

Euclidean domains in complex manifolds

Franc Forstnerič

Univerza v Ljubljani



The Conference on Complex Geometric Analysis
POSTECH, Pohang, Republic of Korea
12 January 2022

Admissible pairs in complex manifolds

It is of major interest to understand which properties of a complex manifold make it a complex Euclidean spaces \mathbb{C}^n (**the recognition problem in complex analysis**), or an open subset thereof. In this talk, I will discuss recent results concerning the second question.

[Euclidean domains in complex manifolds, *J. Math. Anal. Appl.* 506(1), 2022]

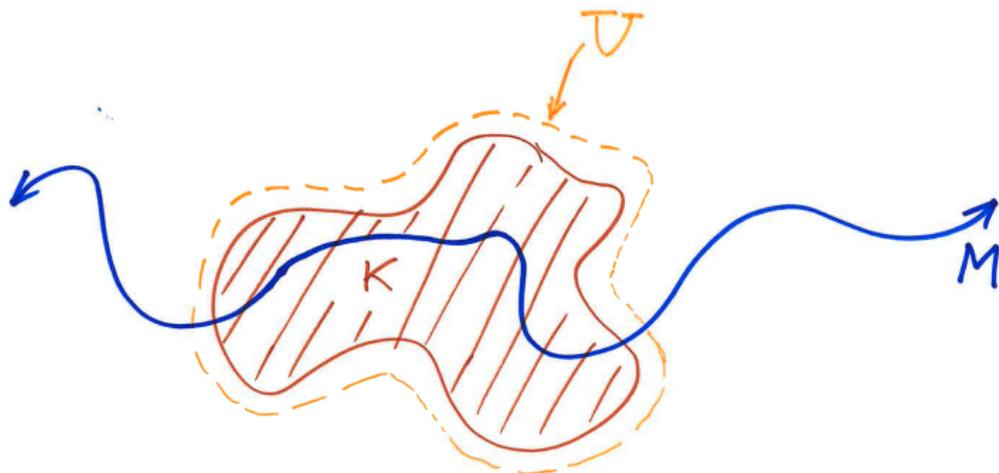
We shall be considering pairs of sets of the following kind.

Definition

Let X be a complex manifold. A pair (K, M) of subsets of X is an **admissible pair** if the following conditions hold.

- (a) K is compact and has a Stein neighbourhood $U \subset X$ such that K is $\mathcal{O}(U)$ -convex. (Such K is said to be **holomorphically convex**.)
- (b) M is a not necessarily closed embedded Stein submanifold of X .
- (c) $K \cap M$ is a compact $\mathcal{O}(M)$ -convex subset of M .

An admissible pair



An admissible pair (K, M)

Stein neighbourhood bases

Recall the following result of Siu (1976; the case $K = \emptyset$) and myself (2005). In this theorem, both M and X may have singularities.

Theorem (Stein neighbourhoods of admissible pairs)

If (K, M) is an admissible pair in a complex space X , then $K \cup M$ has a basis of open Stein neighbourhoods $\Omega \subset X$ such that M is closed in Ω and K is $\mathcal{O}(\Omega)$ -convex.

This is false if $K \cap M$ fails to be $\mathcal{O}(M)$ -convex. An example is

$$X = \mathbb{C}^2, \quad M = \mathbb{C} \times \{0\}, \quad K = \{(z, w) : 1 \leq |z| \leq 2, |w| \leq 1\}.$$

Every Stein neighbourhood of $K \cup M$ contains $\{|z| \leq 2, |w| \leq 1\}$.

It is easily seen that for every admissible pair (K, M) in X and neighbourhood $V \supset K \cup M$ there is an admissible pair (K', M) such that $K' \cup M \subset V$ and K' is the closure of compact strongly pseudoconvex Stein domain such that bK' intersects M transversely. We shall call such (K', M) a **regular admissible pair**.

Stein neighbourhood bases, II

Here is a related recent result for complex curves with boundaries.

Theorem

Assume that X is a complex space and M is a compact complex curve in X with embedded \mathcal{C}^2 boundary having no irreducible components without boundary.

