Schwarz–Pick lemma for harmonic maps which are conformal at a point

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18 May 2021

Abstract

- We prove a sharp estimate on the norm of the differential of a harmonic map from the unit disc $\mathbb{D}$ in $\mathbb{C}$ into the unit ball $\mathbb{B}^n$ of $\mathbb{R}^n$, $n \geq 2$, at any point where the map is conformal.

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- We then give a differential-geometric interpretation, showing that every conformal harmonic immersion $M \to \mathbb{B}^n$ from a hyperbolic conformal surface is distance-decreasing in the Poincaré metric on M and the Cayley–Klein metric $\mathcal{CK}$ on the ball $\mathbb{B}^n$. The extremal maps are precisely the conformal embeddings of the disc $\mathbb{D}$ onto affine discs in $\mathbb{B}^n$.

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- We then give a differential-geometric interpretation, showing that every conformal harmonic immersion $M \to \mathbb{B}^n$ from a hyperbolic conformal surface is distance-decreasing in the Poincaré metric on M and the Cayley–Klein metric $\mathcal{CK}$ on the ball $\mathbb{B}^n$. The extremal maps are precisely the conformal embeddings of the disc $\mathbb{D}$ onto affine discs in $\mathbb{B}^n$.
- Using these results, we lay foundations of the hyperbolicity theory for domains in $\mathbb{R}^n$ based on minimal surfaces.

F. F. & David Kalaj, Hyperbolicity theory for conformal minimal surfaces in $\mathbb{R}^n$.
https://arxiv.org/abs/2102.12403

The classical Schwarz–Pick Lemma

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disc.

The following result is due to **Karl Hermann Amandus Schwarz (1869)**, **Henri Poincaré (1884)**, and **Georg Alexander Pick (1915)**.

Theorem (Schwarz–Pick lemma for holomorphic maps)

If $f : \mathbb{D} \to \mathbb{D}$ is a holomorphic map, then for every $z \in \mathbb{D}$ we have that

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with equality at one point if and only if f is a biholomorphism of the disc $\mathbb{D}$.

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Using pre- and postcompositions by holomorphic automorphisms of $\mathbb{D}$, the proof reduces to the case $z = 0$, $f(0) = 0$. In this case, it follows from the maximum principle applied to the holomorphic function $g(z) = f(z)/z$ on $\mathbb{D}$.

This is the most fundamental rigidity result in complex analysis which leads to the theory of **Kobayashi hyperbolic manifolds.**

Differential-theoretic interpretation

Let $\mathcal{P}$ denote the **Poincaré metric** on the disc $\mathbb{D} = \{|z| < 1\}$:

$$\mathcal{P}(z, \xi) = \frac{|\xi|}{1 - |z|^2}, \qquad z \in \mathbb{D},\ \xi \in T_z\mathbb{C} \cong \mathbb{C}.$$

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The Schwarz–Pick lemma is equivalent to the statement that for any holomorphic map $f : \mathbb{D} \to \mathbb{D}$ we have

$$\mathcal{P}(f(z), df_z(\xi)) \leq \mathcal{P}(z, \xi), \quad z \in \mathbb{D},\ \xi \in \mathbb{C},$$

with equality at one point if and only if f is an automorphism of $\mathbb{D}$,

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That is, **holomorphic maps $\mathbb{D} \to \mathbb{D}$ are distance-decreasing in the Poincaré metric, and orientation-preserving isometries are precisely the elements of $\mathrm{Aut}(\mathbb{D})$.** The analogus conclusion holds for the Poincaré distance function

$$\mathrm{dist}_{\mathcal{P}}(z, w) = \frac{1}{2} \log \left(\frac{|1 - z\overline{w}| + |z - w|}{|1 - z\overline{w}| - |z - w|} \right), \quad z, w \in \mathbb{D}.$$

The following special case of our main result gives the same conclusion at a given point for a much bigger class of maps.

