The Calabi-Yau problem for minimal surfaces of finite genus and countably many ends

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What is a minimal surface

1774 **Lagrange** A variational formula for the area of a surface. A smooth graph $(x, y, f(x, y)) \subset \overline{\Omega} \times \mathbb{R}$ over a smoothly bounded domain $\Omega \subset \mathbb{R}^2$ is a critical point of the area functional with prescribed boundary values iff it satisfies the equation of minimal graphs:

$$\mathrm{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) = 0.$$

1776 **Meusnier** A smooth surface $S \subset \mathbb{R}^3$ satisfies locally Lagrange's equation iff its mean curvature function vanishes identically:

$$\mathrm{H} = \frac{\kappa_1 + \kappa_2}{2} = 0.$$

Here, κ_1, κ_2 denote the **principal curvatures**. The **mean curvature vector** of S is

$$\mathbf{H} = \frac{\kappa_1 + \kappa_2}{2}\mathbf{N}$$

where $\mathbf{N}$ is the unit normal vector field to S (the **Gauss map**) of S.

Lagrange's first variational formula for the area

Assume that M is a smooth compact surface with boundary bM and $X : M \to \mathbb{R}^n$ $(n \geq 3)$ is a $\mathscr{C}^2$ immersion.

For any smooth variation $X^t : M \to \mathbb{R}^n$ of $X^0 = X$ fixed on bM we have

$$\frac{d}{dt}\bigg|_{t=0} \mathrm{Area}(X^t(M)) = -2 \int_M E \cdot \mathbf{H}\, dA,$$

where

- $E = \partial x^t / \partial t|_{t=0}$ is the variational vector field at $t = 0$,
- $\mathbf{H}$ is the mean curvature vector field of X, and
- dA is the element of the surface area in the metric $X^*(ds^2)$ on M.

Hence, $\frac{d}{dt}\big|_{t=0} \mathrm{Area}(X^t(M)) = 0$ for all variations iff $\mathbf{H} = 0$.

Riemann surfaces are everywhere

Assume that M is a surface embedded or immersed into a Riemannian manifold (N, ds^2). Let g be the induced metric on M.

1922 **Gauss,...** At every point of M there exists an **isothermal coordinate** $z = x + \imath y$ in which

$$g = \lambda(dx^2 + dy^2) = \lambda|dz|^2$$

for some positive function $\lambda > 0$. Thus, g is a **Kähler metric**.

The transition map between any pair of isothermal coordinates is a conformal diffeomorphism between plane domains, hence holomorphic or antiholomorphic. Hence, if M is orientable, the Riemannian metric g determines on M the structure of a **Riemann surface** such that the immersion $M \to N$ is conformal.

1971 **Rüedy** Every Riemann surface (either open or closed) admits a proper conformal embedding into $\mathbb{R}^3$.

Conformal minimal surfaces in $\mathbb{R}^n$

Assume now that M is an **open Riemann surface**. Let $X : M \to \mathbb{R}^n$ be a **conformal immersion** and $g = X^* ds^2$ the induced metric. Then,

$$\Delta_g X = \frac{1}{\lambda}\Delta X = 2\,\mathbf{H}.$$

where Δ_g is the metric Laplacian and $\mathbf{H} : M \to \mathbb{R}^n$ is the mean curvature vector. **Hence, a conformal immersion is minimal iff it is harmonic.**

There is a deeper reason for considering conformal parameterizations of minimal surfaces. Given an immersion $X : \overline{\mathbb{D}} \to \mathbb{R}^n$, its **Dirichlet (energy) integral** is

$$\mathscr{D}(X) = \int_{\mathbb{D}} |\nabla X|^2 du\, dv = \int_{\mathbb{D}} \left(|X_u|^2 + |X_v|^2\right) du\, dv.$$

Then, $2\mathrm{Area}(X) \leq \mathscr{D}(X)$ and the equality holds if and only if X is conformal. Consider maps X whose restriction to the circle $\mathbb{T} = b\mathbb{D}$ is a monotone parameterization of a given smooth oriented Jordan curve Γ. A map in this class minimizing $\mathscr{D}(X)$ also minimizes the area and provides a conformally parameterized minimal surface with boundary Γ. Hence,

a conformal parameterization gives a least energy spreading of the surface over a geometric configuration of least area in $\mathbb{R}^n$.

