $\widetilde{f}$ and $\widetilde{g}_j$ for each j (we shrink P as needed), and Proposition 4.3 lets us glue $\widetilde{f}$ with the sprays $\widetilde{g}_j$ into a spray F of class $\mathscr{A}^r(D')$ over a domain $D' \subset S$ as in Lemma 5.2. By the construction, F approximates $\widetilde{f}$ in the $\mathscr{C}^r(\bar{D} \times P)$-topology, and it agrees with $\widetilde{f}$ to order k at the points $z_j \in K$. The core map $f' = F(\cdot\,, 0)\colon \bar{D}' \to X$ then satisfies the conclusion of the lemma.

A word is in order regarding the application of Proposition 4.3. Unlike in that proposition, the final domain D' in our present situation depends on the choices of the sprays $\widetilde{g}_j$ (since the size of their z-domains in S depends on the rate of approximation). We can choose from the outset a fixed domain $D_1 \subset S$ such that (D, D_1) is a Cartan pair in S satisfying $\overline{D \cap D_1} \subset \bigcup_{j=1}^{m}(\bar{D} \cap U_j)$. Applying Theorem 3.2 gives maps α and β over $\bar{D}$ (resp., $\bar{D}_1$); the new spray F is defined as $\widetilde{f}(z, \alpha(z,t))$ for $z \in \bar{D}$ and by $\widetilde{g}_j(z, \beta(z,t))$ for $z \in \bar{D}_1 \cap U_j$. Thus we are not using the map β on its entire domain of existence but only over the domain of the sprays $\widetilde{g}_j$. $\square$

We continue with the proof of Theorem 5.1. Let $f'\colon \bar{D}' \to X$ be a map furnished by Lemma 5.2. In each boundary curve $C_j \subset bD$, we choose a closed arc $\lambda_j \subset C_j$ such that $C_j \backslash \lambda_j \subset D'$. (This is possible since D' contains the point $p_j \in C_j$.) Let ξ_j be a holomorphic vector field in a neighborhood of λ_j in S such that $\xi(z)$ points to the interior of D for every $z \in \lambda_j$. More precisely, if $D = \{v < 0\}$, with $dv \neq 0$ on bD, we ask that $\Re(\xi_j \cdot v) < 0$ on λ_j; such fields clearly exist.

Choose a domain $D_0 \subset S$ with $\bar{D}' \subset D_0$ such that $\bar{D}$ is holomorphically convex in D_0. (This holds when $D_0 \backslash \bar{D}$ is connected.) The union of K with all the arcs λ_j is a compact holomorphically convex set in D_0. The tangent bundle of D_0 is trivial, which lets us identify vector fields with functions. Hence there exists a holomorphic vector field ξ on D_0 which approximates the field ξ_j sufficiently closely on λ_j so that it remains inner radial to D there, and ξ vanishes to order k at the points $z_j \in K$. For sufficiently small $t > 0$, the flow ϕ_t of ξ carries each of the arcs λ_j into D, and hence $\phi_t(\bar{D}) \subset D'$, provided that $t > 0$ is small enough. (Recall that $C_j \backslash \lambda_j \subset D'$; hence the points of $\bar{D}$ which may be carried out of $\bar{D}$ by the flow ϕ_t along $C_j \backslash \lambda_j$ remain in D' for small $t > 0$.)

Since the set $\sigma' = \{z \in D'\colon f'(z) \in X_{\mathrm{sing}}\}$ is discrete, a generic choice of $t > 0$ also ensures that $\phi_t(bD) \cap \sigma' = \emptyset$. For such t, the map $f' \circ \phi_t$ is holomorphic in an open neighborhood of $\bar{D}$, it maps bD to X_{reg}, it approximates f in the $\mathscr{C}^r(\bar{D})$-topology, and it agrees with f to order k at each point $z_j \in K$. This provides a sequence f_ν satisfying Theorem 5.1. $\square$

Remark 5.3

D. Chakrabarti proved the following approximation result in [9, Theorem 1.1.4] (see also [10]). *If D is a domain in $\mathbb{C}$ bounded by finitely many Jordan curves and X is a complex manifold, then every continuous map $f\colon \bar{D} \to X$ which is holomorphic on D can be approximated uniformly on $\bar{D}$ by maps that are holomorphic in open neighborhoods of $\bar{D}$ in $\mathbb{C}$.* A comparison with Theorem 5.1 shows that there is a stronger hypothesis on X but a weaker hypothesis on the map.

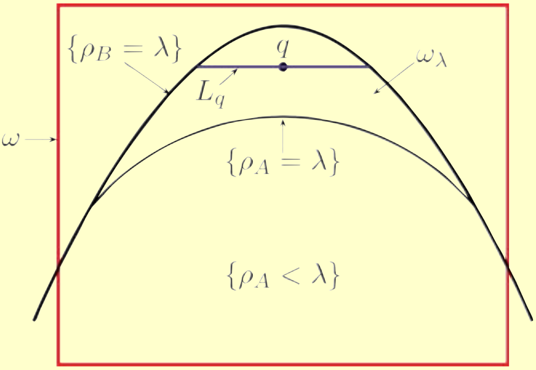

Figure 2. A 2-convex bump

6. Proof of Theorem 1.1

We begin with the two main lemmas. The induction step in the proof of Theorem 1.1 is provided by Lemma 6.3, and the key local step is furnished by Lemma 6.2.

We denote by $d_{1,2}$ the partial differential with respect to the first two complex coordinates on $\mathbb{C}^n$.