Corollary (of our main result)

Let $f : \mathbb{D} \to \mathbb{D}$ be a harmonic map. If f is conformal at a point $z \in \mathbb{D}$, then at this point we have that

$$|f'(z)| = \|df_z\| \leq \frac{1 - |f(z)|^2}{1 - |z|^2},$$

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By using precompositions by automorphisms of $\mathbb{D}$, the proof reduces to the case $z = 0$. On the other hand, postcompositions of harmonic maps $\mathbb{D} \to \mathbb{D}$ by holomorphic automorphism of $\mathbb{D}$ need not be harmonic, so we cannot exchange $f(0)$ and 0. Also, $f(z)/z$ need not be harmonic. The standard proof of the classical Schwarz–Pick lemma breaks down at this point.

Without conformality, this fails for some harmonic diffeomorphisms of $\mathbb{D}$.

Schwarz–Pick lemma for harmonic maps into balls

Theorem (1)

Let $f : \mathbb{D} \to \mathbb{B}^n$ is a harmonic map for some $n \geq 2$ which is conformal at a point $z \in \mathbb{D}$. Denote by $\theta \in [0, \pi/2]$ the angle between the vector $f(z)$ and the plane $df_z(\mathbb{R}^2)$. Then at this point we have that

$$\|df_z\| \;\leq\; \frac{1-|f(z)|^2}{1-|z|^2}\,\frac{1}{\sqrt{1-|f(z)|^2\sin^2\theta}},$$

with equality if and only if f is a conformal diffeomorphism onto the affine disc

$$\Sigma = (f(z) + df_z(\mathbb{R}^2)) \cap \mathbb{B}^n.$$

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The number $R = \sqrt{1-|f(z)|^2 \sin^2\theta}$ is the radius of the disc Σ.

If $n = 2$ then $\theta = 0$, $R = 1$, so the previous corollary is a special case.

If $f(z) = 0$ then the angle θ is not defined, but it is irrelevant.

An estimate without conformality

For a fixed value of $|f(z)| \in [0,1)$, the maximum of the right hand side over angles $\theta \in [0, \pi/2]$ equals $\frac{\sqrt{1-|f(z)|^2}}{1-|z|^2}$ and is reached at $\theta = \pi/2$ when $f(z)$ is orthogonal to the 2-plane $\Lambda = df_z(\mathbb{R}^2)$, unless $f(z) = 0$ in which case it equals $\frac{1}{1-|z|^2}$ for all θ.

We show that this weaker estimate holds for all harmonic maps $\mathbb{D} \to \mathbb{B}^n$.

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For every harmonic map $f : \mathbb{D} \to \mathbb{B}^n$ $(n \geq 2)$ we have that

$$\frac{1}{\sqrt{2}}|\nabla f(z)| \ \leq \ \frac{\sqrt{1-|f(z)|^2}}{1-|z|^2}, \qquad z \in \mathbb{D}.$$

Equality holds for some $z_0 \in \mathbb{D}$ if $f(z_0)$ is orthogonal to the 2-plane $\Lambda = df_{z_0}(\mathbb{R}^2)$ and f is a conformal diffeomorphism onto the affine disc $(f(z_0) + \Lambda) \cap \mathbb{B}^n$.

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Unlike the main result, this weaker estimate is simple consequence of the assumption that $f(\mathbb{D}) \subset \mathbb{B}^n$ and hence

$$\int_0^{2\pi} |f(e^{it})|^2 \frac{dt}{2\pi} \leq 1.$$

Discussion

The precise upper bound on the size of the gradient of harmonic maps $f : \mathbb{D} \to \mathbb{B}^n$ with a given centre $f(0) = \mathbf{x} \in \mathbb{B}^n \setminus \{0\}$ seems unknown, except for $n = 1$.

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The **harmonic Schwarz lemma** says that any harmonic function $f : \mathbb{B}^m \to (-1, +1)$ for $m \geq 2$ satisfies the estimate

$$|\nabla f(0)| \ \leq \ 2\frac{\mathrm{Vol}(\mathbb{B}^{m-1})}{\mathrm{Vol}(\mathbb{B}^m)},$$

with equality if and only if f is a harmonic function on $\mathbb{B}^m$ whose boundary values equal ± 1 on a pair of opposite hemispheres.