Connection with complex analysis

Let $X(z)$ be a smooth function of a complex variable $z = x + \mathrm{i}y$ (which we think of as a local coordinate on a Riemann surface M). Set

$$\partial X = \frac{1}{2}\left(\frac{\partial X}{\partial x} - \mathrm{i}\frac{\partial X}{\partial y}\right) dz, \qquad \bar{\partial} X = \frac{1}{2}\left(\frac{\partial X}{\partial x} + \mathrm{i}\frac{\partial X}{\partial y}\right) d\bar{z}.$$

Then,

- X is **holomorphic** iff $\bar{\partial} X = 0$; equivalently, if $dX = \partial X$.
- X is **harmonic** iff

$$\Delta X = 2\mathrm{i}\,\partial\bar{\partial} X = -2\mathrm{i}\,\bar{\partial}\partial X = 0 \iff \partial X \text{ is holomorphic.}$$

- An immersion $X = (X_1, X_2, \cdots, X_n) : M \to \mathbb{R}^n$ is **conformal** iff

$$X_x \cdot X_y = 0,\ |X_x|^2 = |X_y|^2 \iff \sum_{k=1}^{n} (\partial X_k)^2 = 0$$

The Enneper-Weierstrass formula

This shows that a smooth immersion $X = (X_1, X_2, \cdots, X_n) : M \to \mathbb{R}^n$ is a conformal minimal immersion (a conformal minimal surface) if and only if

$$\partial X = (\partial X_1, \ldots \partial X_n) \text{ is holomorphic and } \sum_{k=1}^{n} (\partial X_k)^2 = 0.$$

Fix a nowhere vanishing holomorphic 1-form θ on M. The quotient

$$f = 2\partial X/\theta : M \to \mathcal{A}_* = \mathcal{A} \setminus \{0\}$$

is a holomorphic map with values in the (punctured) **null quadric**:

$$\mathcal{A} = \left\{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{j=1}^{n} z_j^2 = 0 \right\}.$$

Hence, every conformal minimal immersion $X : M \to \mathbb{R}^n$ is of the form

$$X = \int \Re(f\theta),$$

where $f \colon M \to \mathcal{A}_*$ is a holomorphic map such that

$$\int_C \Re(f\theta) = 0 \in \mathbb{R}^n \quad \text{for all closed curves } C \text{ in } M.$$

The Weierstrass formula in dimension $n = 3$

Let $X = (X_1, X_2, X_3) : M \to \mathbb{R}^3$ be a conformal minimal immersion. Set

$$\phi = 2\,\partial X = (\phi_1, \phi_2, \phi_3), \qquad \sum_{j=1}^{3} \phi_j^2 = 0.$$

The stereographic projection of the real Gauss map $N = (N_1, N_2, N_3) : M \to S^2$ of X is the (holomorphic) **complex Gauss map**

$$\mathfrak{g} = \frac{N_1 + \mathrm{i} N_2}{1 - N_3} = \frac{\phi_2 - \mathrm{i}\,\phi_1}{\mathrm{i}\,\phi_3} = \frac{\phi_3}{\phi_1 - \mathrm{i}\phi_2} : M \to \mathbb{CP}^1.$$

Then, X is recovered from the pair $(\mathfrak{g}, \phi_3)$ by the formula

$$X = \Re \int \left(\frac{1}{2}\left(\frac{1}{\mathfrak{g}} - \mathfrak{g}\right), \frac{\mathrm{i}}{2}\left(\frac{1}{\mathfrak{g}} + \mathfrak{g}\right), 1 \right) \phi_3.$$

2019 **Alarcón, F., López** Every meromorphic function on an open Riemann surface M is the Gauss map of a conformal minimal surface $X : M \to \mathbb{R}^3$.