Definition 6.1
Let A and B be relatively compact open sets in a complex space X. We say that B *is a 2-convex bump on* A (see Figure 2) if there exist an open set $\Omega \subset X_{\mathrm{reg}}$ containing $\bar{B}$, a biholomorphic map Φ from Ω onto a convex open set $\omega \subset \mathbb{C}^n$, and smooth real functions $\rho_B \le \rho_A$ on ω such that

$$\Phi(A \cap \Omega) = \big\{x \in \omega \colon \rho_A(x) < 0\big\},$$
$$\Phi\big((A \cup B) \cap \Omega\big) = \big\{x \in \omega \colon \rho_B(x) < 0\big\},$$

ρ_A and ρ_B are strictly convex with respect to the first two complex coordinates, and $d_{1,2}(t\rho_A + (1-t)\rho_B)$ is nondegenerate on ω for each $t \in [0, 1]$.

Let $\rho \colon X \to \mathbb{R}$ be a smooth function that is $(n-1)$-convex on an open subset $U \subset X$. If the set $\{x \in U \colon c_0 \le \rho(x) \le c_1\}$ is compact, contained in X_{reg}, and contains no critical points of ρ, then the set $\{x \in U \colon \rho(x) \le c_1\}$ is obtained from $\{x \in U \colon \rho(x) \le c_0\}$ by a finite process in which every step is an attachment of a 2-convex bump (see [44, Lemma 12.3]). The essential ingredient in the proof is Narasimhan's lemma on local convexification.

The following lemma was proved in [21] in the case when X is a complex manifold and D is the disc and for holomorphic maps instead of sprays. Its proof in [21, Lemma 3.1] was based on the solution of the nonlinear Cousin problem in [69]. This does not seem to suffice in the case of a complex space with singularities and an arbitrary bordered Riemann surface. Instead, we use Proposition 4.3.

Since the complex space X is paracompact, it is metrizable. Fix a complete distance function d on X.

LEMMA 6.2

Let X be an irreducible complex space of $\dim X \geq 2$. Let $A \Subset X$ be a relatively compact open subset of X, and let B be a 2-convex bump on A (see Definition 6.1). Let D be a bordered Riemann surface with smooth boundary, let P be a domain in $\mathbb{C}^N$ containing 0, and let $k \geq 0$ be an integer. Assume that $f\colon \bar{D} \times P \to X$ is a spray of maps of class $\mathscr{A}^2(D)$ with the exceptional set σ of order k (see Definition 4.1) such that $f_0(bD) \cap \bar{A} = \emptyset$. (Here $f_0 = f(\cdot\,, 0)$ is the core map of the spray.) Further, assume that K is a compact subset of A and U is an open subset of D such that $f_0(\bar{D} \backslash U) \cap K = \emptyset$.

Given $\epsilon > 0$, there are a domain $P' \subset P$ containing $0 \in \mathbb{C}^N$ and a spray of maps $g\colon \bar{D} \times P' \to X$ of class $\mathscr{A}^2(D)$, with the exceptional set σ of order k, such that g_0 is homotopic to f_0 and the following hold for all $t \in P'$:

(i) $g_t(bD) \cap \overline{A \cup B} = \emptyset$,

(ii) $d(g_t(z), f_t(z)) < \epsilon$ *for* $z \in \overline{U}$,

(iii) $g_t(\bar{D} \backslash U) \cap K = \emptyset$, *and*

(iv) *the maps f_0 and g_0 have the same k-jets at every point in σ.*

Proof

Let $\Phi\colon X \supset \Omega \to \omega \subset \mathbb{C}^n$ be a biholomorphic map as in Definition 6.1. By enlarging the set $U \Subset D$, we may assume that $\sigma \subset U$. For small $\lambda > 0$, set

$$\omega_\lambda = \bigl\{x \in \omega\colon \rho_B(x) < \lambda,\ \ \rho_A(x) > \lambda\bigr\}, \qquad \Omega_\lambda = \Phi^{-1}(\omega_\lambda).$$

Then $\omega_\lambda \Subset \omega$, and $\Omega_\lambda \Subset \Omega$.

Since $f_0(bD) \cap \bar{A} = \emptyset$, we have $\rho_A\bigl(\Phi(f_0(z))\bigr) > \lambda$ for every sufficiently small $\lambda > 0$ and for all $z \in bD$ with $f_0(z) \in \Omega$. A transversality argument shows that for almost every small $\lambda > 0$, the set $bD \cap f_0^{-1}(\overline{\Omega}_\lambda)$ is a finite union $\bigcup_{j=1}^{m'} I_j$ of pairwise disjoint closed arcs I_j ($j = 1, \ldots, m$) and simple closed curves I_j ($j = m+1, \ldots, m'$). Fix a λ for which the above hold.

If I_j is an arc, we choose a smooth simple closed curve $\Gamma_j \subset \bar{D} \backslash U$ such that $\Gamma_j \cap bD$ is a neighborhood of I_j in bD, and Γ_j bounds a simply connected domain $U_j \subset D \backslash \overline{U}$ (see Figure 3). Choose a smooth diffeomorphism $h_j\colon \bar{\triangle} \to \overline{U}_j$ which is holomorphic on $\triangle$, and choose a compact set $V_j \subset \overline{U}_j$ containing a neighborhood of I_j in $\bar{\triangle}$.

