Hyperbolic domains in real Euclidean spaces

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Abstract

Let Ω be a domain in $\mathbb{R}^n$, $n \geq 3$. We introduce a Kobayashi-type Finsler pseudometric $g_\Omega : T\Omega = \Omega \times \mathbb{R}^n \to \mathbb{R}_+$ defined in terms of conformal harmonic discs. Such discs parameterize minimal surfaces in $\mathbb{R}^n$.

The integrated form of this pseudometric is the **minimal pseudodistance** $\rho_\Omega : \Omega \times \Omega \to \mathbb{R}_+$, which is also defined by chains of conformal harmonic discs.

On the unit ball $\mathbb{B}^n$, $g_{\mathbb{B}^n}$ coincides with the **Cayley–Klein metric**, one of the classical models of hyperbolic geometry.

We obtain several sufficient conditions for a domain Ω to be (complete) hyperbolic, meaning that g_Ω is a (complete) metric; equivalently, ρ_Ω is a (complete) distance function. In particular, we show that a convex domain is complete hyperbolic iff it does not contain any affine 2-plane.

F. F. & David Kalaj, Hyperbolicity theory for conformal minimal surfaces in $\mathbb{R}^n$. `https://arxiv.org/abs/2102.12403`**, March 2021**

Barbara Drinovec Drnovšek and F. F., Hyperbolic domains in real Euclidean spaces. `https://arxiv.org/abs/2109.06943`**, Sept 2021**

The minimal pseudodistance on a domain in $\mathbb{R}^n$

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disc, and let Ω be a domain in $\mathbb{R}^n$.

Denote by $\mathrm{CH}(\mathbb{D}, \Omega)$ the space of all harmonic discs $f : \mathbb{D} \to \Omega$ (every component of f is a harmonic function) which are conformal:

$$f_x \cdot f_y = 0, \qquad |f_x| = |f_y|; \qquad z = x + \imath y \in \mathbb{D}.$$

Fix a pair of points $\mathbf{x}, \mathbf{y} \in \Omega$ and consider finite chains of conformal harmonic discs $f_i \in \mathrm{CH}(\mathbb{D}, \Omega)$ and points $a_i \in \mathbb{D}$ $(i = 1, \ldots, k)$ such that

$$f_1(0) = \mathbf{x}, \quad f_{i+1}(0) = f_i(a_i) \text{ for } i = 1, \ldots, k-1, \quad f_k(a_k) = \mathbf{y}.$$

To any such chain we associate the number

$$\sum_{i=1}^{k} \frac{1}{2} \log \frac{1 + |a_i|}{1 - |a_i|} \geq 0.$$

The pseudodistance $\rho_\Omega : \Omega \times \Omega \to \mathbb{R}_+$ is the infimum of the numbers obtained in this way. Clearly it satisfies the triangle inequality.

If $\Omega \subset \mathbb{C}^n$ and we use only holomorphic discs, we get the Kobayashi pseudodistance $\mathcal{K}_\Omega$. Hence, $\rho_\Omega \leq \mathcal{K}_\Omega$. These pseudodistances agree on domains in $\mathbb{R}^2 = \mathbb{C}$, but strict inequality holds if $n > 1$.

The minimal pseudometric

Define a Finsler pseudometric $g_\Omega : \Omega \times \mathbb{R}^n \to \mathbb{R}_+$ on $(\mathbf{x}, \mathbf{u}) \in \Omega \times \mathbb{R}^n$ by

$$g_\Omega(\mathbf{x}, \mathbf{v}) = \inf\big\{1/r > 0 : \exists f \in \mathrm{CH}(\mathbb{D}, \Omega),\ f(0) = \mathbf{x},\ f_x(0) = r\mathbf{v}\big\}.$$

Clearly, g_Ω is upper-semicontinuous and absolutely homogeneous:

$$g_\Omega(\mathbf{x}, t\mathbf{v}) = |t|\, g_\Omega(\mathbf{x}, \mathbf{v}) \quad \text{for} \quad t \in \mathbb{R}.$$

If $\Omega \subset \mathbb{C}^n$ and using only holomorphic disc gives the Kobayashi pseudometric.

Theorem

The minimal pseudodistance ρ_Ω is obtained by integrating the pseudometric g_Ω:

$$\rho_\Omega(\mathbf{x}, \mathbf{y}) = \inf_\gamma \int_0^1 g_\Omega(\gamma(t), \dot\gamma(t))\, dt, \qquad \mathbf{x}, \mathbf{y} \in \Omega,$$

where the infimum is over all piecewise smooth paths $\gamma : [0,1] \to \Omega$ with $\gamma(0) = \mathbf{x}$ and $\gamma(1) = \mathbf{y}$.