The Gauss map and Gaussian curvature

Many important quantities of the minimal surface $X : M \to \mathbb{R}^3$ can be computed from its Gauss map $\mathfrak{g} : M \to \mathbb{CP}^1$, in particular:

$$g = \frac{(1 + |\mathfrak{g}|^2)^2}{4|\mathfrak{g}|^2} |\phi_3|^2 \qquad \text{(the induced Riemannian metric)}$$

$$Kg = -\frac{4|d\mathfrak{g}|^2}{(1 + |\mathfrak{g}|^2)^2} = -\mathfrak{g}^*(\sigma^2_{\mathbb{CP}^1}) \quad \text{(the Gaussian curvature),}$$

and the **total Gaussian curvature**

$$\mathrm{TC}(X) = \int_M K dA = -\mathrm{Area}_{\mathbb{CP}^1}(\mathfrak{g}(M)).$$

Assuming that $(M, g = X^* ds^2)$ is complete, we have $\mathrm{TC}(X) > -\infty$ iff M is a compact Riemann surface $\overline{M}$ punctured at finitely many points and the Gauss map of X extends to a holomorphic map $\mathfrak{g} : \overline{M} \to \mathbb{CP}^1$. In this case,

$$\mathrm{TC}(X) = -4\pi \deg(\mathfrak{g}).$$

Holomorphic null curves

A holomorphic immersion $Z = (Z_1, \ldots, Z_n) : M \to \mathbb{C}^n$ $(n \geq 3)$ is said to be a **holomorphic null curve** if

$$\sum_{k=1}^{n} (\partial Z_k)^2 = 0.$$

Every such curve is of the form

$$Z = \int f\theta,$$

where $f : M \to \mathcal{A}_*$ is a holomorphic map such that

$$\int_C f\theta = 0 \quad \text{for all closed curves } C \text{ in } M.$$

Hence, the real and the imaginary part of a null curve are conformal minimal surfaces. Conversely, every conformal minimal surfaces is locally (on simply connected domains) the real part of a holomorphic null curve.

Catenoid and helicoid are conjugate minimal surfaces

Example

The **catenoid** and the **helicoid** are conjugate minimal surfaces — the real and the imaginary part of the same null curve $Z\colon \mathbb{C} \to \mathbb{C}^3$ given by

$$Z(z) = (\cos z, \sin z, -\mathrm{i}z), \qquad z = x + \mathrm{i}y \in \mathbb{C}.$$

Consider the family of minimal surfaces ($t \in \mathbb{R}$):

$$\begin{aligned} X_t(z) &= \Re\left(e^{\mathrm{i}t} Z(z)\right) \\ &= \cos t \begin{pmatrix} \cos x \cdot \cosh y \\ \sin x \cdot \cosh y \\ y \end{pmatrix} + \sin t \begin{pmatrix} \sin x \cdot \sinh y \\ -\cos x \cdot \sinh y \\ x \end{pmatrix}. \end{aligned}$$

At $t = 0$ we have a parametrization of a catenoid, and at $t = \pm\pi/2$ we have a (left or right handed) helicoid.

The catenoid and the helicoid

The Calabi-Yau problem for minimal surfaces

An immersed surface $X : M \to \mathbb{R}^n$ is **complete** if $g = X^* ds^2$ is a complete metric on M, i.e., the boundary is at infinite distance.

1965 **Calabi's Conjecture** There does not exist a complete minimal surface in $\mathbb{R}^3$ (and hypersurface in $\mathbb{R}^n$, $n \geq 3$) with a bounded coordinate function.

1980 **Jorge and Xavier** Calabi's Conjecture is false: there is a complete minimal disc $\mathbb{D} \to \mathbb{R}^2 \times (-1, +1)$ with the third coordinate $\Re z$.

1996 **Nadirashvili** There is a complete bounded minimal disc in $\mathbb{R}^3$.

2000 **S.T. Yau** Review of geometry and analysis ("the millenium lecture"). The problem became known as the **Calabi-Yau problem for minimal surfaces**.