If I_j is a simple closed curve, there is a collar neighborhood $\overline{U}_j \subset \bar{D} \backslash \overline{U}$ of I_j in $\bar{D}$ whose boundary $bU_j = I_j \cup I_j'$ consists of two smooth simple closed curves. For consistency of notation, we set $\Gamma_j = I_j$. There are an open subset W_j of $\triangle$ and a diffeomorphism $h_j\colon \bar{\triangle} \backslash W_j \to \bar{U}_j$ which is holomorphic on $\triangle \backslash \overline{W}_j$ such that $h_j(b\triangle) = \Gamma_j$. Choose a compact annular neighborhood V_j of Γ_j in $U_j \cup \Gamma_j$.

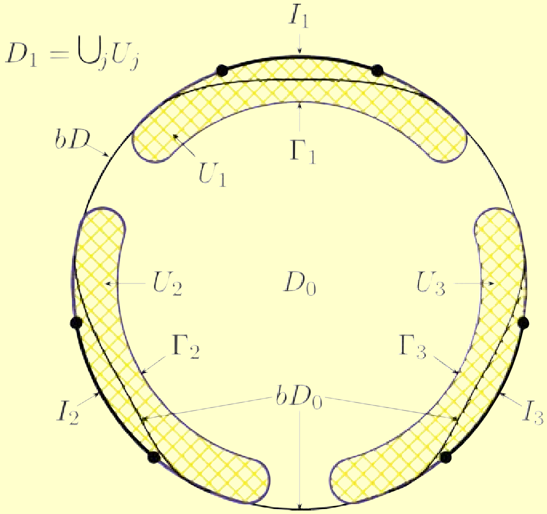

Figure 3. Cartan pair (D_0, D_1)

By choosing the sets $U_1, \ldots, U_{m'}$ sufficiently small, we can ensure that their closures are pairwise disjoint and do not intersect $\overline{U}$, and we have

$$f_0(\overline{U}_j) \subset \big\{x \in \Omega \colon \rho_A\big(\Phi(x)\big) > \lambda\big\}, \quad j = 1, \ldots m'.$$

Denote by D_1 the union $\bigcup_{j=1}^{m'} U_j$. There is a smoothly bounded open set D_0, with $D \backslash D_1 \subset D_0 \subset D \backslash \bigcup_{j=1}^{m'} V_j$, such that (D_0, D_1) is a Cartan pair (see Definition 3.1; see also Figure 3). Let $D_{0,1} = D_0 \cap D_1$.

Our goal is to approximate f in the $\mathscr{C}^2$-topology on $\bar{D}_{0,1}$ by a spray f' over $\bar{D}_1$ so that the maps f'_t satisfy properties (i) and (iii) on its domain. (The final spray g over $\bar{D}$ is obtained by gluing the restriction of f to $\bar{D}_0$ with the spray f', using Proposition 4.3.) To this end, we now find a suitable family of holomorphic discs that are used to increase the value of $\rho \circ f_0$ on the part of bD which is mapped by f_0 into Ω_λ.

Consider the homotopy $\rho_s \colon \omega \to \mathbb{R}$ defined by

$$\rho_s = (1 - s)(\rho_A - \lambda) + s(\rho_B - \lambda), \quad s \in [0, 1].$$

The function ρ_s is strictly convex with respect to the first two coordinates (since it is a convex combination of functions with this property), and $d_{1,2}\rho_s$ is nondegenerate on ω by the definition of a 2-convex bump. As the parameter s increases from $s = 0$ to $s = 1$, the sets $\{\rho_s \le 0\}$ increase smoothly from $\{\rho_A \le \lambda\}$ to $\{\rho_B \le \lambda\}$. (Inside ω_λ, these sets are strictly increasing.) For each point $q \in \omega_\lambda$, we have $\rho_A(q) > \lambda$, while $\rho_B(q) < \lambda$; hence there is a unique $s \in [0, 1]$ such that $\rho_s(q) = 0$. Write

$q = (q_1, q_2, q'')$ with $q'' \in \mathbb{C}^{n-2}$. The set

$$M_{s,q''} = \big\{(x_1, x_2, q'') \in \omega\colon \rho_s(x_1, x_2, q'') = 0\big\}$$

is a real three-dimensional submanifold of $\mathbb{C}^2 \times \{q''\}$. Let $T_q M_{s,q''}$ denote its real tangent space at q; then $E_q = T_q M_{s,q''} \cap i\, T_q M_{s,q''}$ is a complex line in $T_q\mathbb{C}^n = \mathbb{C}^n$. By strict convexity of ρ_B with respect to the first two variables, the intersection

$$L_q = (q + E_q) \cap \big\{x \in \omega\colon \rho_B(x) \le \lambda\big\}$$

is a compact, connected, smoothly bounded convex subset of $q + E_q$ with $bL_q \subset \{\rho_B = \lambda\}$ (see Figure 2). The sets L_q depend smoothly on $q \in \omega_\lambda$ and degenerate to the point $L_q = \{q\}$ for $q \in b\,\omega_\lambda \cap \{\rho_A > \lambda\}$. We set $L_q = \{q\}$ for all points $q \in \omega$ with $\rho_B(q) \ge \lambda$.

Given a point $z \in \Gamma_j \subset bD_1$ for some $j \in \{1, \dots, m'\}$, we set

$$\widetilde{L}_z = L_q \quad \text{with } q = \Phi\big(f_0(z)\big).$$

The definition is good since $\rho_A\big(\Phi(f_0(z))\big) > \lambda$ for all $z \in \bar{D}_1$.

An elementary argument (see, e.g., [35, §4]) gives for each $j \in \{1, \dots m'\}$ a continuous map $H_j\colon \Gamma_j \times \bar{\Delta} \to \omega$ such that for each $z \in I_j$, the map $\bar{\Delta} \ni \eta \mapsto H_j(z, \eta) \in \widetilde{L}_z$ is a holomorphic parametrization of $\widetilde{L}_z$ and $H_j(z, 0) = \Phi(f_0(z))$; if $z \in \Gamma_j \backslash I_j$, then $H_j(z, \eta) = \Phi(f_0(z))$ for all $\eta \in \bar{\Delta}$.

