Mergelyan approximation theorem for holomorphic Legendrian curves

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25 February 2020

Complex contact manifolds

Kobayashi 1959 A **complex contact manifold** is a pair (X, ξ) where:

- X is a complex manifold of odd dimension $2n + 1 \geq 3$,
- ξ is a holomorphic hyperplane subbundle of the tangent bundle TX which is **maximally nonintegrable**, in the sense that the O'Neill tensor

$$O : \xi \times \xi \to TX/\xi = L, \quad (v, w) \mapsto [v, w] \mod \xi$$

  (also called the **Frobenius obstruction**) is nondegenerate.
- Equivalently, $\xi = \ker \alpha$ where α is a holomorphic 1-form on X with values in the normal bundle $L = TX/\xi$, realising the quotient projection

$$0 \longrightarrow \xi \hookrightarrow TX \xrightarrow{\alpha} L \longrightarrow 0$$

  and satisfying

$$\alpha \wedge (d\alpha)^n \neq 0.$$

Such ξ is a **holomorphic contact structure**, and α is a **contact form**.

Darboux's theorem and Gray's stability theorem

Two complex contact manifolds (X, ξ) and (X', ξ') are said to be **contactomorphic** if there exists a biholomorphism $F : X \to X'$ satisfying

$$dF_x(\xi_x) = \xi'_{F(x)} \quad \text{for all } x \in X.$$

Example (Model complex contact space)

$$(\mathbb{C}^{2n+1}, \xi_0 = \ker \alpha_0), \quad \alpha_0 = dz + \sum_{j=1}^{n} x_j dy_j.$$

Darboux 1882, Moser 1965 Every complex contact manifold (X^{2n+1}, ξ) is locally contactomorphic to $(\mathbb{C}^{2n+1}, \xi_0)$.

Gray's stability theorem, 1959 If (X, ξ) is a compact contact manifold then any small contact perturbation ξ' of ξ is contactomorphic to ξ.

LeBrun & Salamon 1994 Any two complex contact structures on a simply connected compact complex manifold are contactomorphic.

A couple of examples

Example (A contact structure on $\mathbb{CP}^{2n+1}$)

Let $z_0, \ldots, z_{2n+1}$ be complex coordinates on $\mathbb{C}^{2n+2}$ and

$$\eta = z_0 dz_1 - z_1 dz_0 + \cdots + z_{2n} dz_{2n+1} - z_{2n+1} dz_{2n}.$$

Let η_j be the pull-back of η to $\mathbb{C}^{2n+1} \cong H_j = \{z_j = 1\} \subset \mathbb{C}^{2n+2}$; e.g.

$$\eta_0 = dz_1 + z_2 dz_3 - z_3 dz_2 + \cdots$$

Then (H_j, η_j) is contactomorphic to $(\mathbb{C}^{2n+1}, \alpha_0)$ for each j, and this collection defines a contact structure on $\mathbb{CP}^{2n+1}$.

Example (A contact structure on T^*Z)

Let Z^{n+1} be a complex manifold. The cotangent bundle T^*Z carries the tautological 1-form which is given in any set of local coordinates $z_0, \ldots, z_n$ on Z and the induced fibre coordinates $\zeta_0, \ldots, \zeta_n$ on T^*Z by

$$\eta = \zeta_0 dz_0 + \ldots + \zeta_n dz_n \quad (= \mathbf{p} d\mathbf{q} \text{ in classical notation}).$$

Then, η is a contact form on $\mathbb{P}(T^*Z)$.

Isotropic and Legendrian submanifolds

A smooth map $f : M \to (X, \xi)$ is said to be **isotropic** if

$$df_p(T_pM) \subset \xi_{f(p)}, \quad p \in M.$$

If $\xi = \ker \alpha$ then $f : M \to X$ is isotropic iff $f^*\alpha = 0$.
An isotropic immersion is **Legendrian** if $\dim_{\mathbb{R}} M = 2n$ is maximal.