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For $m = 2$ the inequality reads

$$|\nabla f(0)| \;\leq\; \frac{4}{\pi}.$$

A simple proof was given by **Kalaj and Vuorinen (2012)** who obtained it from the classical Schwarz–Pick lemma applied to the holomorphic function $\phi \circ F : \mathbb{D} \to \mathbb{D}$, where $F = f + \mathrm{i}g : \mathbb{D} \to \Omega = (-1, +1) \times \mathrm{i}\mathbb{R}$ is a holomorphic extension of f and $\phi : \Omega \to \mathbb{D}$ is a biholomorphism.

Comparison with the Schwarz lemma in the complex ball

It is well known that the extremal holomorphic discs in the complex ball $\mathbb{B}^n_{\mathbb{C}} \subset \mathbb{C}^n$ are the holomorphic parameterizations of complex affine discs in $\mathbb{B}^n_{\mathbb{C}}$. The standard proof strongly uses the fact that the group of holomorphic automorphisms of $\mathbb{B}^n_{\mathbb{C}}$ acts transitively.

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The biggest group preserving the set of all conformal minimal discs in $\mathbb{B}^n$ (under postcomposition) is the orthogonal group, which does not act transitively. Our proof also gives a new proof of the complex Schwarz lemma without using the group $\mathrm{Aut}(\mathbb{B}^n_{\mathbb{C}})$. As will be shown presently, it is just as elementary as the standard proof.

Proof of the main theorem, 1

It suffices to prove Theorem 1 for $z = 0$. Indeed, with f and z as in the theorem, let $\phi_z \in \mathrm{Aut}(\mathbb{D})$ be such that $\phi_z(0) = z$. The harmonic map $\tilde{=} f \circ \phi_z : \mathbb{D} \to \mathbb{B}^n$ is then conformal at the origin. Since $|\phi_z'(0)| = 1 - |z|^2$, the estimate follows from the same estimate for the map $\tilde{f}$ applied at $z = 0$.

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We find an explicit conformal parameterization of affine discs in $\mathbb{B}^n$. Fix a point $\mathbf{q} \in \mathbb{B}^n$ and a 2-plane $0 \in \Lambda \subset \mathbb{R}^n$, and consider the affine disc $\Sigma = (\mathbf{q} + \Lambda) \cap \mathbb{B}^n$. Let $\mathbf{p} \in \Sigma$ be the closest point to the origin.

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If $n = 2$ then $\mathbf{p} = 0$ and $\Sigma = \mathbb{D}$. Suppose now that $n = 3$; the case $n > 3$ will be the same. By an orthogonal rotation on $\mathbb{R}^3$ we may assume that

$$\mathbf{p} = (0, 0, p) \quad \text{and} \quad \Sigma = \left\{ (x, y, p) : x^2 + y^2 < 1 - p^2 \right\}.$$

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Let $\mathbf{q} = (b_1, b_2, p) \in \Sigma$, and let θ denote the angle between $\mathbf{q}$ and Σ. Set

$$R = \sqrt{1 - p^2} = \sqrt{1 - |\mathbf{q}|^2 \sin^2 \theta}, \qquad a = \frac{b_1 + \imath b_2}{R} \in \mathbb{D}, \qquad |a| = \frac{|\mathbf{q}| \cos \theta}{R}.$$

We orient Σ by the pair of tangent vectors ∂_x, ∂_y.

Proof, 2

Every orientation preserving conformal parameterization $f : \mathbb{D} \to \Sigma$ with $f(0) = \mathbf{q}$ is then of the form

$$f(z) = \left(R \cdot \Re \frac{e^{it} z + a}{1 + \bar{a} e^{it} z}, R \cdot \Im \frac{e^{it} z + a}{1 + \bar{a} e^{it} z}, p \right) = \left(R \, \frac{e^{it} z + a}{1 + \bar{a} e^{it} z}, p \right)$$

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We have that

$$\begin{aligned} \|df_0\| &= R\,(1 - |a|^2) = \frac{R^2 - R^2|a|^2}{R} = \frac{1 - |\mathbf{q}|^2 \sin^2\theta - |\mathbf{q}|^2 \cos^2\theta}{R} \\ &= \frac{1 - |\mathbf{q}|^2}{\sqrt{1 - |\mathbf{q}|^2 \sin^2\theta}} = \frac{1 - |f(0)|^2}{\sqrt{1 - |f(0)|^2 \sin^2\theta}}. \end{aligned}$$

This shows that the conformal parameterizations of the proper affine discs in the ball satisfy the equality in the theorem at every point.