The proof is similar to the one for the Kobayashi pseudometric.

Metric decreasing properties

A conformal (or Riemann) surface M is **hyperbolic** if its universal covering space is the disc $\mathbb{D}$. Such a surface carries the **Poincaré metric**, $\mathcal{P}_M$, the unique Riemannian metric such that any conformal covering map $h : \mathbb{D} \to M$ is an isometry from $(\mathbb{D}, \mathcal{P}_{\mathbb{D}})$ onto $(M, \mathcal{P}_M)$. The Poincaré metric on $\mathbb{D}$ is

$$\mathcal{P}_{\mathbb{D}}(z, \xi) = \frac{|\xi|}{1 - |z|^2}, \qquad z \in \mathbb{D},\ \xi \in \mathbb{C}.$$

It follows from the definition of g_Ω that for any conformal harmonic map $f : \mathbb{D} \to \Omega$ we have that

$$g_\Omega(f(z), df_z(\xi)) \le \mathcal{P}_{\mathbb{D}}(z, \xi), \qquad z \in \mathbb{D},\ \xi \in \mathbb{C},$$

and g_Ω is the largest pseudometric on Ω with this property.

The same holds for conformal harmonic maps $(M, \mathcal{P}_M) \to (\Omega, g_\Omega)$.

Any rigid map $R : \mathbb{R}^n \to \mathbb{R}^m$ $(n \le m)$ with $R(\Omega) \subset \Omega'$ is metric-decreasing:

$$g_{\Omega'}(R(\mathbf{x}), R(\mathbf{v})) \le g_\Omega(\mathbf{x}, \mathbf{v}), \qquad \mathbf{x} \in \Omega,\ \mathbf{v} \in \mathbb{R}^n.$$

A Finsler pseudometric on the Grassmanian of 2-planes

We also introduce a Finsler pseudometric on $\Omega \times \mathbb{G}_2(\mathbb{R}^n)$, where $\mathbb{G}_2(\mathbb{R}^n)$ denotes the Grassmann manifold of 2-planes in $\mathbb{R}^n$, by

$$\mathcal{M}_\Omega(\mathbf{x}, \Lambda) = \inf\left\{1/\|df_0\| : f \in \mathrm{CH}(\mathbb{D}, \Omega),\ f(0) = \mathbf{x},\ df_0(\mathbb{R}^2) = \Lambda\right\}.$$

Here, $\|df_0\|$ denotes the operator norm of the differential $df_0 : \mathbb{R}^2 \to \mathbb{R}^n$.

It clearly follows that for any vector $\mathbf{v} \in \mathbb{R}^n$, $|\mathbf{v}| = 1$ we have

$$g_\Omega(\mathbf{x}, \mathbf{v}) = \inf\left\{\mathcal{M}_\Omega(\mathbf{x}, \Lambda) : \Lambda \in \mathbb{G}_2(\mathbb{R}^n), \mathbf{v} \in \Lambda\right\}.$$

Note that the set of 2-planes $\Lambda \subset \mathbb{R}^n$ containing $\mathbf{v}$ is parameterized by the unit sphere S^{n-2} in the normal space $\mathbf{v}^\perp \cong \mathbb{R}^{n-1}$.

This illuminates the main difference between the minimal metric and the Kobayashi metric on a domain in $\mathbb{C}^n$: a given nonzero vector $\mathbf{v} \in \mathbb{C}^n$ determines a unique complex line $\mathbb{C}\mathbf{v}$, and for the Kobayashi metric we only consider complex discs tangent to that line.

The Cayley–Klein metric on the ball

Theorem (F.–Kalaj 2021)

The minimal metric $g^2_{\mathbb{B}^n}$ on the unit ball $\mathbb{B}^n$ equals the **Cayley–Klein metric***:*

$$\mathcal{CK}(\mathbf{x},\mathbf{v})^2 = \frac{(1-|\mathbf{x}|^2)|\mathbf{v}|^2 + |\mathbf{x}\cdot\mathbf{v}|^2}{(1-|\mathbf{x}|^2)^2} = \frac{|\mathbf{v}|^2}{1-|\mathbf{x}|^2} + \frac{|\mathbf{x}\cdot\mathbf{v}|^2}{(1-|\mathbf{x}|^2)^2}.$$

We also have

$$\mathcal{CK}(\mathbf{x},\mathbf{v}) = \frac{\sqrt{1-|\mathbf{x}|^2\sin^2\phi}}{1-|\mathbf{x}|^2}\,|\mathbf{v}|, \qquad \mathbf{x}\in\mathbb{B}^n,\ \mathbf{v}\in\mathbb{R}^n,$$

where $\phi\in[0,\pi/2]$ is the angle between the vector $\mathbf{v}$ and the line $\mathbb{R}\mathbf{x}\subset\mathbb{R}^n$.