If $\dim X = 2n+1$ and f is an isotropic immersion, then $\dim_R M \leq 2n$; if $\dim_{\mathbb{R}} M = 2n$ then $f(M)$ is an **immersed complex submanifold** of X.

Example

Let $(\mathbb{C}^{2n+1}, \xi_0 = \ker \alpha_0)$ with $\alpha_0 = dz + \sum_{j=1}^n x_j dy_j$. Given a holomorphic function $z = z(y_1, \ldots, y_n)$, the formula

$$dz - \sum_{j=1}^{n} \frac{\partial z}{\partial y_j} dy_j = 0$$

shows that $y \mapsto (-\partial z/\partial y, y, z(y))$ is a complex Legendrian submanifold.

A few (non-) existence and approximation results

Segre 1926, Bryant 1981 Every compact Riemann surface embeds as a complex Legendrian curve in $\mathbb{CP}^3$.

Alarcón, F., López, 2017 Every open Riemann surface embeds as a proper complex Legendrian curve in $(\mathbb{C}^{2n+1}, \alpha_0)$. Furthermore, Runge approximation (and Weierstrass interpolation) theorem holds for such curves.

F., 2017 There is a contact form α on $\mathbb{C}^3$ which is **Kobayashi hyperbolic**. In particular, there are no nonconstant holomorphic Legendrian lines $\mathbb{C} \to (\mathbb{C}^3, \alpha)$. Hence, $\xi = \ker \alpha$ is a **nonstandard contact structure** on $\mathbb{C}^3$.

Lárusson, F., 2019 If Z is an Oka manifold of dimension > 1, then every open Riemann surface admits a holomorphic Legendrian immersion into $\mathbb{P}(T^*Z)$. If Z is Stein with the holomorphic density property, then there exist proper Legendrian immersions into $\mathbb{P}(T^*Z)$.

Alarcón, F., Lárusson, 2019 Let M be a Riemann surface, open or compact, and let K be a compact subset of M. Then, every holomorphic Legendrian map f from a neighbourhood of K to $\mathbb{CP}^3$ can be approximated uniformly on K by holomorphic Legendrian maps $M \to \mathbb{CP}^3$. The approximants can be taken to agree with f to any finite order at each point of any finite subset of K. The analogous result holds for Legendrian immersions.

The new result — Mergelyan theorem for Legendrian immersions

Let M be a connected Riemann surface.

A compact subset $S \subsetneq M$ is **admissible** if $S = K \cup E$, where K is a finite union of pairwise disjoint compact domains in M with piecewise $\mathscr{C}^1$ boundaries and $E = \overline{S \setminus K}$ is a union of finitely many pairwise disjoint smooth Jordan arcs and closed Jordan curves meeting K transversely (only) in their endpoints.

Theorem (F., 2020)

Let (X, ξ) be a complex contact manifold, and let S be an admissible set in a Riemann surface M. Every Legendrian immersion $f : S \to X$ of class $\mathscr{A}^r(S, X)$, $r \geq 2$, can be approximated in the $\mathscr{C}^r(S, X)$ topology by holomorphic Legendrian immersions $\tilde{f} : U \to X$ from open neighbourhoods of S in M.

Furthermore, $\tilde{f}$ can be chosen to agree with f on any given finite subset A of S (to any given finite order at the points of $A \cap \mathring{S}$), and it can be chosen an embedding provided that f is injective on A.

A few corollaries

Given $\rho > 1$ we set $A_\rho = \{z \in \mathbb{C} : \rho^{-1} < |z| < \rho\}$.

Corollary (Approximating Legendrian loops by annuli)

Let X be a complex contact manifold. Every Legendrian loop $S^1 \to X$ of class $\mathscr{C}^r(S^1, X)$ $(r \geq 2)$ can be approximated in $\mathscr{C}^r(S^1, X)$ by embedded holomorphic Legendrian annuli $A_\rho \hookrightarrow X$, where $\rho > 1$ may depend on the map.