Proof, 3

Let $f : \mathbb{D} \to \mathbb{B}^3$ be as above, where we may assume that $t = 0$.

Suppose that $g : \mathbb{D} \to \mathbb{B}^3$ is a harmonic map such that $g(0) = f(0)$, g is conformal at 0, and $dg_0(\mathbb{R}^2) = df_0(\mathbb{R}^2)$. Up to replacing g by $g(e^{\mathrm{i}t}z)$ or $g(e^{\mathrm{i}t}\bar{z})$ for some $t \in \mathbb{R}$, we may assume that

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We must prove that $r \leq 1$, and that $r = 1$ if and only if $g = f$.

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Consider the holomorphic map $F : \mathbb{D} \to \Omega = \mathbb{B}^3 \times \mathrm{i}\mathbb{R}^3$ with $f = \Re F$, given by

$$F(z) = \left(R \cdot \frac{z + a}{1 + \bar{a}z}, -R \cdot \mathrm{i}\frac{z + a}{1 + \bar{a}z}, p \right), \quad z \in \mathbb{D}.$$

Let $G : \mathbb{D} \to \Omega$ be the holomorphic map with $\Re G = g$ and $G(0) = F(0)$.

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By the Cauchy–Riemann equations, the condition $dg_0 = r\, df_0$ implies

$$G'(0) = r\, F'(0).$$

Proof, 4

It follows that the map $(F(z) - G(z))/z$ is holomorphic on $\mathbb{D}$ and

$$\lim_{z \to 0} \frac{F(z) - G(z)}{z} = F'(0) - G'(0) = (1 - r)F'(0).$$

Since $g : \mathbb{D} \to \mathbb{B}^3$ is a bounded harmonic map, it has a nontangential boundary value at almost every point of the circle $\mathbb{T} = b\mathbb{D}$. Since the Hilbert transform is an isometry on the Hilbert space $L^2(\mathbb{T})$, the same is true for G.

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Denote by $\langle \cdot, \cdot \rangle$ the complex bilinear form on $\mathbb{C}^n$ given by

$$\langle z, w \rangle = \sum_{i=1}^{n} z_i w_i$$

for $z, w \in \mathbb{C}^n$.

Proof, 5

For each $z = e^{\mathrm{i}t} \in b\mathbb{D}$ the vector $f(z) \in b\mathbb{B}^3$ is the unit normal vector to the sphere $b\mathbb{B}^3$ at the point $f(z)$. Since $\mathbb{B}^3$ is strongly convex, we have that

$$\Re \left\langle F(z) - G(z), f(z) \right\rangle = \left\langle f(z) - g(z), f(z) \right\rangle \geq 0 \quad a.e.\ z \in b\mathbb{D},$$

and the value is positive for almost every $z \in b\mathbb{D}$ if and only if $g \neq f$.

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Consider the following function on the circle $b\mathbb{D}$:

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A calculation, taking into account $z\bar{z} = 1$, shows that

$$\tilde{f}(z) = \begin{pmatrix} \frac{c}{2}\left(1 + a^2 + 4(\Re a)z + (1 + \bar{a}^2)z^2\right) \\ \frac{c}{2}\left(\mathrm{i}(1 - a^2) + 4(\Im a)z + \mathrm{i}(\bar{a}^2 - 1)z^2\right) \\ p\,(z + a)(1 + \bar{a}z) \end{pmatrix}, \qquad |z| = 1.$$

Conclusion of the proof

We extend $\tilde{f}$ to all $z \in \mathbb{C}$ by letting it equal the holomorphic polynomial map on the right hand side above. Since $|1+\bar{a}z|^2 > 0$ for $z \in \overline{\mathbb{D}}$, we have

$$\begin{aligned} h(z) &:= \Re \left\langle F(z) - G(z), |1+\bar{a}z|^2 f(z) \right\rangle \\ &= \left\langle f(z) - g(z), |1+\bar{a}z|^2 f(z) \right\rangle \geq 0 \quad a.e.\ z \in b\mathbb{D}, \end{aligned}$$

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and $h > 0$ almost everywhere on $b\mathbb{D}$ if and only if $g \neq f$.