If K is a compact holomorphically convex set in X such that $K \cap M$ is $\mathcal{O}(M)$ -convex, then $K \cup M$ has a basis of open Stein neighbourhoods.

This fails in general if M is a complex submanifold of dimension $m > 1$ with $bM \cap K \neq \emptyset$. For example, let M be a closed ball in $\Sigma = \mathbb{C}^2 \times \{0\} \subset \mathbb{C}^3 = X$, and let K be a closed ball in \mathbb{C}^3 centred at 0. If M is not contained in K but bM intersects K , then $(K \cup M) \cap \Sigma$ is a union of two intersecting balls in $\Sigma = \mathbb{C}^2$ with a nontrivial envelope of holomorphy, so $K \cup M$ is not a Stein compact.

On the other hand, it was shown by Starčič (2008) that the result holds if M is a compact complex subvariety with Stein interior whose embedded strongly pseudoconvex boundary bM of class \mathcal{C}^2 does not intersect K .

The main theorem

Theorem

Assume that X is a complex manifold of dimension n , (K, M) is an admissible pair in X such that $TX|_M$ is a trivial bundle, $\Omega_0 \subset X$ is an open neighbourhood of K , and $\Phi_0 : \Omega_0 \xrightarrow{\cong} \Phi_0(\Omega_0) \subset \mathbb{C}^n$ is a biholomorphic map such that $\Phi_0(K)$ is polynomially convex in \mathbb{C}^n . (Such K is called a polynomially convex compact in X .)

If $\dim X \geq 2 \dim M + 1$ then for any $\epsilon > 0$ there are a Stein neighbourhood $\Omega \subset X$ of $K \cup M$ and a biholomorphic map

$\Phi : \Omega \xrightarrow{\cong} \Phi(\Omega) \subset \mathbb{C}^n$ such that $\Phi(K)$ is polynomially convex,

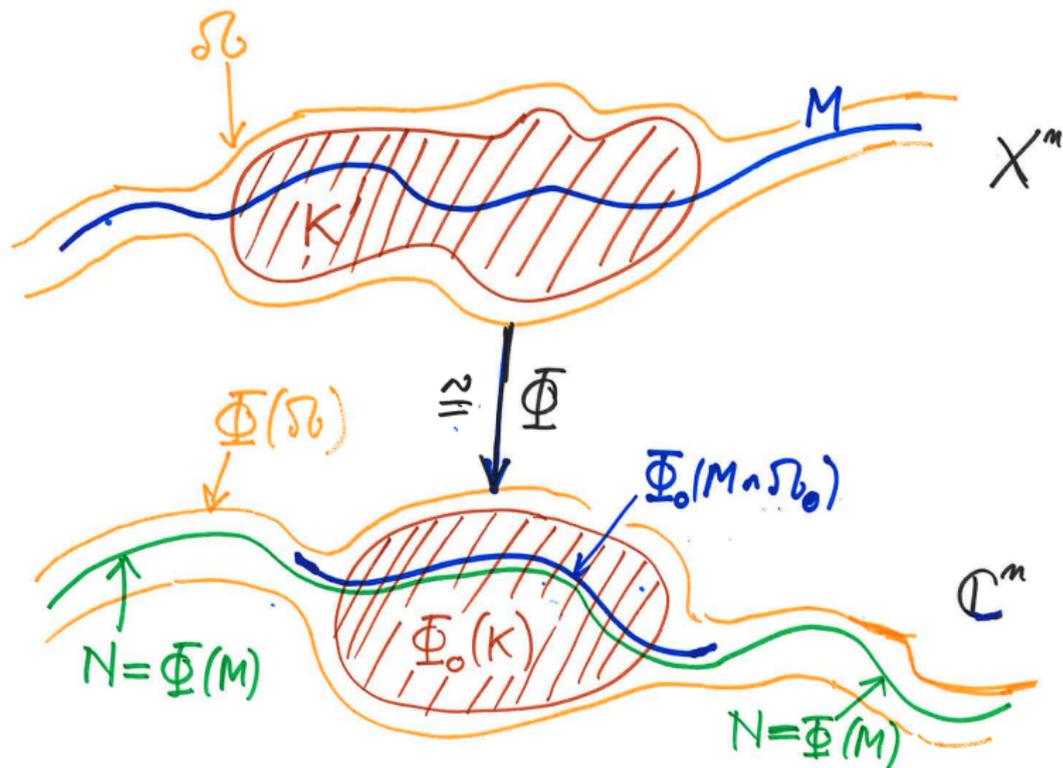
$$\sup_{x \in K} |\Phi(x) - \Phi_0(x)| < \epsilon, \quad \text{and}$$

$\Phi(M)$ is a closed complex submanifold of \mathbb{C}^n .