2007 **Martín & Nadirashvili 2007** There is a continuous map $X : \overline{\mathbb{D}} \to \mathbb{R}^3$ such that $X : \mathbb{D} \to \mathbb{R}^3$ is a complete conformal minimal immersion.

2008 **Colding & Minicozzi 2008** Calabi's conjecture holds for **embedded** minimal surfaces in $\mathbb{R}^3$ of finite topological type — any such is proper in $\mathbb{R}^3$.

Complete minimal surfaces with Jordan boundaries

2015 **Alarcón, Drinovec, F., López, Proc. LMS 2015** Let M be a compact bordered Riemann surface. Every conformal minimal immersion $X_0 : M \to \mathbb{R}^n$ $(n \geq 3)$ can be uniformly approximated by continuous maps $X \colon M \to \mathbb{R}^n$ (topological embeddings if $n \geq 5$) such that

$X : \mathring{M} \to \mathbb{R}^n$ is a **complete conformal minimal immersion, and the boundary $X(bM) \subset \mathbb{R}^n$ is a union of Jordan curves.**

2019 **Alarcón and F., Rev. Mat. Iberoam., to appear**

The same is true for every bordered Riemann surface of finite genus and with countably many boundary curves:

$$M = R \setminus \cup_{i=1}^{\infty} D_i$$

where R is a compact Riemann surface and $\overline{D}_i$ are pairwise disjoint discs in R such that $R \setminus \cup_i \overline{D}_i = \mathring{M}$ is an open domain in R.

2015 **Meeks, Pérez, and Ros**

If M is as above then every complete conformal minimal embedding $M \hookrightarrow \mathbb{R}^3$ is proper in $\mathbb{R}^3$, hence unbounded.

Novelties and techniques behind our theorems

The main novelties:

(a) There is no change of the conformal structure on M

(b) The boundary of the image surface consists of Jordan curves

(c) Countably many (nonpoint) boundary components are allowed

(d) The analogous result holds for holomorphic curves, null holomorphic curves, and holomorphic Legendrian curves

The principal methods: This is proved by a fast uniformly convergent sequence of spiralling modifications which inductively increase the metric on M and make the limit minimal surface complete. It is reminiscent of the construction of $\mathscr{C}^1$ isometric immersions by **Nash 1956**. The main tools:

(a) **Exposing boundary points of Riemann surfaces**

(b) **The Riemann-Hilbert boundary value problem for minimal surfaces**

(c) **A lower estimate on the metric under uniformly small deformations**

Enlarging the intrinsic diameter of a minimal surface

Lemma (The Main Lemma)

Let M be a compact bordered conformal surface with smooth boundary, and let $X : M \to \mathbb{R}^n$ $(n \geq 3)$ be a conformal minimal immersion.
Given a point $p_0 \in \mathring{M}$ and numbers $\epsilon > 0$ (small) and $\tau > 0$ (big), there is a conformal minimal immersion $Y : M \to \mathbb{R}^n$ satisfying the following conditions.

1. $|Y(p) - X(p)| < \epsilon$ *for all* $p \in M$.
2. $\mathrm{dist}_Y(p_0, bM) > \tau$.
3. $\mathrm{Flux}_Y = \mathrm{Flux}_X$.
4. $Y|_{bM} : bM \to \mathbb{R}^n$ *is an embedding.*
5. *If* $n \geq 5$ *then* $Y : M \to \mathbb{R}^n$ *is an embedding.*

This lemma was proved by **A. Alarcón, B. Drinovec Drnovšek, F. Forstnerič, F.J. López**, Proc. London Math Soc. 2015. It shows that, when inductively constructing a conformal minimal immersion $M \to \mathbb{R}^n$ satisfying some desired conditions, guaranteeing completeness comes for free.

The Riemann-Hilbert method

The proof uses the **Riemann-Hilbert boundary value problem** adapted to null curves and minimal surfaces.