Corollary (Legendrian immersions into $\mathbb{CP}^3$)

Let S be an admissible set in a Riemann surface M. Every Legendrian immersion $S \to \mathbb{CP}^3$ of class $\mathscr{A}^r(S, \mathbb{CP}^3)$ $(r \geq 2)$ can be approximated in $\mathscr{C}^r(S, \mathbb{CP}^3)$ by holomorphic Legendrian immersions $M \to \mathbb{CP}^3$ (embeddings if M is an open Riemann surface and S has no holes in M).

Corollary (Superminimal surfaces in S^4)

Let K be a compact domain with piecewise $\mathscr{C}^1$ boundary in a Riemann surface M. Every conformal superminimal immersion $K \to S^4$ of class $\mathscr{C}^2(K, S^4)$ can be approximated in $\mathscr{C}^2(K, S^4)$ by complete superminimal immersions $M \to S^4$.

Comparison with the real analytic case

It is known that any smooth compact isotropic submanifold M in a real analytic contact manifold (X, ξ) can be approximated with a real analytic contact submanifold.

A proof can be found in the book **Cieliebak and Eliashberg, From Stein to Weinstein and Back, 2012**. Unfortunately, this proof does not generalize to the complex case. Among other things, it relies on Gray's stability theorem for real contact structures which does not hold (not even locally) for real analytic deformations of complex contact structures.

The space of k-dimensional distributions on $\mathbb{R}^n$ (i.e., vector subbundles of rank k of the tangent bundle $T\mathbb{R}^n$) has functional dimension $k(n-k)$, the dimension of the Grassmanian manifold of k-planes in $\mathbb{R}^n$. The symmetry group of diffeomorphisms is given by n functions, so the functional dimension of k-distributions on $\mathbb{R}^n$ up to diffeomorphisms is $k(n-k)-n$.

The only cases when this number is non-positive (and there is a unique local normal form) are $(k, n) \in \{(1, n), (n-1, n), (2, 4)\}$, corresponding to vector fields, real contact and even-contact structures, and the Engel case. Any other pair (k, n) produces distributions which have functional moduli and thus are not be stable.

Holomorphic differential equations

Assume that M is an open Riemann surface. Fix a holomorphic immersion $z : M \to \mathbb{C}$. Consider the ordinary differential equation

$$dw = V(p, w, t)dz, \quad w(p_0, t) = w_0, \tag{1}$$

where the independent variable is $p \in S \subset M$, the dependent variable w belongs to some disc $\triangle \subset \mathbb{C}$ around the origin, the differentials dz and dw are taken with respect to $p \in M$, $t = (t_1, \ldots, t_l)$ is a complex parameter in a ball $B \subset \mathbb{C}^l$ around the origin, and V is a function of class $\mathscr{C}^r$ on $S \times \triangle \times B \subset M \times \mathbb{C}^{l+1}$ for some $r \geq 1$ which is holomorphic in the interior.

On a neighbourhood of a point $p_0 \in S$, using the holomorphic immersion $z : M \to \mathbb{C}$ as a local coordinate near p_0 and setting $z_0 = z(p_0) \in \mathbb{C}$, this equation assumes the more familiar form

$$\frac{dw}{dz} = V(z, w, t), \quad w(z_0, t) = w_0.$$

Around any interior point of S, one may find a local solution in terms of the power series expansion

$$w(z, t) = w_0 + \sum_{k=1}^{\infty} c_k(z_0, w_0, t)(z - z_0)^k.$$

Holomorphic differential equations, 2

This method does not apply on the boundary of an admissible set S. Here is an alternative approach. Write the variables and the vector field in the form

$$z = x + iy, \quad w = w_1 + iw_2, \quad V = V_1 + iV_2$$

with real components. The above holomorphic ODE is equivalent to the CR system of two real partial differential equations for $w = (w_1, w_2)$:

$$\begin{aligned} \frac{\partial w_1}{\partial x} &= V_1, & \frac{\partial w_2}{\partial x} = V_2, \\ \frac{\partial w_1}{\partial y} &= -V_2, & \frac{\partial w_2}{\partial y} = V_1. \end{aligned}$$