If $\dim X = 2 \dim M$ then Φ can be chosen an immersion which is proper on M and satisfies

$$\Phi(\Omega \setminus K) \subset \mathbb{C}^n \setminus \Phi(K).$$

A Euclidean neighbourhood of $K \cup M$



Euclidean neighbourhoods of complex curves attached to a polynomially convex set

Recall that any holomorphic vector bundle on an open Riemann surface M is holomorphically trivial, and M admits a proper holomorphic embedding in \mathbb{C}^n for any $n \geq 3$. This gives

Corollary

Assume that X is a complex manifold of dimension $n \geq 3$, $K \subset X$ is a polynomially convex compact, and $M \subset X$ is an open embedded complex curve such that $K \cap M$ is compact and $M \setminus K$ has no holes in M . Then $K \cup M$ has a Euclidean neighbourhood in which M corresponds to a properly embedded complex curve in \mathbb{C}^n .

If $\dim M > 1$ then the condition that $TX|_M$ be a trivial bundle may impose a topological restriction on the embedding $M \hookrightarrow X$. For example, if TM is nontrivial, then the vector bundle $TM \oplus (M \times \mathbb{C}^k)$ is nontrivial for every $k \in \mathbb{N}$ (stable range), and hence $M \times \{0\}^k$ does not admit a Euclidean neighbourhood in $X = M \times \mathbb{C}^k$.

An addendum to the main theorem

Assume in addition that the manifold X is Stein, M is a closed complex submanifold of X , and K is $\mathcal{O}(X)$ -convex. (This holds if we replace X by a suitable Stein neighbourhood of $K \cup M$.)

Then there is a holomorphic map $\Phi : X \rightarrow \mathbb{C}^n$ ($n = \dim X$) which is injective on a neighbourhood $\Omega \subset X$ of $M \cup K$, it approximates Φ_0 on K , $\Phi(M)$ is a closed complex submanifold of \mathbb{C}^n , and

$$\Phi(X \setminus K) \subset \mathbb{C}^n \setminus \Phi(K).$$

Thus, $\Phi : X \rightarrow \mathbb{C}^n$ covers the set $\Phi(K) \subset \mathbb{C}^n$ in a schlicht way.

If $4 \leq 2 \dim M = n$ then Φ can be chosen an immersion along M .

This generalization uses the following result of Kusakabe (2020):

For any compact polynomially convex set $L \subset \mathbb{C}^n$ the complement $\mathbb{C}^n \setminus L$ is an Oka manifold.

Ingredients in the proof

WLOG: X is Stein, M is closed in X , and K is $\mathcal{O}(X)$ -convex.

- Ⓐ **An embedding theorem for Stein manifolds with control of the image** (Drinovec and F., 2010; Ritter and F., 2014) gives an injective holomorphic map $F : U \cup M \rightarrow \mathbb{C}^n$, where $U \subset X$ is a neighbourhood of K , such that F approximates Φ_0 on a neighbourhood of K and $N = F(M) \subset \mathbb{C}^n$ is a properly embedded complex submanifold of \mathbb{C}^n such that

$$F(M \setminus K) \subset \mathbb{C}^n \setminus F(K).$$

- Ⓑ By topological reasons, the biholomorphism $F|_M : M \xrightarrow{\cong} N \subset \mathbb{C}^n$ is covered by an isomorphism $\Theta : \nu_{M,X} = TX|_M / TM \rightarrow \nu_{N,\mathbb{C}^n}$ of the normal bundles which is given on $K \cap M$ by dF .
- Ⓒ **A generalized Docquier–Grauert tubular neighbourhood theorem.** This new result of possible independent interest gives a biholomorphism $\Phi : \Omega \rightarrow \Phi(\Omega) \subset \mathbb{C}^n$ as in the theorem such that $\Phi = F$ holds on M , and Φ is close to F on K .