Let $X : M \to \mathbb{R}^n$ be a conformal minimal immersion (CMI). Choose an arc $I \subset bM$. Let $G : I \times \overline{\mathbb{D}} \to \mathbb{R}^n$ be a smooth map such that for every $p \in I$,

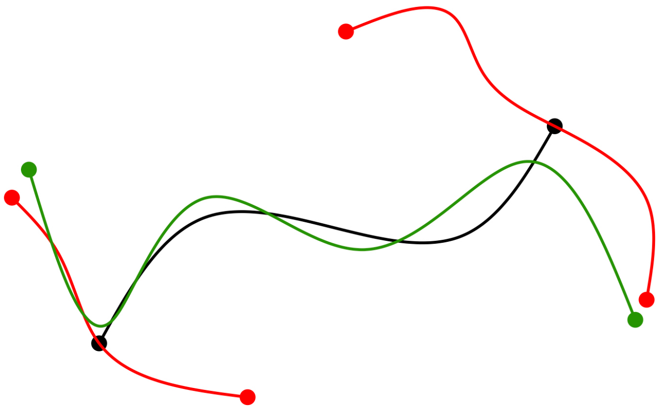
$$G(p, \cdot) : \overline{\mathbb{D}} \to \mathbb{R}^n \text{ is a conformal minimal disc and } G(p, 0) = X(p).$$

If $n > 3$, assume in addition that these discs lie in parallel planes (this condition is not necessary for $n = 3$).

Then, there exists a CMI $Y : M \to \mathbb{R}^n$ with the following properties:

- $Y(p)$ lies close to $G(p, b\mathbb{D})$ for $p \in I$.
- Y is uniformly close to X on $M \setminus U$, where $U \subset M$ is a given small neighbourhood of the arc $I \subset bM$.
- $Y(U)$ lies close to $G(I \times \overline{\mathbb{D}})$.
- $\mathrm{Flux}_Y = \mathrm{Flux}_X$.
- We can make Y agree with X at finitely many points of $M \setminus I$.

Illustration of the Riemann-Hilbert deformation



The effect of the Riemann-Hilbert deformation

The effect of this deformation is that for any path $\gamma : [0,1] \to M$ from $\gamma(0) = p_0 \in \mathring{M}$ to a point $\gamma(1) = p \in I$, we have

$$\mathrm{length}(Y \circ \gamma) \approx \mathrm{length}(X \circ \gamma) + \mathrm{diam}\, G(p, \overline{\mathbb{D}}).$$

Let $\Gamma : bM \to \mathbb{R}^n$ be a fixed map such that $X(p) \neq \Gamma(p)$ for all $p \in bM$. By a suitable choice of the discs $G(p, \cdot\,)$ orthogonal to the position vector $X(p) - \Gamma(p)$ of size $r(p) > 0$, the intrinsic distance from p_0 to $I \subset bM$ increases by a given fixed amount $\delta > 0$, while the extrinsic diameter increases only by the order δ^2 by Pythagoras' theorem:

$$\|Y(p) - \Gamma(p)\| \approx \sqrt{r(p)^2 + \delta^2} \approx r(p) + \delta^2/2r(p), \quad p \in M.$$

Repeating this deformation on other arcs for a sequence $\delta_j > 0$ with

$$\sum_j \delta_j = +\infty \quad \text{and} \quad \sum_j \delta_j^2 < \epsilon,$$

we can arrange that the boundaries $Y_j(bM)$ spiral around $\Gamma(bM)$ and the intrinsic diameter of $(M, Y_j^* ds^2)$ grows to infinity, while the distance of Y_j to X remains uniformly small.

A lower bound on the intrinsic diameter

Let $p_0 \in \mathring{M}$. We denote by $\Gamma_{\mathrm{qd}}(M, p_0)$ the space of all **quasi-divergent paths** $\gamma : [0,1) \to M$ with $\gamma(0) = p_0$, i.e., such that the image of γ is not contained in any compact subset of $\mathring{M}$ (equivalently, γ clusters on bM).