The **Beltrami–Cayley–Klein model of hyperbolic geometry** was introduced by **Arthur Cayley (1859)**, **Eugenio Beltrami (1868)**, and **Felix Klein (1871–73)**. The underlying space is the unit ball, geodesics are straight line segments with endpoints on the boundary sphere, and the distance between points on a geodesic is given by cross ratio. This metric is the restriction of the Kobayashi metric on the complex ball $\mathbb{B}^n_{\mathbb{C}}\subset\mathbb{C}^n$ to points in $\mathbb{B}^n=\mathbb{B}^n_{\mathbb{C}}\cap\mathbb{R}^n$ and vectors in $\mathbb{R}^n$. It is a special case of the metric on convex domains in $\mathbb{R}^n$ introduced by **David Hilbert** in 1895.

This theorem is a corollary to the following Schwarz–Pick lemma.

Theorem (F.–Kalaj 2021)

Let $f : \mathbb{D} \to \mathbb{B}^n$ is a harmonic map for some $n \geq 2$ which is conformal at a point $z \in \mathbb{D}$. Denote by $\theta \in [0, \pi/2]$ the angle between the vector $f(z)$ and the plane $df_z(\mathbb{R}^2)$. Then at this point we have that

$$(*) \qquad \|df_z\| \leq \frac{1-|f(z)|^2}{1-|z|^2} \frac{1}{\sqrt{1-|f(z)|^2 \sin^2 \theta}}.$$

Equality holds if and only if f is a conformal diffeomorphism onto the affine disc

$$\Sigma = (f(z) + df_z(\mathbb{R}^2)) \cap \mathbb{B}^n.$$

The number $R = \sqrt{1-|f(z)|^2 \sin^2 \theta}$ is the radius of the affine disc Σ.

The Schwarz–Pick lemma implies the theorem

Let us see how this Schwarz–Pick lemma implies the theorem. Take $z = 0$ and

$$\mathbf{x} = f(0) \in \mathbb{B}^n, \quad f_x(0) = r\mathbf{v} \in \mathbb{R}^n, \quad df_0(\mathbb{R}^2) = \Lambda,$$

where $\mathbf{v}$ is a unit vector contained in Λ. Let θ denote the angle between $\mathbf{x}$ and Λ. The inequality (*) is equivalent to

$$\frac{\sqrt{1-|\mathbf{x}|^2\sin^2\theta}}{1-|\mathbf{x}|^2} \le \frac{1}{\|df_0\|}.$$

The infimum of the right over all discs f with the given data equals $\mathcal{M}_{\mathbb{B}^n}(\mathbf{x},\Lambda)$, so we obtain

$$\mathcal{M}_{\mathbb{B}^n}(\mathbf{x},\Lambda) = \frac{\sqrt{1-|\mathbf{x}|^2\sin^2\theta}}{1-|\mathbf{x}|^2}.$$

Note that $0 \le \theta \le \phi \le \pi/2$ where ϕ is the angle between $\mathbf{x}$ and the line $\mathbb{R}\mathbf{v} \subset \Lambda$. Taking the infimum over all Λ containing $\mathbf{v}$ gives

$$g_{\mathbb{B}^n}(\mathbf{x},\mathbf{v}) = \frac{\sqrt{1-|\mathbf{x}|^2\sin^2\phi}}{1-|\mathbf{x}|^2} = \mathcal{CK}(\mathbf{x},\mathbf{v}).$$

Schwarz–Pick Lemma for harmonic self-map maps of the disc which are conformal at a point

If $n = 2$ then $\theta = 0$, $R = 1$, so the theorem implies the following corollary generalizing the classical Schwarz–Pick lemma for holomorphic maps $\mathbb{D} \to \mathbb{D}$ due to **Karl Hermann Amandus Schwarz (1869)**, **Henri Poincaré (1884)**, and **Georg Alexander Pick (1915)**.

Corollary

Let $f : \mathbb{D} \to \mathbb{D}$ be a harmonic map. If f is conformal at a point $z \in \mathbb{D}$, then at this point we have that

$$|f'(z)| = \|df_z\| \leq \frac{1 - |f(z)|^2}{1 - |z|^2},$$

with equality if and only if f is a conformal diffeomorphism of the disc $\mathbb{D}$.