From the definition of $\tilde{f}$ we see that

$$h(z) = \Re \left\langle \frac{F(z) - G(z)}{z}, \tilde{f}(z) \right\rangle \quad a.e.\ z \in b\mathbb{D}$$

Since the maps $(F(z) - G(z))/z$ and $\tilde{f}(z)$ are holomorphic on $\mathbb{D}$, h extends to a nonnegative harmonic function on $\mathbb{D}$ which is positive on $\mathbb{D}$ unless $f = g$.

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At $z = 0$ we have

$$h(0) = \Re \left\langle F'(0) - G'(0), \tilde{f}(0) \right\rangle = (1-r)\, \Re \left\langle F'(0), \tilde{f}(0) \right\rangle \geq 0,$$

with equality if and only if $g = f$. Applying this to the constant map $g(z) = f(0)$ gives $\Re \langle F'(0), \tilde{f}(0) \rangle > 0$. It follows that $r \leq 1$, with equality if and only if $g = f$. This completes the proof.

Discussion

The above proof is motivated by the seminal work of **László Lempert (1981)** on Kobayashi extremal holomorphic discs in bounded strongly convex domains $\Omega \subset \mathbb{C}^n$ with smooth boundaries.

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In Lempert's terminology, a proper holomorphic disc $F : \mathbb{D} \to \Omega$ extending continuously to $\overline{\mathbb{D}}$ is a **stationary disc** if, denoting by $\nu(z)$ the unit normal to $b\Omega$ along the circle $F(b\mathbb{D})$, there is a positive function $q > 0$ on $b\mathbb{D}$ such that the function

$$z\, q(z)\overline{\nu(z)}$$

extends from the circle $|z| = 1$ to a holomorphic function $\tilde{f}(z)$ on $\mathbb{D}$. The use of such a function, along with the convexity of the domain, enables the arguments used above to show that a stationary disc F is the unique Kobayashi extremal disc in Ω through the point $F(a)$ in the tangent direction $F'(a)$ for every $a \in \mathbb{D}$.

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In our case, $\nu(z) = f(z)$ is real-valued, and a suitable function is

$$\tilde{f}(z) = z|1 + \bar{a}z|^2 f(z), \qquad |z| = 1.$$

The fact that $\Omega = \mathbb{B}^n \times i\mathbb{R}^n$ is unbounded does not matter.

We can interpret Theorem 1 as the metric-decreasing property of conformal harmonic maps $\mathbb{D} \to \mathbb{B}^n$ with respect to the **Cayley–Klein metric** on $\mathbb{B}^n$:

$$\mathcal{CK}(\mathbf{x}, \mathbf{v}) = \frac{\sqrt{1 - |\mathbf{x}|^2 \sin^2 \phi}}{1 - |\mathbf{x}|^2} \, |\mathbf{v}|, \qquad \mathbf{x} \in \mathbb{B}^n, \ \mathbf{v} \in \mathbb{R}^n,$$

where $\phi \in [0, \pi/2]$ is the angle between the vector $\mathbf{v}$ and the line $\mathbb{R}\mathbf{x} \subset \mathbb{R}^n$.

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$$\mathcal{CK}(\mathbf{x}, \mathbf{v})^2 = \frac{(1 - |\mathbf{x}|^2)|\mathbf{v}|^2 + |\mathbf{x} \cdot \mathbf{v}|^2}{(1 - |\mathbf{x}|^2)^2} = \frac{|\mathbf{v}|^2}{1 - |\mathbf{x}|^2} + \frac{|\mathbf{x} \cdot \mathbf{v}|^2}{(1 - |\mathbf{x}|^2)^2}.$$

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The **Cayley–Klein model**, also called the **Beltrami–Klein model** of hyperbolic geometry was introduced by **Arthur Cayley (1859)** and **Eugenio Beltrami (1868)**, and it was developed by **Felix Klein (1871, 1873)**. The underlying space is the n-dimensional unit ball, geodesics are straight line segments with endpoints on the boundary sphere, and the distance between points on a geodesic is given by a cross ratio. This is a special case of the **Hilbert metric** on convex domains in $\mathbb{R}^n$, introduced by **David Hilbert** in 1895.