Lemma

Let M be a compact smooth manifold with nonempty boundary bM, and let $Y : M \to \mathbb{R}^n$ be a $\mathscr{C}^1$ immersion. Given a point $p_0 \in \mathring{M}$ and a number $\eta > 0$, there exists a number $\epsilon > 0$ such that for every continuous map $X : M \to \mathbb{R}^n$ with

$$\|X - Y\|_{\mathscr{C}^0(M)} := \max\{|X(p) - Y(p)| : p \in M\} < \epsilon$$

we have that

$$\inf\{\mathrm{length}(X \circ \gamma) : \gamma \in \Gamma_{\mathrm{qd}}(M, p_0)\} \geq \mathrm{dist}_Y(p_0, bM) - \eta.$$

The same result holds for maps to any Riemannian manifold.

Proof of the main theorem, I

Assume that R is a compact Riemann surface and

$$M = R \setminus \bigcup_{i=0}^{\infty} D_i \quad \text{is an open domain in } R,$$

where $\{D_i\}_{i\in\mathbb{Z}_+}$ is a countable family of pairwise disjoint closed discs in R. Denote by d a Riemannian distance function on R. Let

$$M_i = R \setminus \bigcup_{k=0}^{i} \mathring{D}_k, \qquad i = 0, 1, 2, \ldots.$$

Clearly, M_i is a compact bordered Riemann surface with boundary $bM_i = \bigcup_{k=0}^{i} bD_k$, and

$$M_0 \supset M_1 \supset M_2 \supset \cdots \supset \bigcap_{i=1}^{\infty} M_i = \overline{M}.$$

Let $X_0 : M_0 \to \mathbb{R}^n$ be a smooth conformal minimal immersion with vanishing flux and such that $X_0|_{bM_0} : bM_0 \to \mathbb{R}^n$ is injective.

Proof, II

Pick a number $\epsilon_0 > 0$ and a point $p_0 \in M$. An inductive application of both lemmas furnishes a sequence of CMIs $X_i : M_i \to \mathbb{R}^n$ and numbers $\epsilon_i > 0$ satisfying the following conditions for every $i \in \mathbb{N}$.

(a$_i$) $\mathrm{dist}_{X_i}(p_0, bM_i) > i$.

(b$_i$) $X_i : bM_i \to \mathbb{R}^n$ is injective; if $n \geq 5$ then $X_i : M_i \to \mathbb{R}^n$ is injective.

(c$_i$) $\sup_{p \in M_i} |X_i(p) - X_{i-1}(p)| < \epsilon_{i-1}$.

(d$_i$) For every continuous map $Y : M_i \to \mathbb{R}^n$ with $\|Y - X_i\|_{\mathscr{C}(M_i)} < 2\epsilon_i$,

$$\inf\{\mathrm{length}(Y \circ \gamma) : \gamma \in \Gamma_{\mathrm{qd}}(M_i, p_0)\} > \mathrm{dist}_{X_i}(p_0, bM_i) - 1 > i - 1.$$

(e$_i$) $0 < \epsilon_i < \frac{1}{2} \min\{\epsilon_{i-1}, \delta_i\}$, where

$$\delta_i := \frac{1}{i^2} \inf\left\{|X_i(p) - X_i(q)| : p, q \in bM_i,\ \mathrm{d}(p, q) > \frac{1}{i}\right\} > 0.$$

Proof, III — the induction step

Assume that for some $i \in \mathbb{N}$ we have maps $X_0, \ldots, X_{i-1}$ and numbers $\epsilon_0, \ldots, \epsilon_{i-1}$ satisfying these conditions. This holds for $i = 1$ with X_0 and ϵ_0.

The lemma on enlarging the intrinsic diameter, applied to $X_{i-1}|_{M_i}$, furnishes the next conformal minimal immersion $X_i : M_i \to \mathbb{R}^n$ satisfying conditions (a$_i$), (b$_i$), and (c$_i$).

Pick a number $\epsilon_i > 0$ satisfying condition (e$_i$); such exists since $X_i|_{bM_i}$ is injective by (b$_i$). (If $n \geq 5$, we modify the definition of δ_i so as to include all pairs of points $p, q \in M_i$ with $\mathrm{d}(p, q) > \frac{1}{i}$.)