Precomposing f by an automorphism of $\mathbb{D}$ we may assume that $z = 0$. On the other hand, postcompositions of harmonic maps $\mathbb{D} \to \mathbb{D}$ by automorphism of $\mathbb{D}$ are not harmonic in general, so we cannot interchange $f(0)$ and 0. Also, $f(z)/z$ need not be harmonic. Hence, the standard proof of the classical Schwarz–Pick lemma breaks down. The estimate fails for some nonconformal harmonic diffeomorphisms of $\mathbb{D}$.

Proof of the Schwarz–Pick lemma, 1

Precomposing by an automorphism of $\mathbb{D}$, we may assume that $z = 0$.

Fix a point $\mathbf{x} \in \mathbb{B}^n$ and a 2-plane $0 \in \Lambda \subset \mathbb{R}^n$, and consider the affine disc $\Sigma = (\mathbf{x} + \Lambda) \cap \mathbb{B}^n$. Let $\mathbf{p} \in \Sigma$ be the closest point to the origin.

If $n = 2$ then $\mathbf{p} = 0$ and $\Sigma = \mathbb{D}$. Suppose now that $n = 3$; the case $n > 3$ will be the same. By an orthogonal rotation on $\mathbb{R}^3$ we may assume that

$$\mathbf{p} = (0, 0, p) \quad \text{and} \quad \Sigma = \left\{ (x, y, p) : x^2 + y^2 < 1 - p^2 \right\}.$$

Let $\mathbf{x} = (b_1, b_2, p) \in \Sigma$, and let θ denote the angle between $\mathbb{R}\mathbf{x}$ and Σ. Set

$$R = \sqrt{1 - p^2} = \sqrt{1 - |\mathbf{x}|^2 \sin^2 \theta}, \qquad a = \frac{b_1 + \mathrm{i} b_2}{R} \in \mathbb{D}, \qquad |a| = \frac{|\mathbf{x}| \cos \theta}{R}.$$

Proof, 2

The map $f : \mathbb{D} \to \Sigma$ given by

$$f(z) = \left(R \cdot \Re \frac{z+a}{1+\bar{a}z}, R \cdot \Im \frac{z+a}{1+\bar{a}z}, p \right) = \left(R \, \frac{z+a}{1+\bar{a}z}, p \right)$$

is a conformal parameterization of Σ with $f(0) = \mathbf{x}$. We have that

$$\begin{aligned} \|df_0\| &= R\,(1-|a|^2) = \frac{R^2 - R^2|a|^2}{R} = \frac{1 - |\mathbf{x}|^2 \sin^2\theta - |\mathbf{x}|^2 \cos^2\theta}{R} \\ &= \frac{1-|\mathbf{x}|^2}{\sqrt{1-|\mathbf{x}|^2 \sin^2\theta}}. \end{aligned}$$

This shows that the conformal parameterizations of the proper affine discs in the ball satisfy the equality in the theorem at every point.

Proof, 3

Suppose that $g : \mathbb{D} \to \mathbb{B}^3$ is a harmonic map such that

$$g(0) = f(0) = \mathbf{x}, \quad g \text{ is conformal at } 0, \text{ and } dg_0(\mathbb{R}^2) = df_0(\mathbb{R}^2).$$

Up to replacing g by $g(e^{\mathrm{i}t}z)$ or $g(e^{\mathrm{i}t}\bar{z})$ for some $t \in \mathbb{R}$, we may assume that

$$dg_0 = r\, df_0 \quad \text{for some } r > 0.$$

We must prove that $r \leq 1$, and that $r = 1$ if and only if $g = f$.

Consider the holomorphic map $F : \mathbb{D} \to \Omega = \mathbb{B}^3 \times \mathrm{i}\mathbb{R}^3$ with $f = \Re F$, given by

$$F(z) = \left(R \cdot \frac{z+a}{1+\bar{a}z}, -R \cdot \mathrm{i}\frac{z+a}{1+\bar{a}z}, p \right), \quad z \in \mathbb{D}.$$

Let $G : \mathbb{D} \to \Omega$ be the holomorphic map with $\Re G = g$ and $G(0) = F(0)$.

By the Cauchy–Riemann equations, the condition $dg_0 = r\, df_0$ implies

$$G'(0) = r\, F'(0).$$

Proof, 4

It follows that the map $(F(z) - G(z))/z$ is holomorphic on $\mathbb{D}$ and

$$\lim_{z \to 0} \frac{F(z) - G(z)}{z} = F'(0) - G'(0) = (1 - r)F'(0).$$

Since $g : \mathbb{D} \to \mathbb{B}^3$ is a bounded harmonic map, it has a nontangential boundary value at almost every point of the circle $\mathbb{T} = b\mathbb{D}$. Since the Hilbert transform is an isometry on the Hilbert space $L^2(\mathbb{T})$, the same is true for G.