Recall that h_j is a parametrization of $\overline{U}_j$ by a $\bar{\Delta}$ if $j \in \{1, \dots, m\}$ (resp., by an annular region in $\bar{\Delta}$ if $j \in \{m + 1, \dots, m'\}$). Let $G_j : b\Delta \times \bar{\Delta} \to \mathbb{C}^n$ be defined by

$$G_j(\zeta, \eta) = H_j\big(h_j(\zeta), \eta\big) - \Phi\big(f_0(h_j(\zeta))\big), \quad \zeta \in b\Delta,\ \eta \in \bar{\Delta}.$$

Observe that $G_j(\zeta, \eta) = 0$ if $\zeta \in h_j^{-1}(\Gamma_j \backslash I_j)$ and $\eta \in \bar{\Delta}$.

Let $\mathbb{B} \subset \mathbb{C}^n$ denote the unit ball and $\delta\,\mathbb{B}$ the ball of radius δ. For each $j \in \{1, \dots, m'\}$ and each $\delta > 0$, we solve approximately the Riemann-Hilbert problem for the map G_j, using [35, Lemma 5.1], to obtain a holomorphic polynomial map $Q_{\delta,j} : \mathbb{C} \to \mathbb{C}^n$ satisfying the following properties:

$$Q_{\delta,j}(\zeta) \in G_j(\zeta, b\Delta) + \delta\,\mathbb{B} \quad \text{for } \zeta \in b\Delta, \tag{6.1}$$

$$|D^2 Q_{\delta,j}(\zeta)| < \delta \quad \text{for } \zeta \in h_j^{-1}(\overline{U_j \backslash V_j}), \tag{6.2}$$

$$Q_{\delta,j}(\zeta) \in G_j(b\Delta, \bar{\Delta}) + \delta\,\mathbb{B} \quad \text{for } \zeta \in h_j^{-1}(\overline{U}_j). \tag{6.3}$$

Here $D^2 Q = (Q, Q', Q'')$ is the second-order jet of Q. Although [35, Lemma 5.1] only gives a uniform estimate in (6.2), we can apply it to a larger disc containing $h_j^{-1}(\overline{U_j \backslash V_j})$ in its interior to obtain the estimates of derivatives.

Define a map $Q_\delta\colon \bar D_1 = \bigcup_{j=1}^{m'} \overline{U}_j \to \mathbb{C}^n$ by

$$Q_\delta(z) = Q_{\delta,j}\big(h_j^{-1}(z)\big), \quad z \in \overline{U}_j.$$

By (6.2), the map Q_δ and its first two derivatives have modulus bounded by δ on $\bigcup_{j=1}^{m'} \overline{U_j \backslash V_j}$ and hence on $\bar D_{0,1}$. If $z \in \Gamma_j \cap bD$, then (6.1) gives

$$\big|Q_\delta(z) + \Phi\big(f_0(z)\big) - H_j(z,\eta)\big| < \delta \quad \text{for some } \eta \in b\triangle,$$

and hence the point $Q_\delta(z) + \Phi(f_0(z))$ is contained in the δ-neighborhood of $b\widetilde{L}_z$. Recall that for $z \in I_j$, we have $b\widetilde{L}_z \subset \{\rho_B = \lambda\}$, and for $z \in \Gamma_j \backslash I_j$, we have $\widetilde{L}_z = \{\Phi(f_0(z))\}$. By choosing $\delta_0 > 0$ sufficiently small, we ensure that

$$\rho_B\big(Q_\delta(z) + \Phi(f(z,t))\big) > 0$$

for all $z \in \Gamma_j \cap bD$, $j = 1, \dots, m'$, $0 < \delta < \delta_0$, and all t in a certain neighborhood $P_0 \subset P$ of $0 \in \mathbb{C}^N$. For such choices (and a fixed $\delta \in (0, \delta_0)$), the map $f' = f'_\delta\colon \bar D_1 \times P_0 \to X$, defined by

$$f'(z,t) = \Phi^{-1}\big(Q_\delta(z) + \Phi(f(z,t))\big), \quad z \in \bar D_1,\ t \in P_0,$$

is a spray of maps of class $\mathscr{A}^2(D_1)$, with trivial (empty) exceptional set, whose boundary values on $bD_1 \cap bD$ lie outside of $\overline{A \cup B}$. By choosing $\delta > 0$ small enough, we ensure that f' approximates the spray f as closely as desired in the $\mathscr{C}^2$-norm on $\bar D_{0,1} \times P_0$.