Finally, decreasing $\epsilon_i > 0$ if necessary we may assume that (d$_i$) holds as well in view of the lemma on lower bound of the intrinsic diameter.

The induction may proceed.

Conclusion of the proof

Conditions (c_i) and (e_i) give a continuous limit map

$$X = \lim_{i\to\infty} X_i : \overline{M} \to \mathbb{R}^n$$

such that $X : M \to \mathbb{R}^n$ is a conformal minimal immersion with

$$|X(p) - X_i(p)| \le \sum_{k=i}^{\infty} |X_{k+1}(p) - X_k(p)| < \sum_{k=i}^{\infty} \epsilon_k < 2\epsilon_i, \quad p \in \overline{M}. \qquad (1)$$

We claim that $X : M \to \mathbb{R}^n$ is **complete**. Indeed, consider any divergent path $\gamma : [0,1) \to M$ with $\gamma(0) = p_0$. There is a sequence $0 < t_1 < t_2 < \cdots$ with

$$\lim_{j\to\infty} t_j = 1 \ \text{ and } \ \lim_{j\to\infty} \gamma(t_j) = p \in bM.$$

Then, $p \in bD_{i_0}$ for some $i_0 \in \mathbb{Z}_+$, and hence $p \in bM_i$ for all $i \ge i_0$, so γ is a quasidivergent path in M_i for any $i \ge i_0$. Conditions (a_i), (d_i), (e_i), and (**??**) imply

$$\mathrm{length}(X(\gamma)) > \mathrm{dist}_{X_i}(p_0, bM_i) - 1 > i - 1, \quad i \ge i_0.$$

Letting $i \to +\infty$ shows that $\mathrm{length}(X(\gamma)) = +\infty$.

Injectivity of $X : bM \to \mathbb{R}^n$

Pick a pair of points $p, q \in bM$, $p \neq q$. Take $i_0 > 1$ such that $\frac{1}{i} < \mathrm{d}(p, q)$ for all $i \geq i_0$. By (c_i) and (e_i) ($2\epsilon_i < \delta_i$), given $i \geq i_0$ we have

$$\begin{aligned} \|X_i(p) - X_i(q)\| &\leq \|X_{i+1}(p) - X_i(p)\| + \|X_{i+1}(q) - X_i(q)\| \\ &\quad + \|X_{i+1}(p) - X_{i+1}(q)\| \\ &< \delta_i + \|X_{i+1}(p) - X_{i+1}(q)\| \\ &\leq \frac{1}{i^2}\|X_i(p) - X_i(q)\| + \|X_{i+1}(p) - X_{i+1}(q)\|, \end{aligned}$$

and hence $\|X_{i+1}(p) - X_{i+1}(q)\| > \left(1 - \frac{1}{i^2}\right)\|X_i(p) - X_i(q)\|$.
It follows that

$$\|X_{i_0+k}(p) - X_{i_0+k}(q)\| > \|X_{i_0}(p) - X_{i_0}(q)\| \prod_{i=i_0}^{i_0+k-1} \left(1 - \frac{1}{i^2}\right) \quad \text{for all } k \in \mathbb{N}.$$

Letting $k \to \infty$ we obtain

$$\|X(p) - X(q)\| \geq \frac{1}{2}\|X_{i_0}(p) - X_{i_0}(q)\| > 0.$$

This completes the proof.

Some open problems

(a) Let M be an open domain in a compact Riemann surface such that M admits a nonconstant bounded harmonic function which does not extend to a harmonic function on any bigger domain.
Is M the conformal structure of a complete bounded minimal surface in $\mathbb{R}^3$?

(b) Find an example of a complete bounded minimal surface in $\mathbb{R}^3$ whose underlying complex structure is $\mathbb{C} \setminus K$, where K is a Cantor set in $\mathbb{C}$.

(c) Let $f : M \hookrightarrow \mathbb{C}^2$ be a holomorphically embedded compact bordered Riemann surface. Can we approximate it by a complete holomorphically embedded $g : \mathring{M} \hookrightarrow \mathbb{C}^2$, possibly with Jordan boundary?