

# Holomorphic immersions and embeddings of Stein manifolds into Stein manifolds with the density property

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# Embeddings and immersions into Euclidean spaces

**Stein 1951** A Stein manifold is a complex manifold  $S$  which admits many holomorphic maps  $S \rightarrow \mathbb{C}^n$ .

**Remmert 1956; Narasimhan 1960, Bishop 1961**

Every Stein manifold  $S$  admits

- a proper holomorphic embedding  $S \hookrightarrow \mathbb{C}^{2 \dim S + 1}$ ;
- a proper holomorphic immersion  $S \rightarrow \mathbb{C}^{2 \dim S}$ ;
- a proper holomorphic map  $S \rightarrow \mathbb{C}^{\dim S + 1}$ ;
- an almost proper holomorphic map  $S \rightarrow \mathbb{C}^{\dim S}$ .

**Eliashberg and Gromov 1992; Schürmann 1997**

A Stein manifold  $S^d$  immerses properly holomorphically into  $\mathbb{C}^{\lfloor \frac{3d+1}{2} \rfloor}$  and, if  $d > 1$ , it embeds properly holomorphically into  $\mathbb{C}^{\lfloor \frac{3d}{2} \rfloor + 1}$ .

**Question:** Which Stein manifolds, besides the Euclidean spaces, contain all Stein manifolds of appropriate dimension as closed complex submanifolds?

# A soft Oka principle for embeddings

If a manifold  $X$  is **Brody hyperbolic** (i.e., it does not admit any nonconstant holomorphic images of  $\mathbb{C}$ ) then clearly no complex manifold containing a nontrivial holomorphic image of  $\mathbb{C}$  can be embedded into  $X$ . In order to get positive results, one must therefore assume that  $X$  enjoys a suitable holomorphic flexibility (anti-hyperbolicity) property.

However, anything is possible up to a change of the Stein structure:

## Theorem (F., 2017)

*Let  $(S, J)$  and  $X$  be Stein manifolds, where  $J: TS \rightarrow TS$  denotes the complex structure operator on  $S$ . If  $\dim X > 2 \dim S$  then for every continuous map  $f: S \rightarrow X$  there exists a Stein structure  $J'$  on  $S$ , homotopic to  $J$ , and a proper holomorphic embedding  $f': (S, J') \hookrightarrow X$  homotopic to  $f$ . If  $\dim X \geq 2 \dim S$  then  $f'$  can be chosen to be a proper holomorphic immersion having only simple double points.*

In other words,  $X$  contains a closed complex submanifold which is diffeomorphic (but not necessarily biholomorphic) to the given manifold  $S$ .

# Proper embeddings of strongly pseudoconvex domains

Another general embedding result for holomorphically 'small' source manifolds is the following.

## Theorem (Drinovec Drnovšek & F., 2010)

*Let  $X$  be an  $n$ -dimensional Stein manifold, and let  $D$  be a relatively compact, smoothly bounded, strongly pseudoconvex domain in a Stein manifold  $S$  of dimension  $d$ .*

*If  $2d < n$  then every holomorphic map  $f: \bar{D} \rightarrow X$  can be approximated uniformly on compacts in  $D$  by proper holomorphic embeddings  $D \hookrightarrow X$  (immersions with simple double points if  $2d = n$ ).*

This result, which is used in the proof of our main theorem, is obtained by using holomorphic peak functions to push the image of  $\partial D$  to infinity in  $X$ . Earlier results were obtained by many authors (**Fornæss, Løv, F., Globevnik, Hakim, Stenønes, Dor,...**).

# The (volume) density property

**Varolin 2000** A complex manifold  $X$  enjoys the **density property (DP)** if every holomorphic vector field on  $X$  can be approximated by Lie combinations of  $\mathbb{C}$ -complete holomorphic vector fields.

If  $\omega$  is a holomorphic volume form on  $X$ , the pair  $(X, \omega)$  enjoys the **volume density property (VDP)** if the analogous property holds for the holomorphic vector fields with vanishing  $\omega$ -divergence.

**Andersén 1990; Andersén & Lempert 1992**  $\mathbb{C}^n$  enjoys DP for  $n > 1$ , and VDP for the volume form  $\omega = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$  for  $n \geq 1$ .