An embedding theorem with control of the image

Theorem (A; Ritter & F., Math. Z., 2014)

Let L be a compact polynomially convex set in \mathbb{C}^n for some $n > 1$. Let M be a Stein manifold with $2 \dim M \leq n$, K' be a compact $\mathcal{O}(M)$ -convex set, $V \subset M$ be an open set containing K' , and $f : V \rightarrow \mathbb{C}^n$ be a holomorphic map such that $f(bK') \cap L = \emptyset$.

Then for every $\epsilon > 0$ there exists a proper holomorphic immersion $F : M \rightarrow \mathbb{C}^n$ (embedding if $2 \dim M + 1 \leq n$) satisfying

$$(a) F(M \setminus K') \subset \mathbb{C}^n \setminus L, \quad (b) \|F - f\|_{K'} < \epsilon.$$

In our situation, M is a closed Stein submanifold in a Stein manifold X^n , $f : U \rightarrow f(U) \subset \mathbb{C}^n$ is a biholomorphic map on an open neighbourhood of K , and the set $L = f(K) \subset \mathbb{C}^n$ is polynomially convex. Pick a compact set $K_1 \subset U$ with $K \subset \overset{\circ}{K}_1$ such that $L_1 = f(K_1)$ is also poly. convex, and take $K' = K_1 \cap M$ and $V = U \cap M$. Essentially the same proof shows that, in addition to the above, F can be chosen holomorphic on $K_1 \cup M$ and close to f on K_1 , so F is injective holomorphic on K .

Sketch of proof

- We exhaust M by compact strongly pseudoconvex domains $D_1 \subset D_2 \subset \cdots \subset \bigcup_{i=1}^{\infty} D_i = M$ such that $K' \subset D_1 \subset V$, and every D_i is a sublevel set $\{\rho \leq c_i\}$ of a smooth strongly psh exhaustion function on M .
- We also exhaust \mathbb{C}^n by $f(K_1) = L_1 \subset L_2 \subset \cdots \subset \bigcup_i L_i = \mathbb{C}^n$ where L_i for $i \geq 2$ may be chosen to be closed balls.
- We construct $F = \lim_{i \rightarrow \infty} F_i : K_1 \cup M \rightarrow \mathbb{C}^n$ inductively, taking $F_1 = f$. At the inductive step, F_i is holomorphic on $K_1 \cup D_i$, $F|_{D_i}$ is an embedding, and $F_i(bD_i) \subset \mathbb{C}^n \setminus L_i$. We obtain F_{i+1} in two steps:
 - Ⓐ Approximate F_i on $K_1 \cup D_i$ by a holomorphic map $G_i : K_1 \cup D_{i+1} \rightarrow \mathbb{C}^n$ such that $G_i(\overline{D_{i+1}} \setminus D_i) \subset \mathbb{C}^n \setminus L_i$. This is possible by methods of Oka theory since $\mathbb{C}^n \setminus L_i$ is an Oka domain.
 - Ⓑ Approximate G_i on $K_1 \cup D_i$ by $F_{i+1} : K_1 \cup D_{i+1} \rightarrow \mathbb{C}^n$ such that

$$F_{i+1}(\overline{D_{i+1}} \setminus D_i) \subset \mathbb{C}^n \setminus L_i \quad \text{and} \quad F_{i+1}(bD_{i+1}) \subset \mathbb{C}^n \setminus L_{i+1}.$$

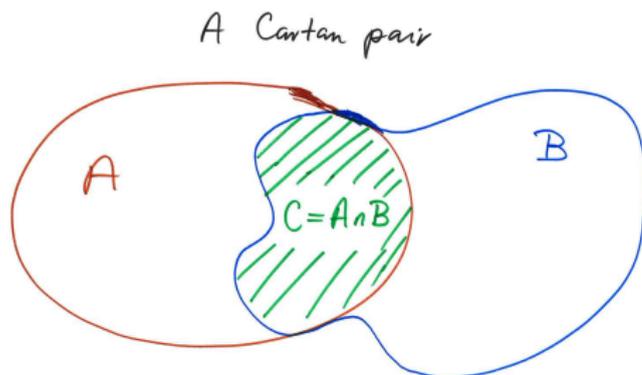
This is possible by methods developed by Drinovec Drnovšek and myself in 2010 (Amer. J. Math.). By general position, F_{i+1} can be chosen an embedding on D_{i+1} .