Comments on the Cayley–Klein metric

The Cayley–Klein metric $\mathcal{CK}$ is the restriction of the Kobayashi metric on the unit ball $\mathbb{B}^n_{\mathbb{C}} \subset \mathbb{C}^n$ to points $\mathbf{x} \in \mathbb{B}^n = \mathbb{B}^n_{\mathbb{C}} \cap \mathbb{R}^n$ and vectors in $T_{\mathbf{x}}\mathbb{R}^n \cong \mathbb{R}^n$.

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It also equals $1/\sqrt{n+1}$ times the Bergman metric on $\mathbb{B}^n_{\mathbb{C}}$ restricted to $\mathbb{B}^n$ and real tangent vectors. (On the ball of $\mathbb{C}^n$, most holomorphically invariant metrics coincide up to scalar factors.)

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The metric $\mathcal{CK}$ is not conformally equivalent to the Euclidean metric on $\mathbb{B}^n$. It coincide with the Poincaré metric on $\mathbb{B}^n$, given by $\frac{|\mathbf{v}|}{1-|\mathbf{x}|^2}$, in the radial direction parallel to the base point $\mathbf{x} \in \mathbb{B}^n$, but is strictly smaller in the direction perpendicular to $\mathbf{x}$. We have that

$$\frac{|\mathbf{v}|}{\sqrt{1-|\mathbf{x}|^2}} \le \mathcal{CK}(\mathbf{x},\mathbf{v}) \le \frac{|\mathbf{v}|}{1-|\mathbf{x}|^2},$$

with the upper bound reached for $\phi = 0$ and the lower bound for $\phi = \pi/2$.

The inequality in Theorem 1 can be rewritten as

$$\frac{\sqrt{1-|f(z)|^2\sin^2\theta}}{1-|f(z)|^2}\,|df_z(\xi)| \;\le\; \frac{|\xi|}{1-|z|^2}, \qquad \xi \in T_z\mathbb{D} = \mathbb{R}^2,$$

where $\theta \in [0, \pi/2]$ is the angle between $f(z)$ and the plane $\Lambda = df_z(\mathbb{R}^2)$.

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If $\phi\in[\theta,\pi/2]$ is the angle between $\mathbb{R}f(z)$ and the vector $df_z(\xi)\in\Lambda$, then

$$\begin{aligned}
\mathcal{CK}\big(f(z),df_z(\xi)\big) &= \frac{\sqrt{1-|f(z)|^2\sin^2\phi}}{1-|f(z)|^2}\,|df_z(\xi)| \\
&\le \frac{\sqrt{1-|f(z)|^2\sin^2\theta}}{1-|f(z)|^2}\,|df_z(\xi)| \\
&\le \frac{|\xi|}{1-|z|^2} = \mathcal{P}_{\mathbb{D}}(z,\xi).
\end{aligned}$$

The first inequality is equality if and only if $\phi=\theta$.

Metric decreasing property of conformal harmonic maps

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\end{aligned}$$

The first inequality is equality if and only if $\phi=\theta$.

The second inequality is equality of and only if f is a conformal diffeomorphism onto the linear disc $\big(f(z)+df_z(\mathbb{R}^2)\big)\cap\mathbb{B}^n$.

Conformal harmonic maps are metric-decreasing

Corollary

If $f : \mathbb{D} \to \mathbb{B}^n$ is a conformal harmonic map then for every $z \in \mathbb{D}$, $\xi \in \mathbb{R}^2$:

$$\mathcal{CK}\big(f(z), df_z(\xi)\big) \le \frac{|\xi|}{1-|z|^2} = \mathcal{P}_{\mathbb{D}}(z,\xi), \tag{1}$$

with equality for some $z \in \mathbb{D}$ and $\xi \in \mathbb{R}^2 \setminus \{0\}$ if and only if f is a conformal diffeomorphism onto the affine disc

$$\Sigma = (f(z) + df_z(\mathbb{R}^2)) \cap \mathbb{B}^n$$

and the vector $df_z(\xi)$ is tangent to the diameter of Σ through the point $f(z)$.