Denote by $\langle \cdot, \cdot \rangle$ the complex bilinear form on $\mathbb{C}^n$ given by

$$\langle \mathbf{z}, \mathbf{w} \rangle = \sum_{i=1}^{n} z_i w_i, \qquad \mathbf{z}, \mathbf{w} \in \mathbb{C}^n.$$

For each $z \in b\mathbb{D}$ the vector $f(z) \in b\mathbb{B}^3$ is the unit normal vector to the sphere $b\mathbb{B}^3$ at the point $f(z)$. Since $\mathbb{B}^3$ is strongly convex, we have that

$$\Re \left\langle F(z) - G(z), f(z) \right\rangle = \left\langle f(z) - g(z), f(z) \right\rangle \geq 0 \quad a.e.\ z \in b\mathbb{D},$$

and the value is positive for almost every $z \in b\mathbb{D}$ if and only if $g \neq f$.

Consider the following function on the circle $b\mathbb{D}$:

$$\tilde{f}(z) = z|1 + \bar{a}z|^2 f(z), \qquad |z| = 1.$$

A calculation, taking into account $z\bar{z} = 1$, shows that

$$\tilde{f}(z) = \begin{pmatrix} \frac{c}{2}\left(1 + a^2 + 4(\Re a)z + (1 + \bar{a}^2)z^2\right) \\ \frac{c}{2}\left(\imath(1 - a^2) + 4(\Im a)z + \imath(\bar{a}^2 - 1)z^2\right) \\ p\,(z + a)(1 + \bar{a}z) \end{pmatrix}, \qquad |z| = 1.$$

We extend $\tilde{f}$ to all $z \in \mathbb{C}$ by letting it equal the holomorphic polynomial map on the right hand side above. Since $|1 + \bar{a}z|^2 > 0$ for $z \in \overline{\mathbb{D}}$, we have

$$\begin{aligned} h(z) \quad &:= \quad \Re\left\langle F(z) - G(z), |1 + \bar{a}z|^2 f(z)\right\rangle \\ &= \quad \left\langle f(z) - g(z), |1 + \bar{a}z|^2 f(z)\right\rangle \geq 0 \quad a.e.\ z \in b\mathbb{D}, \end{aligned}$$

and $h > 0$ almost everywhere on $b\mathbb{D}$ if and only if $g \neq f$.

Conclusion of the proof

From the definition of $\tilde{f}$ we see that

$$h(z) = \Re \left\langle \frac{F(z) - G(z)}{z}, \tilde{f}(z) \right\rangle \quad a.e.\ z \in b\mathbb{D}$$

Since the maps $(F(z) - G(z))/z$ and $\tilde{f}(z)$ are holomorphic on $\mathbb{D}$, h extends to a nonnegative harmonic function on $\mathbb{D}$ which is positive on $\mathbb{D}$ unless $f = g$.

At $z = 0$ we have

$$h(0) = \Re \langle F'(0) - G'(0), \tilde{f}(0) \rangle = (1 - r) \Re \langle F'(0), \tilde{f}(0) \rangle \geq 0,$$

with equality if and only if $g = f$.

Applying this to the constant map $g(z) \equiv f(0)$ gives

$$\Re \langle F'(0), \tilde{f}(0) \rangle > 0.$$

Hence $r \leq 1$, with equality if and only if $g = f$. This completes the proof.

Hyperbolic domains

Definition

Let Ω be a domain in $\mathbb{R}^n$, $n \geq 3$.

- (a) Ω is *hyperbolic* if the pseudodistance ρ_Ω is a distance function, and is *complete hyperbolic* if (Ω, ρ_Ω) is a complete metric space.
- (b) Ω is hyperbolic at a point $\mathbf{p} \in \Omega$ if there are a neighbourhood $U \subset \Omega$ of $\mathbf{p}$ and a constant $c > 0$ such that

$$g_\Omega(\mathbf{x}, \mathbf{u}) \geq c|\mathbf{u}|, \qquad \mathbf{x} \in U, \ \mathbf{u} \in \mathbb{R}^n.$$

Example

(A) The ball $\mathbb{B}^n \subset \mathbb{R}^n$, $n \geq 3$, is complete hyperbolic since the Cayley–Klein metric is complete.

(B) Every bounded domain $\Omega \subset \mathbb{R}^n$ is hyperbolic since it is contained in a ball. However, it need not be complete hyperbolic.

(C) Every bounded strongly convex domain in $\mathbb{R}^n$ is complete hyperbolic.

(D) The half-space $\mathbb{H}^n = \{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$ is not hyperbolic since the pseudodistance $\rho_{\mathbb{H}^n}$ vanishes on planes $x_n = const$.

By following the proofs for the Kobayashi metric, one obtains the following.