A Stein manifold  $X$  with DP or VDP is highly symmetric and has a big holomorphic automorphism group (provided  $\dim X > 1$ ). More precisely, we have the following result.

# Approximating biholomorphisms by automorphisms

## Theorem (Andersén-Lempert, Forstnerič-Rosay, Varolin)

Let  $X$  be a Stein manifold with DP. Assume that

$$F_t: \Omega_0 \rightarrow \Omega_t \subset X \quad (t \in [0, 1])$$

is a smooth isotopy of biholomorphic maps between Stein Runge domains in  $X$ , with  $F_0 = \text{Id}|_{\Omega_0}$ . Then,  $F_1: \Omega_0 \rightarrow \Omega_1$  is a limit of holomorphic automorphisms of  $X$ , uniformly on compacts in  $\Omega_0$ .

There is an analogous result for isotopies of biholomorphic maps preserving the volume form on a Stein manifold with VDP.

**Kaliman and Kutzschebauch 2008** A Stein manifold with the (volume) density property is an Oka manifold.

**Varolin's Conjecture** A contractible Stein manifold with DP is biholomorphic to the Euclidean space.

# The main theorem

## Theorem (Andrist, F., Ritter, Wold 2016; F. 2017)

*Assume that  $S$  is a Stein manifold, and  $X$  is a Stein manifold with the density property or the volume density property.*

- (a) *If  $\dim X > 2 \dim S$ , then any continuous map  $S \rightarrow X$  is homotopic to a proper holomorphic embedding  $S \hookrightarrow X$ .*
- (b) *If  $\dim X = 2 \dim S$ , then any continuous map  $S \rightarrow X$  is homotopic to a proper holomorphic immersion with simple double points.*

In addition, we have the approximation and interpolation results for proper holomorphic immersions and embeddings.

## Problem

*Is the same true if  $X$  is a Stein Oka manifold?*

*Does every Stein Oka manifold enjoy the (volume) density property?*

# Examples of manifolds with DP or VDP

- $\mathbb{C}^n$  for  $n \geq 1$  satisfies VDP for  $dz_1 \wedge \cdots \wedge dz_n$  (Andersén).
- $\mathbb{C}^n$  for any  $n > 1$  satisfies DP (Andersén and Lempert).
- $(\mathbb{C}^*)^n$  with the volume form  $\frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}$  satisfies VDP (Varolin). It is not known whether DP holds when  $n > 1$ .
- If  $G$  is a linear algebraic group and  $H \subset G$  is a closed proper reductive subgroup, then  $X = G/H$  is a Stein manifold with the density property, except when  $X = \mathbb{C}$ ,  $(\mathbb{C}^*)^n$ , or a  $\mathbb{Q}$ -homology plane with fundamental group  $\mathbb{Z}_2$  (Kaliman, Donzelli & Dvorsky).
- In particular, a linear algebraic group with connected components different from  $\mathbb{C}$  or  $(\mathbb{C}^*)^n$  has DP (Kaliman and Kutzschebauch).
- If  $p : \mathbb{C}^n \rightarrow \mathbb{C}$  is a holomorphic function with smooth reduced zero fibre, then  $X = \{xy = p(z)\}$  has DP (K&K). The same is true if the source  $\mathbb{C}^n$  of  $p$  is an arbitrary Stein manifold with DP.
- A Cartesian product  $X_1 \times X_2$  of two Stein manifolds  $X_1, X_2$  with DP also has DP. The analogous result holds for VDP (K&K).

# Gizatullin surfaces and the Koras-Russell cubic

- **Andrist 2017** A generic **Gizatullin surface** has the density property. A smooth affine algebraic surface  $X$  is a Gizatullin surface if  $\text{Aut}_{\text{alg}}(X)$  acts transitively on  $X$  up to finitely many points. Every such surface admits a fibration  $\pi: X \rightarrow \mathbb{C}$  whose generic fiber equals  $\mathbb{C}$  and there is only one exceptional fiber. If this exceptional fiber is reduced, then  $X$  has the density property.
- **Leuenberger 2016** DP holds for a family of hypersurfaces

$$X = \{(x, y, z) \in \mathbb{C}^{n+3} : x^2y = a(z) + xb(z)\},$$

where  $x, y \in \mathbb{C}$  and  $a, b \in \mathbb{C}[z]$  are polynomials in  $z \in \mathbb{C}^{n+1}$ . This family includes the **Koras-Russell cubic threefold**

$$C = \{(x, y, z_0, z_1) \in \mathbb{C}^4 : x^2y + x + z_0^2 + z_1^3 = 0\}.$$

This threefold is diffeomorphic to  $\mathbb{R}^6$ , but is not algebraically isomorphic to  $\mathbb{C}^3$  (Makar-Limanov, Dubouloz).