By Proposition 4.3, we can glue f and f' into a spray of maps $g\colon \bar D \times P' \to X$ approximating f on $\bar D_0 \times P'$; hence the central map $g_0 = g(\cdot\,,0)$ satisfies Lemma 6.2(ii) and also property (i) on $bD_0 \cap bD$. For $z \in \bar D_1$, we have $g(z,t) = f'(z, \beta(z,t))$ by (4.4), where the $\mathscr{C}^2$-norm of β is controlled by δ. Choosing $\delta > 0$ sufficiently small, we ensure that for each $z \in bD_1 \cap bD$, we have $g_0(z) = g(z,0) \in X \backslash \overline{A \cup B}$, so (i) holds also on $bD_1 \cap bD$. Similarly, since $f'_t(\bar D_1)$ does not intersect $\bar A \supset K$, we see that g_0 satisfies property (iii). By shrinking P', we obtain the same properties for all maps g_t, $t \in P'$. Finally, property (iv) holds by the construction. (This does not depend on the choice of the constants.) $\square$

LEMMA 6.3

Let X be an irreducible complex space of dimension $n \geq 2$, and let $\rho\colon X \to \mathbb{R}$ be a smooth exhaustion function that is $(n-1)$-convex on $\{x \in X\colon \rho(x) > M_1\}$. Let D be a finite Riemann surface, let P be an open set in $\mathbb{C}^N$ containing the origin, and let $M_2 > M_1$. Assume that $f\colon \bar D \times P \to X$ is a spray of maps of class $\mathscr{A}^2(D)$ with the exceptional set $\sigma \subset D$ of order $k \in \mathbb{Z}_+$, and $U \Subset D$ is an open subset such that $f_0(z) \in \{x \in X_{\mathrm{reg}}\colon \rho(x) \in (M_1, M_2)\}$ for all $z \in \bar D \backslash U$. Given $\epsilon > 0$ and a number

$M_3 > M_2$, there exist a domain $P' \subset P$ containing $0 \in \mathbb{C}^N$ and a spray of maps $g\colon \bar{D} \times P' \to X$ of class $\mathscr{A}^2(D)$, with exceptional set σ of order k, satisfying the following properties:

(i) $g_0(z) \in \{x \in X_{\text{reg}}\colon \rho(x) \in (M_2, M_3)\}$ *for* $z \in bD$,
(ii) $g_0(z) \in \{x \in X\colon \rho(x) > M_1\}$ *for* $z \in \bar{D}\backslash U$,
(iii) $d(g_0(z), f_0(z)) < \epsilon$ *for* $z \in \overline{U}$, *and*
(iv) f_0 *and* g_0 *have the same* k*-jets at each of the points in* σ.

Moreover, g_0 *can be chosen homotopic to* f_0.

Proof

The idea is the following. Lemma 6.2 allows us to push the boundary of our curve out of a 2-convex bump in X. By choosing these bumps carefully, we can ensure that in finitely many steps, we push the boundary of the curve to a given, higher super level set of ρ (see property (i)); at the same time, we take care not to drop it substantially lower with respect to ρ (see property (ii)) and to approximate the given map on the compact subset $\overline{U} \subset D$ (see property (iii)). In the construction, we always keep the boundary of the image curve in the regular part of X. Special care must be taken to avoid the critical points of ρ. We now turn to details.

By [14, Lemma 5], there exists an *almost plurisubharmonic function* v on X (i.e., a function whose Levi form has bounded negative part on each compact in X) which is smooth on X_{reg} and satisfies $v = -\infty$ on X_{sing}. We may assume that $v < 0$ on $\{\rho \le M_3 + 1\}$.

For every sufficiently small $\delta > 0$, the function $\tau_\delta = \rho - M_1 + \delta v$ is $(n-1)$-convex on $\{\rho \le M_3\}$, and its Levi form is positive on the linear span of the eigenspaces corresponding to the positive eigenvalues of the Levi form of ρ at each point. Note that $X_{\text{sing}} \cup \{\rho \le M_1\} \subset \{\tau_\delta < 0\}$. Since $\rho(f_0(z)) > M_1$ and $f_0(z) \in X_{\text{reg}}$ for all $z \in bD$, we have $\tau_\delta(f_0(z)) > 0$ for all $z \in bD$ and all small $\delta > 0$. Fix $\delta > 0$ for which all of the above hold, and write $\tau = \tau_\delta$.

Choose a number $M \in (M_2, M_3)$. (The central map g_0 of the final spray maps bD close to $\{\rho = M,\ \tau > 0\}$.) Since $\tau = -\infty$ on X_{sing}, the set

$$\Omega = \left\{x \in X\colon \rho(x) < M_3,\ \tau(x) > 0\right\}$$

is contained in the regular part of X. By a small perturbation, one can in addition achieve that zero is a regular value of τ, M is a regular value of ρ, and the level sets $\{\rho = M\}$ and $\{\tau = 0\}$ intersect transversely. Denote their intersection manifold by Σ. There is a neighborhood U_Σ of Σ in X with $\overline{U}_\Sigma \subset \{\rho > M_2\} \cap X_{\text{reg}}$.

We are now in the same geometric situation as in [23, §6.5] (see especially [23, proof of Lemma 6.9]; the fact that our X is not necessarily a manifold is unimportant since $\overline{\Omega} \subset X_{\text{reg}}$). For $s \in [0, 1]$, set

$$\rho_s = (1-s)\tau + s(\rho - M), \qquad G_s = \{\rho_s < 0\} \cap \{\rho < M_3\}.$$