Theorem

The following conditions are equivalent for a domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$.

1. *The family $\mathrm{CH}(\mathbb{D}, \Omega)$ of conformal harmonic discs $\mathbb{D} \to \Omega$ is pointwise equicontinuous for some metric ρ on Ω inducing its natural topology.*
2. *Ω is hyperbolic at every point.*
3. *Ω is hyperbolic.*
4. *The minimal distance ρ_Ω induces the topology of Ω.*

A domain $\Omega \subset \mathbb{R}^n$ is called **taut** if $\mathrm{CH}(\mathbb{D}, \Omega)$ is a normal family.

Theorem

The following hold for any domain Ω in $\mathbb{R}^n$, $n \geq 3$:

1. *If Ω is complete hyperbolic, then it is taut.*
2. *If Ω is taut, then it is hyperbolic.*

$$\text{complete hyperbolic} \implies \text{taut} \implies \text{hyperbolic}$$

Strongly minimally convex domains are complete hyperbolic

A domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$ with smooth boundary is said to be **strongly minimally convex** if at every point $\mathbf{p} \in b\Omega$ the principal normal curvatures $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_{n-1}$ of $b\Omega$ satisfy

$$\nu_1 + \nu_2 > 0.$$

Theorem (B. Drinovec Drnovšek & F. F.)

Every bounded strongly minimally convex domain is complete hyperbolic. If $\nu_1 + \nu_2 < 0$ at some $\mathbf{p} \in b\Omega$ then $\mathbf{p}$ is at finite distance from the interior.

This can be seen as an analogue of Graham's theorem that bounded strongly pseudoconvex domains in $\mathbb{C}^n$ are complete Kobayashi hyperbolic.

For bounded strongly convex domains this follows easily from the comparison principle with the minimal metric on the ball.

The proof for non-convex domains is considerably more involved. It uses the existence of a strongly minimally plurisubharmonic defining function and an analogue of the Sibony metric in this category.

Let Ω be a domain in $\mathbb{R}^n$. An upper-semicontinuous function $u : \Omega \to [-\infty, +\infty)$ is said to be **minimal plurisubharmonic, MPSH**, if for every affine 2-plane $L \subset \mathbb{R}^n$ the restriction $u : L \cap \Omega \to [-\infty, +\infty)$ is subharmonic (in any conformal affine coordinates on L). This class of functions was studied by Harvey and Lawson in a series of papers.

Note that every MPSH function on a domain $\Omega \subset \mathbb{R}^{2n} = \mathbb{C}^n$ is also plurisubharmonic in the usual sense.

A function $u \in \mathscr{C}^2(\Omega)$ is MPSH if and only if

$$\mathrm{tr}_\Lambda \mathrm{Hess}_u(\mathbf{x}) \geq 0 \;\; \text{holds for every} \;\; (\mathbf{x}, \Lambda) \in \Omega \times \mathbb{G}_2(\mathbb{R}^n),$$

and this holds if and only if

$$(*) \qquad \lambda_1(\mathbf{x}) + \lambda_2(\mathbf{x}) \geq 0 \;\; \text{for all } \mathbf{x} \in \Omega,$$

where $\lambda_1(\mathbf{x}), \lambda_2(\mathbf{x})$ denote the smallest eigenvalues of the Hessian $\mathrm{Hess}_u(\mathbf{x})$.

We say that $u \in \mathscr{C}^2(\Omega)$ is **strongly minimally plurisubharmonic** if strong inequality holds in (*). Every bounded strongly minimally convex domain in $\mathbb{R}^n$ admits a strongly MPSH defining function.

Here is the key property of MPSH functions pertaining to minimal surfaces.

Proposition

An upper-semicontinuous function $u : \Omega \to [-\infty, +\infty)$ is MPSH if and only if for each conformal harmonic map $f : M \to \Omega$ from a conformal surface the function $u \circ f : M \to \mathbb{R}$ is subharmonic. If $u \in \mathscr{C}^2(\Omega)$ is strongly MPSH and f is an immersion, then $u \circ f$ is strongly subharmonic on M.

For functions $u \in \mathscr{C}^2(\Omega)$ this follows from the following formula, which holds for every conformal harmonic map $f : \mathbb{D} \to \Omega$:

$$\Delta(u \circ f)(z) = \mathrm{tr}_{df_z(\mathbb{R}^2)} \mathrm{Hess}_u(f(z)) \cdot \|df_z\|^2, \quad z \in \mathbb{D}.$$

Lemma

Let $\mathbf{x}$ be the Euclidean coordinate on $\mathbb{R}^n$, $n \geq 3$.