**It remains an open question whether  $C$  is biholomorphic to  $\mathbb{C}^3$ .**

# On the Schoen-Yau conjecture

**Schoen-Yau Conjecture 1997** The disc  $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$  does not admit any proper harmonic maps to  $\mathbb{R}^2$ .

**Božin 1999:** There exists a proper harmonic map  $\mathbb{D} \rightarrow \mathbb{R}^2$ .

**Globevnik & F. 2001** There is a proper holomorphic map  $f = (f_1, f_2): \mathbb{D} \rightarrow \mathbb{C}^2$  whose image is contained in  $(\mathbb{C}^*)^2$ . Hence, the harmonic map  $(\log |f_1|, \log |f_2|): \mathbb{D} \rightarrow \mathbb{R}^2$  is proper.

**Alarcón & López 2012** Every open Riemann surface  $X$  admits a conformal minimal immersion  $u = (u_1, u_2, u_3): X \rightarrow \mathbb{R}^3$  with a proper (harmonic) projection  $(u_1, u_2): X \rightarrow \mathbb{R}^2$ .

Since the space  $(\mathbb{C}^*)^n$  enjoys the volume density property for the volume form  $\omega = \frac{dz_1 \wedge \dots \wedge dz_n}{z_1 \dots z_n}$ , our main theorem implies:

## Corollary

*Every Stein manifold  $X$  of dimension  $n$  admits a proper holomorphic immersion to  $(\mathbb{C}^*)^{2n}$ , and a proper pluriharmonic map into  $\mathbb{R}^{2n}$ .*

# Preliminaries

A compact set  $K$  in a topological space  $S$  is said to be **regular** if  $K$  is the closure of its interior  $\overset{\circ}{K}$ .

## Definition

A pair  $K \subset L$  of compact convex sets in  $\mathbb{R}^N$  is a **simple convex pair** if there are a linear function  $\lambda: \mathbb{R}^N \rightarrow \mathbb{R}$  and constant  $a \in \mathbb{R}$  such that

$$K = \{z \in L : \lambda(z) \leq a\}. \quad (1)$$

The following lemma is obvious.

## Lemma

*Given regular compact convex sets  $C \subset B$  in  $\mathbb{R}^N$  and an open set  $U \subset \mathbb{R}^N$  containing  $C$ , there is a finite sequence of regular compact convex sets  $K_1 \subset K_2 \subset \dots \subset K_{m+1} = B$  such that  $C \subset K_1 \subset U$  and  $(K_i, K_{i+1})$  is a simple convex pair for every  $i = 1, \dots, m$ .*

# Slices and slabs

If  $K$  is a compact (convex) set in  $\mathbb{R}^N$ , then a **slice** of  $K$  is the intersection of  $K$  with a real affine hyperplane:

$$K_a = \{x \in K : \lambda(x) = a\}, \quad a \in \mathbb{R},$$

and a **slab** of  $K$  is a subset of the form

$$K_{a,b} = \{x \in K : a \leq \lambda(x) \leq b\}$$

where  $a < b$  and  $\lambda: \mathbb{R}^N \rightarrow \mathbb{R}$  is a linear function.

The number  $b - a > 0$  is called the **thickness** of the slab  $K_{a,b}$ .

If  $K$  is a compact subset of a manifold  $S$  contained in a local chart, then a slice or a slab of  $K$  will be understood as a subset of the respective type in the given chart.

# (Very) special Cartan pairs

Let  $S$  be a Stein manifold.

## Definition

A pair  $(A, B)$  of compact sets in  $S$  is a **special Cartan pair** if

- (i) the sets  $A, B, D = A \cup B$  and  $C = A \cap B$  are  $\mathcal{O}(S)$ -convex,
- (ii)  $A$  and  $B$  are **separated** in the sense that  $\overline{A} \setminus \overline{B} \cap \overline{B} \setminus \overline{A} = \emptyset$ , and
- (iii) there is a holomorphic coordinate system on a neighborhood of  $B$  in  $S$  in which  $B$  and  $C = A \cap B$  are regular convex sets. The set  $B$  is a **convex bump** and  $C$  is the **attaching set**.