The analogous conclusion holds if $\mathbb{D}$ is replaced by any hyperbolic conformal surface M with the Poincaré metric $\mathcal{P}_M$. Equality can only occur if $M = \mathbb{D}$.

A conformal surface is *hyperbolic* if its universal conformal covering is the disc. One introduces the Poincaré metric on such a surface by asking that the universal covering projection $h : \mathbb{D} \to M$ be a local isometry.

Distance-decreasing property of conformal harmonic maps

It follows that for any $r > 0$, a conformal harmonic map $f : M \to \mathbb{B}^n$ maps the r-ball around a point $z \in M$ in the Poincaré metric into the r-ball around the image point $f(z) \in \mathbb{B}^n$ in the $\mathcal{CK}$ metric, with equality at some point if and only if $M = \mathbb{D}$ and f is a conformal embedding onto an affine disc in $\mathbb{B}^n$.

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An explicit formula for the $\mathcal{CK}$ distance function is

$$\mathrm{dist}_{\mathcal{CK}}(\mathbf{x}, \mathbf{y}) = \frac{1}{2} \log \left(\frac{|1 - \mathbf{x} \cdot \mathbf{y}| + \sqrt{|\mathbf{x} - \mathbf{y}|^2 + |\mathbf{x} \cdot \mathbf{y}|^2 - |\mathbf{x}|^2 |\mathbf{y}|^2}}{|1 - \mathbf{x} \cdot \mathbf{y}| - \sqrt{|\mathbf{x} - \mathbf{y}|^2 + |\mathbf{x} \cdot \mathbf{y}|^2 - |\mathbf{x}|^2 |\mathbf{y}|^2}} \right).$$

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In particular, if $M = \mathbb{D}$ and $f(0) = 0$ then

$$|f(z)| \leq |z| \text{ for every } z \in \mathbb{D},$$

with equality at a point $z \in \mathbb{D} \setminus \{0\}$ if and only if f is a conformal linear embedding onto a linear disc in $\mathbb{B}^n$.

A pseudodistance on a domain in $\mathbb{R}^n$

There is a natural procedure to define a pseudodistance function $\rho = \rho_D$ on any domain $D \subset \mathbb{R}^n$ using conformal minimal discs $\mathbb{D} \to D$. It is motivated by Kobayashi's construction of his pseudometric on complex manifolds.

Fix a pair of points $\mathbf{x}, \mathbf{y} \in D$ and consider finite chains of conformal harmonic discs $f_i : \mathbb{D} \to D$ and points $a_i \in \mathbb{D}$ $(i = 1, \dots, k)$ such that

$$f_1(0) = \mathbf{x}, \quad f_{i+1}(0) = f_i(a_i) \text{ for } i = 1, \dots, k-1, \quad f_k(a_k) = \mathbf{y}.$$

To any such chain we associate the number

$$\sum_{i=1}^{k} \frac{1}{2} \log \frac{1 + |a_i|}{1 - |a_i|} \geq 0.$$

The i-th term in the sum is the Poincaré distance from 0 to a_i in $\mathbb{D}$.

The pseudodistance $\rho_D : D \times D \to \mathbb{R}_+$ is the infimum of the numbers obtained in this way. Clearly it satisfies the triangle inequality.

If D is a domain in $\mathbb{C}^n$ and we use only holomorphic discs, then the corresponding pseudodistance ρ is precisely the one of Kobayashi.

Lemma

(A) Conformal harmonic maps $M \to D$ from any hyperbolic conformal surface are distance-decreasing in the Poincaré distance on M and the pseudodistance ρ_D on D.

(B) ρ_D is the largest pseudodistance function on D for which this holds.