A calculation shows that the vector fields

$$X = \frac{\partial}{\partial x} + V_1 \frac{\partial}{\partial w_1} + V_2 \frac{\partial}{\partial w_2}, \qquad Y = \frac{\partial}{\partial y} - V_2 \frac{\partial}{\partial w_1} + V_1 \frac{\partial}{\partial w_2}$$

commute when the function $V = V_1 + iV_2$ is holomorphic in (z, w), and by continuity this persists up to the boundary of S. Hence, the flow ϕ_x of X commutes with the flow ψ_y of Y on their domains of definition.
The local solution $w = w(z)$ of the initial value problem is then the composition of these two flows, projected onto the w-space:

$$w(z_0 + x + iy) = pr_w \circ \phi_x \circ \psi_y(z_0, w_0) = pr_w \circ \psi_y \circ \phi_x(z_0, w_0).$$

The Poincaré first return map

Assume that C is a closed piecewise smooth Jordan curve in S, and choose a parameterisation $h:[0,1]\to C$ with $h(0)=h(1)=p_0\in C$. In the parameter $s\in[0,1]$, our ODE with the initial condition $w(p_0,t)=w_0$ takes the form

$$\frac{d}{ds}w(h(s),t)=V(h(s),w(h(s),t),t)\frac{d}{ds}z(h(s)),\qquad w(h(0),t)=w_0.$$

Assume that the solution $w(s;w_0,t)$ with $w(0;w_0,t)=w_0$ exists for all $s\in[0,1]$. The number

$$\mathscr{P}_C(p_0,w_0,t)=w(1;w_0,t)-w(0;w_0,t)=w(1;w_0,t)-w_0$$

is called the **period** along C for the data (p_0,w_0,t).

A clear necessary condition for the existence of a single valued solution of the equation along the curve C for the data (p_0,w_0,t) is that

$$\mathscr{P}_C(p_0,w_0,t)=0.$$

Conversely, if this holds then the ODE has a single valued solution on an annulus around $C\subset M$ intersected with S.
Varying the initial value gives the **period map**, or the **Poincaré first return map** of the closed orbit $s\mapsto w(s;w_0,t)$:

$$\zeta\longmapsto\mathscr{P}_C(p_0,\zeta,t)\in\mathbb{C},\qquad \zeta\in\mathbb{C}\text{ near }w_0.$$

Outline of proof of the main theorem

1. We have $\xi = \ker \eta$ where η is a holomorphic contact form on X^{2n+1}. The immersion $f : S \to X$ extends to an immersion $F : S \times \mathbb{B}^{2n} \to X$ of class $\mathscr{A}^r$, and $\beta = F^*\eta$ is a contact form on $S \times \mathbb{B}^{2n}$ of class $\mathscr{A}^{r-1}$.

2. Next, we obtain a partial normal form of β along $S \times \{0\}^{2n}$, which amounts to a suitable change of the immersion F. The main point of the proof is a careful study of the **Poincaré first return map** of solutions of the differential equation for β-Legendrian curves along closed curves in S.

3. We begin by constructing a family of non-single valued solutions which is period dominating along each closed curve in a homology basis of S.

4. Next, approximate F in the $\mathscr{C}^r$ topology by a holomorphic immersion $\widetilde{F} : U \times \rho\mathbb{B}^{2n} \to X$, where $U \subset M$ is a neighbourhood of S and $0 < \rho < 1$, and consider the holomorphic contact form $\tilde{\beta} = \widetilde{F}^*\eta$.
   If the approximation is close enough, then the period domination implies that the differential equation for $\tilde{\beta}$-Legendrian curves has a single-valued solution on a neighbourhood of S. Its $\widetilde{F}$-image is an immersed holomorphic η-Legendrian curve $\tilde{f} : U \to X$ approximating $f : S \to X$.

Step 2 — partial normal form of β

Step 1 gives a contact form $\beta = F^*\eta$ on $S \times \mathbb{B}^{2n}$ of class $\mathscr{A}^r$, $r \geq 1$. Let $z : M \to \mathbb{C}$ be a holomorphic immersion. The following lemma give a partial normal form of β. It is essentially from **Alarcón & F., IMRN 2019**.