(a) *The function $\log|\mathbf{x}|$ is MPSH on $\mathbb{R}^n$.*

(b) *If u is MPSH on $\Omega \subset \mathbb{R}^n$ then for any $\mathbf{p} \in \Omega$ the function $\mathbf{x} \mapsto |\mathbf{x} - \mathbf{p}|^2 e^{u(\mathbf{x})}$ and its logarithm are MPSH on Ω.*

A pseudometric defined by MPSH functions

We define the pseudometric $\mathcal{F}_\Omega : \Omega \times \mathbb{G}_2(\mathbb{R}^n) \to \mathbb{R}_+$ by

$$\mathcal{F}_\Omega(\mathbf{x}, \Lambda) = \frac{1}{2} \sup_u \sqrt{\mathrm{tr}_\Lambda \mathrm{Hess}_u(\mathbf{x})}, \qquad \mathbf{x} \in \Omega,\ \ \Lambda \in \mathbb{G}_2(\mathbb{R}^n),$$

where the supremum is over all MPSH functions $u : \Omega \to [0, 1]$ which are of class $\mathscr{C}^2$ near $\mathbf{x}$ such that $u(\mathbf{x}) = 0$ and $\log u$ is MPSH on Ω. (If there are no such functions, we take $\mathcal{F}_\Omega(\mathbf{x}, \Lambda) = 0$.)

$\mathcal{F}_\Omega$ is an analogue of the **Sibony metric** (1981), the latter being defined in terms of the usual log-psh functions on domains in $\mathbb{C}^n$.

The main point is that $\mathcal{F}_\Omega$ gives a lower bound for the minimal pseudometric:

Proposition

For any domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, we have that

$$\mathcal{F}_\Omega(\mathbf{x}, \Lambda) \leq \mathcal{M}_\Omega(\mathbf{x}, \Lambda) \quad \text{for all } (\mathbf{x}, \Lambda) \in \Omega \times \mathbb{G}_2(\mathbb{R}^n).$$

If for any point $\mathbf{p} \in \Omega$ there is a neighbourhood $\mathbf{p} \in U \subset \Omega$ such that $\mathcal{F}_\Omega(\mathbf{x}, \Lambda) \geq c > 0$ for every $\mathbf{x} \in U$ and $\Lambda \in \mathbb{G}_2(\mathbb{R}^n)$, then Ω is hyperbolic.

Proof of the proposition

Fix $(\mathbf{x}, \Lambda) \in \Omega \times \mathbb{G}_2(\mathbb{R}^n)$. Let $f \in \mathrm{CH}(\mathbb{D}, \Omega)$ be such that $f(0) = \mathbf{x}$ and $df_0(\mathbb{R}^2) = \Lambda$. Let $u : \Omega \to [0,1]$ be as in the definition of $\mathcal{F}_\Omega$. The function $v = u \circ f : \mathbb{D} \to [0,1]$ is then subharmonic, of class $\mathscr{C}^2$ near the origin, $v(0) = 0$, and $\log v = \log u \circ f : \mathbb{D} \to [-\infty, 0)$ is also subharmonic.

By Sibony (1981) we have that

$$\Delta v(0) \le 4.$$

(The unique extremal function with $\Delta v(0) = 4$ is $v(x + \imath y) = x^2 + y^2$.) Hence,

$$\mathrm{tr}_\Lambda \mathrm{Hess}_u(\mathbf{x}) \cdot \|df_0\|^2 = \Delta v(0) \le 4.$$

Equivalently,

$$\frac{1}{2}\sqrt{\mathrm{tr}_\Lambda \mathrm{Hess}_u(\mathbf{x})} \le \frac{1}{\|df_0\|}.$$

The supremum of the left hand side over all admissible functions u equals $\mathcal{F}_\Omega(\mathbf{x}, \Lambda)$, while the infimum of the right hand side over all conformal harmonic discs f as above equals $\mathcal{M}_\Omega(\mathbf{x}, \Lambda)$. Hence, $\mathcal{F}_\Omega \le \mathcal{M}_\Omega$.

Sketch of proof of the theorem on complete hyperbolicity

We use the above proposition with MPSH function

$$\Psi(\mathbf{y}) = \theta\left(r^{-2}|\mathbf{y}-\mathbf{x}|^2\right) e^{\lambda u(\mathbf{y})}, \qquad \mathbf{y} \in \Omega,$$

where $\theta : [0,\infty) \to [0,1]$ is a smooth increasing function such that

$$\theta(t) = t \ \text{ for } 0 \le t \le \frac{1}{2}, \qquad \theta(t) = 1 \ \text{ for } t \ge 1,$$

u is a strongly MPSH defining functions for Ω, $\mathbf{x} \in \Omega$, and $r > 0$ and $\lambda > 0$ are suitably chosen constants.