A special Cartan pair  $(A, B)$  with  $C = A \cap B$  is **very special** if

- (iv) there is a holomorphic coordinate system on a neighborhood of  $B$  in  $S$  in which  $(C, B)$  is a **simple convex pair**.

# Replacing a special Cartan pair by finitely many very special Cartan pairs

**Grauert; Henkin & Leiterer** A Stein manifold  $S$  can be exhausted by an increasing sequence of compact sets  $A_1 \subset A_2 \subset \dots \subset \bigcup_j A_j = X$  such that  $A_{j+1} = A_j \cup B_j$ , where  $(A_j, B_j)$  is a special Cartan pair or  $B_j$  is a handle attached to  $A_j$ . The next lemma says that a special Cartan pair can be replaced by finitely many very special Cartan pairs.

## Lemma

*Assume that  $(A, B)$  is a special Cartan pair in  $S$ . Given an open set  $W \subset S$  containing  $A$ , there is a finite increasing sequence of compact  $\mathcal{O}(S)$ -convex sets*

$$A \subset A_1 \subset A_2 \subset \dots \subset A_{m+1} = A \cup B =: D$$

*such that  $A_1 \subset W$ , and for every  $i = 1, \dots, m$  we have  $A_{i+1} = A_i \cup B_i$  where  $(A_i, B_i)$  is a very special Cartan pair and  $C_i = A_i \cap B_i$  is an arbitrarily thin slab of  $B_i$ .*

# Proof of the lemma

**Proof.** Let  $C = A \cap B$ . By the assumption, there are an open neighborhood  $V_0 \subset S$  of  $B$  and a biholomorphic map

$$\theta: V_0 \rightarrow \tilde{V}_0 \subset \mathbb{C}^d$$

onto an open convex subset  $\tilde{V}_0$  of  $\mathbb{C}^d$  such that  $\theta(C) \subset \theta(B)$  are regular compact convex set in  $\mathbb{C}^d$ . We use the chart  $\theta$  to define the notion of convexity, slices, slabs, and simple convex pairs in  $V_0$ .

Pick an open neighborhood  $U$  of  $C = A \cap B$ , with  $U \subset W$ , and a compact convex set  $\tilde{C} \subset U$  containing  $C$  in its interior.

There is a sequence  $K_1 \subset K_2 \subset \dots \subset K_{m+1} = B$  of regular compact convex sets such that

$$B \cap \tilde{C} \subset K_1 \subset U$$

and  $(K_i, K_{i+1})$  is a simple convex pair for every  $i = 1, \dots, m$ , i.e., there are  $\mathbb{R}$ -linear functions  $\lambda_i: \mathbb{C}^d \rightarrow \mathbb{R}$  and numbers  $b_i \in \mathbb{R}$  such that

$$K_i = \{x \in K_{i+1} : \lambda_i(\theta(x)) \leq b_i\}.$$

## Proof of the lemma, 2

Choose numbers  $a_i < b_i$  close to  $b_i$ . For every  $i = 1, \dots, m$  let

$$A_i = A \cup K_i, \quad B_i = \{x \in K_{i+1} : a_i \leq \lambda_i(\theta(x))\}.$$

Assuming that  $b_i - a_i > 0$  is sufficiently small for each  $i$ , it follows that

$$A \cap B_i = \emptyset \quad \text{for } i = 1, \dots, m.$$

Then

$$A_i \cup B_i = A \cup K_{i+1} = A_{i+1} \subset A \cup B = D$$

and

$$C_i = A_i \cap B_i = \{x \in K_{i+1} : a_i \leq \lambda_i(\theta(x)) \leq b_i\}.$$

Thus,  $C_i$  is a slab of the compact convex set  $K_{i+1}$ .

It is easily verified that each  $(A_i, B_i)$  is a very special Cartan pair.

# The Main Lemma

## Lemma

Assume that  $S$  is a complex manifold of dimension  $d$ , and  $X$  is a Stein manifold of dimension  $n \geq 2d$  with the (volume) density property.