Lemma

There are fibre coordinates $(w, x_2, \ldots, x_n, y = y_1, y_2, \ldots, y_n)$ on $S \times \rho\mathbb{B}^{2n}$ and a nowhere vanishing function $h \in \mathscr{A}^r(S \times \rho\mathbb{B}^{2n})$ for some $0 < \rho < 1$ such that

$$\frac{1}{h}\beta = dw - ydz - \sum_{i=2}^{n} y_i dx_i + \widetilde{\alpha} = \alpha + \widetilde{\alpha} \quad \text{on } S \times \rho\mathbb{B}^{2n}, \tag{2}$$

where

$$\alpha = dw - ydz - \sum_{i=2}^{n} y_i dx_i \tag{3}$$

and the remainder $\widetilde{\alpha}$ contains terms which do not contribute to $\beta \wedge (d\beta)^n$.

The change of coordinates which brings β into this form is a shearlike transformation on $S \times \mathbb{C}^{2n}$ of class $\mathscr{A}^r(S \times \mathbb{C}^{2n})$ preserving the fibres $\{p\} \times \mathbb{C}^{2n}$ and keeping fixed the zero section $S \times \{0\}^{2n}$.

Step 3: Constructing a period dominating spray

We consider the special case when S is a compact connected domain in M and without the interpolation conditions.

Let $\mathcal{C} = \{C_1, \ldots, C_l\}$ be a homology basis of S consisting of smooth oriented Jordan curves with base $p_0 \in \mathring{S}$, such that $C = \bigcup_{i=1}^{l} C_i$ is $\mathscr{O}(S)$-convex and each curve $C_i \in \mathcal{C}$ contains a nontrivial arc I_i disjoint from $\bigcup_{j\neq i} C_j$.

We construct a holomorphic spray of functions

$$y(p,t) = \sum_{i=1}^{l} t_i g_i(p), \qquad p \in S,\ t = (t_1, \ldots, t_l) \in \mathbb{C}^l,$$

where $g_i \in \mathscr{O}(S)$ are holomorphic functions satisfying

$$\int_{C_i} g_j dz = \delta_{i,j}, \qquad i, j = 1, \ldots, l.$$

Inserting the values

$$y = y(p,t), \quad x_2 = \cdots = x_n = y_2 = \ldots = y_n = 0$$

into the 1-form $\alpha = dw - ydz - \sum_{i=2}^{n} y_i dx_i$ gives the equation

$$dw = y(p,t)dz, \qquad p \in S$$

whose local solutions $w = w(p,t)$ are α-Legendrian curves.

Step 3: Constructing a period dominating spray, 2

Since the variable w does not appear on the right hand side, solutions are obtained by integration:

$$w(p,t) = w_0 + \int_{p_0}^{p} y(\cdot\,,t)dz = w_0 + \sum_{i=1}^{l} t_i \int_{p_0}^{p} g_i dz, \qquad p \in S.$$

Any solution satisfying the initial condition $w(p_0,t) = 0$ also satisfies

$$|w(p,t)| = O(|t|), \qquad p \in S,$$

if we integrate along an approximately geodesic curve in S from p_0 to p.

By the construction, the period map $\mathbb{C}^l \ni t \mapsto \mathscr{P}^\alpha(t) \in \mathbb{C}^l$ associated to the 1-form α, the homology basis $\mathcal{C} = \{C_1, \ldots, C_l\}$ of S, and the given spray $y(p,t)$ is the identity map

$$\mathscr{P}^\alpha(t) = \left(\mathscr{P}^\alpha_{C_1}(t), \ldots, \mathscr{P}^\alpha_{C_l}(t)\right) = t.$$

In particular, the only single-valued α-Legendrian curve in this family satisfying the initial condition $w(p_0,t) = 0$ is $w = 0$ (at the parameter value $t = 0$).