Proof of (A) For $M = \mathbb{D}$, this follows from the definition since every conformal harmonic map $f : \mathbb{D} \to D$ is a candidate for computing ρ_D and we are taking the infimum. For general M, the result follows by precomposing f with a universal conformal covering map $h : \mathbb{D} \to M$.

Proof of (B) Suppose that τ is a pseudodistance on D such that every conformal harmonic map $\mathbb{D} \to D$ is distance-decreasing. Let $f_i : \mathbb{D} \to D$ and $a_i \in \mathbb{D}$ for $i = 1, \ldots, k$ be a chain connecting the points $\mathbf{x}, \mathbf{y} \in D$. Then,

$$\tau(\mathbf{x}, \mathbf{y}) \;\leq\; \sum_{i=1}^{k} \tau(f_i(0), f_i(a_i)) \;\leq\; \sum_{i=1}^{k} \frac{1}{2} \log \frac{1+|a_i|}{1-|a_i|}.$$

Taking the infimum over all such chains gives $\tau(\mathbf{x}, \mathbf{y}) \leq \rho_D(\mathbf{x}, \mathbf{y})$.

$\rho_{\mathbb{B}^n} = \mathrm{dist}_{\mathcal{CK}}$

Theorem

On the ball $\mathbb{B}^n$, we have $\rho_{\mathbb{B}^n} = \mathrm{dist}_{\mathcal{CK}}$.

Proof Fix a pair of distinct points $\mathbf{x}, \mathbf{y} \in \mathbb{B}^n$. Let $\mathbf{p}$ be the point on the affine line L through $\mathbf{x}$ and $\mathbf{y}$ which is closest to the origin.

Let $\Lambda \subset \mathbb{R}^n$ be the affine 2-plane containing L and such that $\mathbf{p}$ is orthogonal to Λ (such Λ is unique unless $\mathbf{p} = \mathbf{0}$). Then, $\Sigma := \Lambda \cap \mathbb{B}^n$ is an affine disc, and the points $\mathbf{x}$ and $\mathbf{y}$ lie on the diameter $L \cap \mathbb{B}^n$ of Σ.

These diameters are geodesics for the Cayley–Klein metric on $\mathbb{B}^n$, and $\mathrm{dist}_{\mathcal{CK}}(\mathbf{x}, \mathbf{y})$ equals the Poincaré distance between $\mathbf{x}$ and $\mathbf{y}$ in the affine disc Σ.

By the previous lemma, $\mathrm{dist}_{\mathcal{CK}}(\mathbf{x}, \mathbf{y}) \leq \rho_{\mathbb{B}^n}(\mathbf{x}, \mathbf{y})$. Since the affine disc Σ is a candidate for computing $\rho_{\mathbb{B}^n}(\mathbf{x}, \mathbf{y})$, equality follows.

Hyperbolic domains

Definition (Hyperbolic domains in $\mathbb{R}^n$)

A domain $D \subset \mathbb{R}^n$ $(n \geq 3)$ is *hyperbolic* if the pseudodistance ρ_D is a distance function on D, and is *complete hyperbolic* if (D, ρ_D) is a complete metric space (Cauchy sequences converge).

Example

(A) The ball $\mathbb{B}^n \subset \mathbb{R}^n$ $(n \geq 3)$ is complete hyperbolic since the Cayley–Klein metric is complete.

(B) Every bounded domain $D \subset \mathbb{R}^n$ is hyperbolic since it is contained in a ball. However, it need not be complete hyperbolic.

(C) Every bounded strongly convex domain in $\mathbb{R}^n$ is complete hyperbolic.

(D) The half-space $\mathbb{H}^n = \{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$ is not hyperbolic since the pseudodistance $\rho_{\mathbb{H}^n}$ vanishes on planes $x_n = const$.

Problem

(A) Is the complement of a catenoid in $\mathbb{R}^3$ hyperbolic?

(B) Is every bounded strongly mean-convex domain in $\mathbb{R}^3$ complete hyperbolic?

A domain in $\mathbb{R}^3$ is (strongly) mean-convex if the mean curvature of its boundary is nonnegative (positive) at every point. Such domains are natural domains of existence of proper minimal surfaces conformally parameterized by any bordered Riemann surface.

~ Thank you for your attention ~