In this way, we show that for some $c > 0$ and every $f \in \mathrm{CH}(\mathbb{D}, \Omega)$ we have

$$|\nabla f(z)| \le c\sqrt{|u(f(0))|}, \quad |z| \le \frac{1}{2} \tag{1}$$

provided that the centre $f(0)$ is close enough to $b\Omega$. At $z = 0$ this gives the asymptotic estimate

$$g_\Omega(\mathbf{x},\mathbf{v}) \ge C\frac{|\mathbf{v}|}{\sqrt{\mathrm{dist}(\mathbf{x}, b\Omega)}}, \qquad \mathbf{x} \in \Omega,\ \mathbf{v} \in \mathbb{R}^n,$$

which is the best possible for all vectors $\mathbf{v} \in \mathbb{R}^n$.

Sketch of proof

However, to show completeness we need the stronger estimate

$$g_\Omega(\mathbf{x}, \mathbf{v}) \geq C \frac{|\mathbf{v}|}{\mathrm{dist}(\mathbf{x}, b\Omega)} \tag{2}$$

for vectors $\mathbf{v}$ which are normal to $b\Omega$ at the closest point $\mathbf{p} \in b\Omega$ to $\mathbf{x}$.

We follow Ivashkovich and Rosay (2004). The inequality (1) gives

$$\begin{aligned} |\Delta(u \circ f)(z)| &= |\mathrm{tr}_{df_z(\mathbb{R}^2)} \mathrm{Hess}_u(f(z))| \cdot \|df_z\|^2 \\ &\leq c_1 |\nabla f(z)|^2 \leq C_1 |u(f(0))|, \quad |z| \leq \frac{1}{2} \end{aligned}$$

for some constant $c_1 > 0$ and $C_1 = c_1 c^2 > 0$. We claim that this gives

$$|\nabla(u \circ f)(0)| \leq C_2 |u(f(0))|, \quad f \in \mathrm{CH}(\mathbb{D}, \Omega), \tag{3}$$

which implies (2) and complete hyperbolicity of Ω.

Proof of (3)

By rescaling we may assume that (1) holds for all $z \in \mathbb{D}$.
Set $v = u \circ f : \mathbb{D} \to (-\infty, 0)$, so

$$|\Delta v(z)| \le C_1 |v(0)| = -C_1 v(0), \qquad z \in \mathbb{D}.$$

We extend Δv to $\mathbb{C}$ by setting it equal to 0 on $\mathbb{C} \setminus \overline{\mathbb{D}}$. The function

$$g(z) = v(z) - \left(\frac{1}{2\pi} \log|\cdot| * \Delta v\right)(z) - C_1|v(0)|, \quad z \in \mathbb{D}$$

is then harmonic on $\mathbb{D}$. Note that

$$\left|\frac{1}{2\pi} \log|\cdot| * \Delta v\right| \le C_1|v(0)|.$$

Hence, $g \le v < 0$ on $\mathbb{D}$ and $|g(0)| < (2C_1 + 1)|v(0)|$. The Schwarz lemma for negative harmonic functions gives $|\nabla g(0)| \le 2|g(0)|$, and hence

$$|\nabla v(0)| \le |\nabla g(0)| + \sup_{\mathbb{D}} |\Delta v| \le 2|g(0)| + C_1|v(0)| \le (5C_1 + 2)|v(0)|.$$

This is the desired estimate $(+)$ with the constant $C_2 = 5C_1 + 2$.

Hyperbolicity of convex domains

We have the following characterization of hyperbolic convex domains in $\mathbb{R}^n$, showing in particular that every such domain is also complete hyperbolic.

Theorem (B. Drinovec Drnovšek & F. F.)

The following are equivalent for a (not necessarily bounded) convex domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$.

1. Ω *is complete hyperbolic.*
2. *For any open Riemann surface, M, the family $\mathrm{CH}(M, \Omega)$ of conformal harmonic maps $M \to \Omega$ is a normal family.*
3. Ω *is hyperbolic.*
4. Ω *does not contain any 2-dimensional affine subspaces.*
5. Ω *has $n-1$ linearly independent separating hyperplanes.*

A hyperplane $\Sigma \subset \mathbb{R}^n$ is called *separating* for a domain $\Omega \subset \mathbb{R}^n$ if Ω lies in one of the two half-spaces in $\mathbb{R}^n \setminus \Sigma$.

The corresponding result for Kobayashi hyperbolicity of convex domains in $\mathbb{C}^n$ is due to Barth (1980), Harris (1979), and Bracci and Saracco (2009). Our proof is rather different from theirs.

~ Thank you for your attention ~

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