Let  $(A, B)$  be a special Cartan pair in  $S$ ,  $C := A \cap B$ , and  $D := A \cup B$ . Assume that

- (a)  $L$  is a compact  $\mathcal{O}(X)$ -convex set in  $X$ ,
- (b)  $K$  is a compact set in  $\mathring{A} \setminus C$  such that  $K \cup B$  is  $\mathcal{O}(S)$ -convex, and
- (c)  $f: A \rightarrow X$  is a holomorphic map such that

$$f(A \setminus \mathring{K}) \subset X \setminus L \quad (\iff \quad f^{-1}(L) \subset \mathring{K}).$$

Then we can approximate  $f$  uniformly on  $A$  by a holomorphic immersion  $\tilde{f}: D = A \cup B \rightarrow X$  (embedding if  $n > 2d$ ) such that

$$\tilde{f}(D \setminus \mathring{K}) \subset X \setminus L \quad (\iff \quad \tilde{f}^{-1}(L) \subset \mathring{K}).$$

# Proof, 1: reductions

By the first lemma, we assume that  $(A, B)$  is a very special Cartan pair.

Let  $W \subset S$  be a neighborhood of  $A$  and  $f: W \rightarrow X$  be a holomorphic map such that

$$f(W \setminus \overset{\circ}{K}) \subset X \setminus L.$$

By the theorem by Drinovec Drnovšek and myself we may assume, after shrinking  $W \supset A$  and by approximation, that  $f: W \rightarrow X$  is a **proper holomorphic immersion** whose image is a closed immersed complex submanifold of  $X$  with simple double points (embedded if  $n > 2d$ ).

Hence, the sets  $f(A)$ ,  $f(B)$ ,  $f(C)$ ,  $f(D)$ ,  $f(K \cup C)$  are  $\mathcal{O}(X)$ -convex.

Furthermore, we have  $L \cap f(W) \subset f(K)$ , so the sets  $L' := L \cup f(K)$  and  $L' \cup f(C)$  are also  $\mathcal{O}(X)$ -convex.

## Proof, 2: constructing a spray $F$ over $A$

Recall that the attaching set  $C = A \cap B$  can be chosen an arbitrarily thin slab of the convex set  $B$ .

**A suitable choice of  $C$  ensures that  $f$  (a proper immersion with simple double points) is an embedding on a small convex neighborhood  $\bar{U} \Subset W$  of  $C$ .**

The normal bundle of the immersion  $f: W \rightarrow X$  is holomorphically trivial over  $\bar{U}$  by the Oka-Grauert principle.

Hence, there are a neighborhood  $W_1 \subset W$  of  $A$  and a holomorphic map

$$F: W_1 \times \mathbb{D}^{n-d} \rightarrow X$$

which is injective (an embedding) on  $U \times \mathbb{D}^{n-d}$  and satisfies

$$F(z, 0) = f(z) \quad \text{for all } z \in W_1.$$

Here,  $\mathbb{D}^{n-d}$  denotes the unit polydisc in  $\mathbb{C}^{n-d}$ .

## Proof, 3: an isotopy $r_t: V \rightarrow V$

Recall that the sets  $L' = L \cup f(K)$  and  $L' \cup f(C)$  are  $\mathcal{O}(X)$ -convex.

By shrinking  $U \supset C$  and rescaling the variable  $w \in \mathbb{D}^{n-d}$ , we may assume that the Stein domain

$$\Omega := F(U \times \mathbb{D}^{n-d}) \subset X \setminus L'$$

is sufficiently small around  $f(C)$  such that  $L' \cup \overline{\Omega}$  is  $\mathcal{O}(X)$ -convex.

Hence, there is a Stein neighborhood  $\Omega' \subset X$  of  $L'$  such that

$$\overline{\Omega} \cap \overline{\Omega'} = \emptyset, \text{ and } \Omega_0 := \Omega \cup \Omega' \text{ is a Stein Runge domain in } X.$$

Choose a convex neighborhood  $V \supset B \cup \overline{U}$  and an isotopy  $r_t: V \rightarrow V$  ( $t \in [0, 1]$ ) of injective holomorphic maps (linear contractions) such that

- 1  $r_0 = \text{Id}_V$ ,
- 2  $r_t(U) \subset U$  for all  $t \in [0, 1]$ , and
- 3  $r_1(V) \subset U$ .