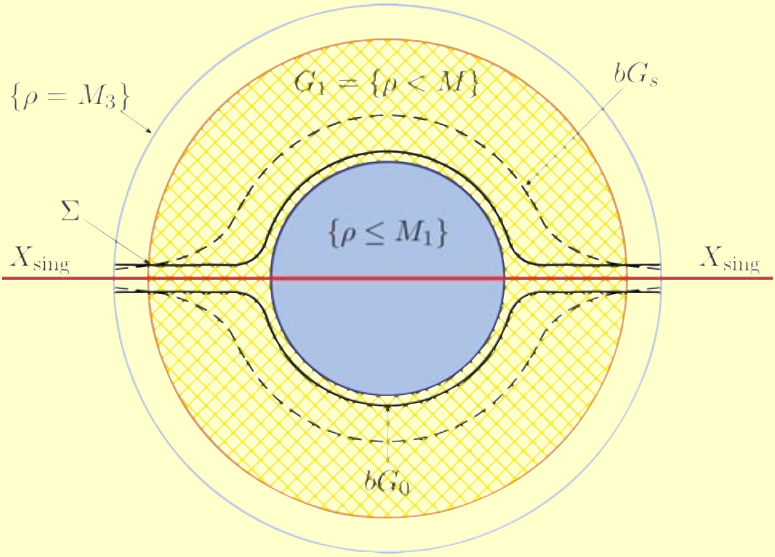

Figure 4. The sets G_s

The Levi form of ρ_s, being a convex combination of the Levi forms of τ and ρ, is positive on the linear span of the eigenspaces corresponding to the positive eigenvalues of the Levi form of ρ. Therefore G_s is strongly $(n-1)$-convex at each smooth boundary point for every $s \in [0,1]$. As the parameter s increases from $s = 0$ to $s = 1$, the domains $G_s \cap \{\rho < M\}$ increase from $\{\tau < 0,\ \rho < M\}$ to $G_1 = \{\rho < M\}$. (The sets $G_s \cap \{M < \rho < M_3\}$ decrease with s, but that part is not used.) All hypersurfaces $\{\rho_s = 0\} = bG_s$ intersect along Σ. Since $d\rho_s = (1-s)\,d\tau + s\,d\rho$ and the differentials $d\tau$ and $d\rho$ are linearly independent along Σ, each hypersurface bG_s is smooth near Σ. By a generic choice of ρ and τ, we can ensure that only for finitely many values of $s \in [0,1]$ does the critical point equation $d\rho_s = 0$ have a solution on $bG_s \cap \Omega$, and in this case, there is exactly one solution. Therefore bG_s has nonsmooth points only for finitely many values of $s \in [0,1]$ (see Figure 4).

Fix two values of the parameter, say, $0 \le s_0 < s_1 \le 1$. Consider first the *noncritical case* when $d\rho_s \neq 0$ on $bG_s \cap \Omega$ for all $s \in [s_0, s_1]$, and hence all boundaries bG_s for $s \in [s_0, s_1]$ are smooth. By attaching to G_{s_0} finitely many small 2-convex bumps of the type used in Lemma 6.2 and contained in $G_1 \cup U_\Sigma$, we cover the set $G_{s_1} \cap \Omega$ (see [23, page 180] for a more detailed description). Using Lemma 6.2 at each bump, we push the boundary of the central map in the spray outside the bump while keeping control on the compact subset $\overline{U} \subset D$. After a finite number of steps, the boundary of the central map lies outside $G_{s_1} \cap \Omega$ and inside $G_1 \cup U_\Sigma$. Up to the end of §6, this is called the *noncritical procedure*.

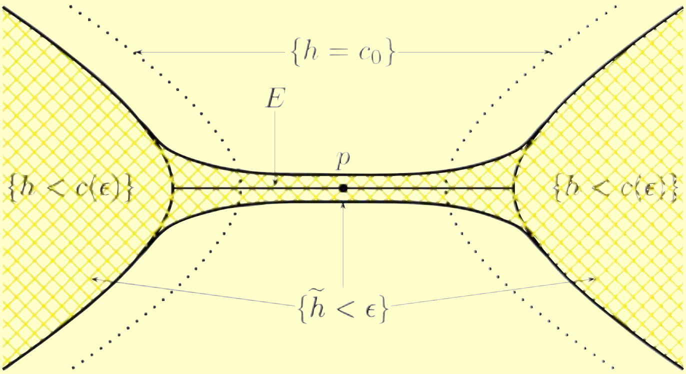

Figure 5. The level sets of $\tilde{h}$

It remains to consider the values $s \in [0, 1]$ for which bG_s has a nonsmooth point (the *critical case*). We begin by discussing the most difficult case, $\dim X = 2$, when there is the least space to avoid the critical points. The functions ρ and τ are then 1-convex and hence strongly plurisubharmonic. As in [23, page 180], we introduce the function

$$h(x) = \frac{\tau(x)}{\tau(x) + M - \rho(x)}, \quad x \in \Omega.$$

A generic choice of τ ensures that h is a Morse function. Note that $\{h = s\} = \{\rho_s = 0\} = bG_s$. The critical points of h coincide with critical points of ρ_s on $\{\rho_s = 0\}$, and the Levi form of h at a critical point is positive definite (see [23, page 180]).

To push the boundary over a critical level of h, we apply [23, Lemma 6.7, page 177] (see also [30, §4]). Let p be a critical point of h with $h(p) = c \in (0, 1)$. (Our h corresponds to ρ in [23].) It suffices to consider the case when the Morse index of p is either 1 or 2 since we cannot approach a minimum of h by the noncritical procedure. Choose a neighborhood $W \subset X$ of p on which h is strongly plurisubharmonic. Lemma 6.7 in [23] furnishes a new function $\widetilde{h}$ (denoted τ in [23]) that is strongly plurisubharmonic on W, while outside of W each level set $\{\widetilde{h} = \epsilon\}$ (for values ϵ close to zero) coincides with a certain level set $\{h = c(\epsilon)\}$ such that $\widetilde{h}$ satisfies the following properties (see Figure 5). The sublevel set $\{\widetilde{h} \le 0\}$ is contained in the union of the sublevel set $\{h \le c_0\}$ for some $c_0 < c$ (close to c) and a totally real disc E (the unstable manifold of the critical point p with respect to the gradient flow of h). Furthermore, for a small $d > 0$ with $c_0 < c - d$, we have

$$\{h \le c + d\} \subset \{\widetilde{h} \le 2d\} \subset \{h < c + 3d\}; \tag{6.4}$$

$\widetilde{h}$ has no critical values on $(0, 3d)$, and h has no critical values on $[c - d, c + 3d]$ except for $h(p) = c$.