Step 3: Constructing a period dominating spray, 3

Inserting the same values

$$y = y(p,t), \quad x_2 = \cdots = x_n = y_2 = \ldots = y_n = 0$$

into the 1-form

$$\frac{1}{h}\beta = \alpha + \widetilde{\alpha},$$

the only nonvanishing terms in $\widetilde{\alpha}$ are those of the form wdy and ydy, possibly multiplied by other normal coordinates and by functions in $\mathscr{A}^r(S)$.

Since y, dy, w are of size $O(t)$, these terms disturb the period map by a term of size $O(|t|^2)$. Hence, the period map of the β-Legendrian curves satisfying the initial condition $w(p_0,t) = 0$ equals

$$\mathscr{P}^\beta(t) = t + O(|t|^2).$$

For every small $\delta > 0$ the map $\mathscr{P}^\beta(t)$ is close enough to t on the polydisc

$$P_\delta = \{(t_1,\ldots,t_l) : |t_i| \le \delta, \; i = 1,\ldots,l\} = \delta\mathbb{D}^l \subset \mathbb{C}^l$$

that it maps bP_δ to $\mathbb{C}^l_*$ and this map has degree one.

Step 4: Holomorphic approximation

Suppose now that $\widetilde{F} : U \times \rho\mathbb{B}^{2n} \to X$ is a holomorphic immersion approximating F in $\mathscr{C}^{r+1}(S \times \rho\mathbb{B}^{2n}, X)$. Such exists by the Mergelyan approximation theorem for maps of class $\mathscr{C}^{r+1}$ from admissible sets in Riemann surfaces into any complex manifold (**Fornæss, F, Wold, Chapter 5 in Advancements in Complex Analysis by Springer-Verlag**).

Then, the pullback

$$\tilde{\beta} := \widetilde{F}^* \eta$$

is holomorphic contact form on $U \times \rho\mathbb{B}^{2n}$ which is $\mathscr{C}^r$-close to β on $S \times \rho\mathbb{B}^{2n}$.

Furthermore, the coefficient $\tilde{h} \in \mathscr{O}(U \times \rho\mathbb{B}^{2n})$ of the differential dw in $\tilde{\beta}$ is close to the corresponding coefficient h of β on $S \times \rho\mathbb{B}^{2n}$ and hence is nonvanishing, perhaps after shrinking $U \supset S$ and decreasing $\rho > 0$ slightly.

We conclude that the holomorphic contact form $\tilde{h}^{-1}\tilde{\beta}$ on $U \times \rho\mathbb{B}^{2n}$ is $\mathscr{C}^r$ close to the form $h^{-1}\beta$ on $S \times \rho\mathbb{B}^{2n}$.

Step 4: Conclusion

We now insert the values

$$y = y(p,t), \quad x_2 = \cdots = x_n = y_2 = \ldots = y_n = 0$$

into $\tilde{h}^{-1}\tilde{\beta}$ and denote by

$$t \mapsto \mathscr{P}^{\tilde{\beta}}(t)$$

the corresponding period map for solutions satisfying $w(p_0,t) = 0$.

Assuming that the approximations are close enough, $\mathscr{P}^{\tilde{\beta}}(t)$ is so close to $\mathscr{P}^{\beta}(t) = t + O(|t|^2)$ on $P_\delta \subset \mathbb{C}^l$ that it maps bP_δ to $\mathbb{C}^l_*$ and this map has degree one. Hence, there is a point $t^0 \in \mathring{P}_\delta$ such that

$$\mathscr{P}^{\tilde{\beta}}(t^0) = 0.$$

For $t = t^0$ we thus obtain a single-valued solution of the differential equation for $\tilde{\beta}$-Legendrian curves satisfying the initial condition $w(p_0, t^0) = 0$.

Its image by $\widetilde{F}$ is a holomorphic $\tilde{\xi}$-Legendrian immersion $\tilde{f} : U \to X$ which approximates the Legendrian immersion $f : S \to X$ in $\mathscr{C}^{r+1}(S, X)$.

THANK YOU

FOR YOUR ATTENTION