## Proof, 4: an isotopy of biholomorphisms $\psi_t: \Omega_0 \rightarrow X$

Define the isotopy  $\phi_t: V \times \mathbb{D}^{n-d} \rightarrow V \times \mathbb{D}^{n-d}$  ( $t \in [0, 1]$ ) by

$$\phi_t(z, w) = (r_t(z), w), \quad z \in V, w \in \mathbb{D}^{n-d}.$$

Since  $r_1(V) \subset U$ , we have that

$$\phi_1(V \times \mathbb{D}^{n-d}) \subset U \times \mathbb{D}^{n-d}.$$

Consider the isotopy  $\psi_t: \Omega_0 \rightarrow X$  ( $t \in [0, 1]$ ) defined by

$$\psi_t = F \circ \phi_t \circ F^{-1} \quad \text{on } \Omega = F(U \times \mathbb{D}^{n-d}); \quad \psi_t = \text{Id} \quad \text{on } \Omega'.$$

Note that  $\psi_0 = \text{Id}$  on  $\Omega_0 = \Omega \cup \Omega'$  and  $\psi_t(\Omega_0)$  is Runge in  $X$  for all  $t$ .

**The crucial point:** Since  $X$  enjoys the density property, we can approximate the biholomorphism  $\psi_1: \Omega_0 \rightarrow \psi_1(\Omega_0) \subset X$  uniformly on compacts in  $\Omega_0$  by holomorphic automorphisms  $\Psi \in \text{Aut}(X)$ .

## Proof, 5: the map $G : V \times \mathbb{D}^{n-d} \rightarrow X$

Recall that  $V \supset B$ . Consider the injective holomorphic map

$$G := \Psi^{-1} \circ F \circ \phi_1 : V \times \mathbb{D}^{n-d} \rightarrow X.$$

Since  $\phi_1(V \times \mathbb{D}^{n-d}) \subset U \times \mathbb{D}^{n-d}$ , we have

$$(F \circ \phi_1)(V \times \mathbb{D}^{n-d}) \subset F(U \times \mathbb{D}^{n-d}) = \Omega \subset X \setminus L'.$$

Since  $\psi_1 = \text{Id}$  on  $\Omega' \supset L'$ ,  $\Psi$  approximates the identity on a neighborhood of  $L'$ , and hence

$$G(V \times \mathbb{D}^{n-d}) \subset \Psi^{-1}(\Omega) \subset X \setminus L'.$$

We also have that  $F \circ \phi_1 = \psi_1 \circ F$  on  $U \times \mathbb{D}^{n-d}$  and hence

$$G = \Psi^{-1} \circ F \circ \phi_1 = \Psi^{-1} \circ \psi_1 \circ F \quad \text{on } U \times \mathbb{D}^{n-d}.$$

Since  $\Psi^{-1} \circ \psi_1 \approx \text{Id}$  on  $\Omega = F(U \times \mathbb{D}^{n-d})$ , we see that

$$G \approx F \text{ uniformly on compacts in } U \times \mathbb{D}^{n-d} \supset C \times \mathbb{D}^{n-d}.$$

## Proof, 6: gluing $F$ and $G$

Hence, we can glue  $F$  and  $G$  into a holomorphic map

$$\tilde{F}: (A \cup B) \times \frac{1}{2}\mathbb{D}^{n-d} \rightarrow X$$

such that

$$\tilde{F} \approx F \text{ on } A \times \frac{1}{2}\mathbb{D}^{n-d}, \quad \tilde{F} \approx G \text{ on } B \times \frac{1}{2}\mathbb{D}^{n-d}.$$

The holomorphic map  $\tilde{f} := \tilde{F}(\cdot, 0): A \cup B = D \rightarrow X$  then satisfies the conclusion of Lemma (after a generic perturbation). Indeed, if the approximations are sufficiently close then

$$\tilde{f}(B) \cap L = \emptyset \quad \text{and} \quad \tilde{f}(D \setminus K) \subset X \setminus L.$$

A similar argument applies if  $X$  enjoys the volume density property.

**Conclusion:** The noncritical case of the theorem follows from the lemma by a standard induction procedure. The analysis near an attached handle (in  $S$ ) reduces to the noncritical case (as in Oka theory).

*THANK YOU*

*FOR YOUR ATTENTION*