By the noncritical procedure applied with the function h, we push the boundary of the central map of the spray into the set $\{c-d<h<c\}$. Let $\widetilde{f}$ denote the new spray. For parameters $t \in \mathbb{C}^N$ sufficiently close to the origin, the map $\widetilde{f}_t$ also has boundary values in $\{c-d<h<c\}$. Since $\dim_{\mathbb{R}} E \leq 2$, we can find t arbitrarily close to the origin such that $\widetilde{f}_t(bD) \cap E = \emptyset$. By translation in the t-variable, we can choose $\widetilde{f}_t$ as the new central map of the spray.

Since $\{\widetilde{h} \leq 0\} \subset \{h \leq c_0\} \cup E \subset \{h \leq c-d\} \cup E$, the above ensures that $\widetilde{h}>0$ on $\widetilde{f}_t(bD)$. Since $\widetilde{h}$ has no critical values on $(0,3d)$, we can use the noncritical procedure with $\widetilde{h}$ to push the boundary of the central map into the set $\{\widetilde{h}>2d\}$, appealing to Lemma 6.2. As $\{\widetilde{h}>2d\} \subset \{h>c+d\}$ by (6.4), we have thus pushed the image of bD across the critical level $\{h=c\}$ and avoided running into the critical point p. Now, we continue with the noncritical procedure applied with h to reach the next critical level of h.

This concludes the proof for $n=2$. The same procedure can be adapted to the case where $n=\dim_{\mathbb{C}} X>2$ by considering the appropriate two-dimensional slices on which the function ρ is strongly plurisubharmonic. Alternatively, we can apply the same geometric construction as in [21] to keep the boundary of the central map at a positive distance from the critical points of ρ. □

Proof of Theorem 1.1

Let d denote a complete distance function on X. We denote the initial map in Theorem 1.1 by $f_0 \colon \bar{D} \to X$. By Theorem 5.1, we may assume that f_0 is holomorphic in a neighborhood of $\bar{D}$ in an open Riemann surface $S \supset \bar{D}$ and $f_0(bD) \subset (X_c)_{\mathrm{reg}}$. Here $X_c=\{\rho>c\}$ is the set on which ρ is assumed to have at least two positive eigenvalues.

Choose an open, relatively compact subset $U \Subset D$ and a number $\epsilon>0$. It suffices to find a proper holomorphic map $g \colon D \to X$ such that $\sup_{z\in U} d(f_0(z), g(z))<\epsilon$ and such that g agrees with f_0 to order k at each of the given points $z_j \in D$; a sequence of proper maps g_ν as in Theorem 1.1 is then obtained by Cantor's diagonal process.

Let σ denote the union of $\{z \in D \colon f_0(z) \in X_{\mathrm{sing}}\}$ and the finite set $\{z_j\} \subset D$ on which we interpolate to order $k \in \mathbb{N}$; thus σ is a finite subset of D. Lemma 4.2 furnishes a spray of maps $f \colon \bar{D} \times P \to X$ of class $\mathscr{A}^2(D)$, with the given central map f_0 and the exceptional set σ of order k, such that $f_t(bD) \subset (X_c)_{\mathrm{reg}}$ for each $t \in P \subset \mathbb{C}^N$.

Set $f^0=f$, set $c=c_0$, and choose an open subset $P_0 \Subset P$ containing the origin $0 \in \mathbb{C}^N$. Choose a number $c_1>c_0$ such that $c_0<\rho(f_t^0(z))<c_1$ for all $z \in bD$ and $t \in P_0$, and then choose an open subset $U_0 \Subset D$ containing $\sigma \cup U$ such that $f_t^0(\bar{D}\backslash U_0) \subset \{x \in X \colon c_0<\rho(x)<c_1\}$ for all $t \in P_0$. Choose a sequence $c_0<c_1<c_2\cdots$ with the given initial numbers c_0 and c_1 such that $\lim_{j\to\infty} c_j=+\infty$. Also, choose a decreasing sequence $\epsilon_j>0$ with $0<\epsilon_1<\epsilon$ such that for each $j \in \mathbb{N}$, we have

$$\bigl(x, y \in X,\ \rho(x) < c_{j+1},\ d(x,y) < \epsilon_j\bigr) \Rightarrow |\rho(x) - \rho(y)| < 1.$$

We inductively find a sequence of sprays $f^j \colon \bar{D} \times P_j \to X$ of class $\mathscr{A}^2(D)$ with the exceptional set σ of order k, with $P = P_0 \supset P_1 \supset P_2 \supset \cdots$, and a sequence of open sets $U_0 \subset U_1 \subset \cdots \subset \bigcup_{j=1}^{\infty} U_j = D$ satisfying the following properties for each $j \in \mathbb{Z}_+$ and $t \in P_j$:

(i) $f_t^j(bD) \subset \{x \in X_{\mathrm{reg}} \colon c_j < \rho(x) < c_{j+1}\}$,
(ii) $f_t^j(\bar{D} \backslash U_j) \subset \{x \in X \colon c_j < \rho(x) < c_{j+1}\}$,
(iii) $f_t^j(\bar{D} \backslash U_{j-1}) \subset \{x \in X \colon c_{j-1} < \rho(x) < c_{j+1}\}$,
(iv) $d(f_0^j(z), f_0^{j-1}(z)) < \epsilon_j 2^{-j}$ for $z \in U_{j-1}$, and
(v) f_0^j and f_0^{j-1} are homotopic, and they have the same k-jets at each of the points in σ.

For $j = 0$, properties (i) and (ii) hold, while the remaining properties are vacuous. (In (iii), we take $U_{-1} = U_0$ and $c_{-1} = c_0$.) Assuming that we already have sprays $f^0, \ldots, f^j$ satisfying these properties, Proposition 6.3 applied to $f = f^j$ furnishes a new spray f^{j+1} (called g in the statement of that proposition) satisfying (i), (iii), (iv), and (v). Choose an open set $U_{j+1} \Subset D$ with $U_j \subset U_{j+1}$ such that (ii) holds. (This is possible by continuity since (i) already holds, and we are allowed to shrink the parameter set P_{j+1}.) Hence the induction proceeds. When choosing the sets U_j, we can easily ensure that they exhaust D.

Conditions (i)–(v) imply that the sequence of central maps $f_0^j \colon \bar{D} \to X$ ($j \in \mathbb{Z}_+$) converges uniformly on compacts in D to a proper holomorphic map $g \colon D \to X$ satisfying $d(f_0(z), g(z)) < \epsilon$ ($z \in \overline{U}_0$) and such that the k-jet of g agrees with the k-jet of f_0 at every point of σ. In addition, we can combine the homotopies from f_0^j to f_0^{j+1} ($j = 0, 1, \ldots$) to obtain a homotopy from $f_0|_D$ to g. This completes the proof of Theorem 1.1. □

Appendix. Approximation of holomorphic vector subbundles

In the proof of Lemma 4.4, we used the following approximation result.

THEOREM A.1 (Heunemann [45, Theorem 1, page 275])
If D is a relatively compact strongly pseudoconvex domain in a Stein manifold S and $E \subset \bar{D} \times \mathbb{C}^n$ is a continuous complex vector subbundle of the trivial bundle over $\bar{D}$ such that E is holomorphic over D, then E can be uniformly approximated by holomorphic vector subbundles $\widetilde{E} \subset U \times \mathbb{C}^n$ over small open neighborhoods $U \subset S$ of $\bar{D}$.

Proof
We offer a simple proof of this useful result. Choose a complementary to E subbundle $G \subset \bar{D} \times \mathbb{C}^n$ of the same class $\mathscr{A}(D)$ (the existence of such G follows from Cartan's Theorem B for vector bundles of class $\mathscr{A}(D)$; see [46], [53]). Let $\Pi\colon \bar{D} \times \mathbb{C}^n \to E$ denote the fiberwise $\mathbb{C}$-linear projection with kernel G and image E. By the Oka-Weil theorem, we approximate Π uniformly on $\bar{D}$ by a holomorphic fiberwise linear map $\Pi'\colon U' \times \mathbb{C}^n \to U' \times \mathbb{C}^n$ over an open set $U' \supset \bar{D}$. In general, Π' fails to be a projection map on the fibers, but this can be corrected by the following simple device (see, e.g., [36]).

Let C be a positively oriented simple closed curve in $\mathbb{C}$, and let $L \in \mathrm{Lin}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^n)$ be a linear map with no eigenvalues on C. Then $\mathbb{C}^n = V_+ \oplus V_-$, where V_+ (resp., V_-) are L-invariant subspaces of $\mathbb{C}^n$ spanned by the generalized eigenvectors of L corresponding to the eigenvalues inside (resp., outside) of C. The map

$$\mathscr{P}(L) = \frac{1}{2\pi i} \int_C (\zeta I - L)^{-1}\, d\zeta$$

is a projection onto V_+ with kernel V_-.

Choose a curve $C \subset \mathbb{C}$ which encircles 1 but not zero; for instance, $C = \{\zeta \in \mathbb{C}\colon |\zeta - 1| = 1/2\}$. Let $\mathscr{P}$ denote the associated projection operator. If $L \in \mathrm{Lin}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^n)$ is a projection, then $\mathscr{P}(L) = L$. If L' is near a projection L, then each eigenvalue of L' is either near zero or near 1, and hence $\mathscr{P}(L')$ is a projection that is close to L and has the same rank as L.

Assuming that Π' is sufficiently close to Π on $\bar{D}$, it follows that for each point z in an open set U' with $\bar{D} \subset U \subset U'$, the map $\widetilde{\Pi}_z = \mathscr{P}(\Pi'_z) \in \mathrm{Lin}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^n)$ is a projection of the same rank as Π_z, and it depends holomorphically on $z \in U$. The map $\widetilde{\Pi}\colon U \times \mathbb{C}^n \to U \times \mathbb{C}^n$ with fibers $\widetilde{\Pi}_z$ is then a projection onto a holomorphic vector subbundle $\widetilde{E} \subset U \times \mathbb{C}^n$ whose restriction to $\bar{D}$ is uniformly close to E, and $\widetilde{G} = \ker \widetilde{\Pi}$ is a holomorphic vector subbundle of $U \times \mathbb{C}^n$ whose restriction to $\bar{D}$ is uniformly close to G. □

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Drinovec Drnovšek
Institute of Mathematics, Physics, and Mechanics, University of Ljubljana, Jadranska 19,
1000 Ljubljana, Slovenia; barbara.drinovec@fmf.uni-lj.si

Forstnerič
Institute of Mathematics, Physics, and Mechanics, University of Ljubljana, Jadranska 19,
1000 Ljubljana, Slovenia; franc.forstneric@fmf.uni-lj.si