Sketch of step (a)

We obtain the extension G_i in a finite number of steps. Let us explain the noncritical case when ρ has no critical points on $D_{i+1} \setminus D_i$. Then, D_{i+1} is obtained by successively attaching convex bumps.

A pair (A, B) of compact subsets in a complex manifold X is a **Cartan pair** if it satisfies the following two conditions:

- (i) The sets $D = A \cup B$ and $C = A \cap B$ are Stein compacta (i.e., they have bases of open Stein neighbourhoods), and
- (ii) A and B are *separated* in the sense that $\overline{A \setminus B} \cap \overline{B \setminus A} = \emptyset$.
- (iii) If B is convex in some holomorphic coordinates on its neighbourhood, then it is called a **convex bump on A** .



Extension to a convex bump

The main step is given by the following lemma.

Lemma

Assume that Y is a complex manifold, $\Omega \subset Y$ is an Oka domain, and $F : A \rightarrow Y$ is a holomorphic map such that $F(C) \subset \Omega$. Then, F can be approximated uniformly on A by a holomorphic map $G : A \cup B \rightarrow Y$ such that $G(B) \subset \Omega$.

This is proved by forming a dominating spray \tilde{F} of maps $A \rightarrow Y$ with the core F , approximating \tilde{F} over C by another spray \tilde{H} of maps $B \rightarrow \Omega$ (this is possible if Ω is an Oka manifold), and then gluing the two sprays in the standard way of modern Oka theory:

$$\tilde{F} = \tilde{H} \circ \gamma; \quad \gamma = \beta \circ \alpha^{-1}; \quad \tilde{F} \circ \alpha = \tilde{H} \circ \beta = \tilde{G}.$$

By the gluing construction, the new map G (a member of the above amalgamated spray \tilde{G}) satisfies the stated properties.

In our case, this is applied to the Oka domain $\Omega = \mathbb{C}^n \setminus L_i$ in $Y = \mathbb{C}^n$.

The restricted tangent bundle and the normal bundle

Theorem (B)

If M is a Stein submanifold of the manifold X and

$$\dim X \geq \left\lceil \frac{3 \dim M + 1}{2} \right\rceil = \dim M + \left\lceil \frac{\dim M + 1}{2} \right\rceil$$

then the normal bundle $\nu_{M,X} = TX|_M / TM$ is uniquely determined by the restricted tangent bundle $TX|_M$.

The reason is that if $E \rightarrow M$ and $E' \rightarrow M$ are holomorphic vector bundles over a Stein manifold M such that

$$\text{rank } E' \geq \text{rank } E + \left\lceil \frac{\dim M + 1}{2} \right\rceil$$

then there is a unique (up to isotopy) holomorphic vector bundle embedding $E \hookrightarrow E'$, and hence a unique isomorphism class of the quotient E'/E . Taking $E = TM$ and $E' = TX|_M$ gives the conclusion.

A generalization of the Docquier–Grauert theorem

Theorem (C)

Let (K, M) be an admissible pair in a complex manifold X . Assume that Y is a complex manifold with $\dim Y = \dim X$, $\Omega_0 \subset X$ is an open neighbourhood of K , and $F : \Omega_0 \cup M \rightarrow Y$ is an injective map such that

$F|_{\Omega_0} : \Omega_0 \xrightarrow{\cong} F(\Omega_0) \subset Y$ is a biholomorphism, and

$f = F|_M : M \rightarrow f(M) = N \subset Y$ is a holomorphic embedding.

Assume that there is a topological isomorphism $\Theta : \nu_{M,X} \xrightarrow{\cong} \nu_{N,Y}$ over f which is given over $K \cap M$ by the differential of F .

Given $\epsilon > 0$ there are an open Stein neighbourhood $\Omega \subset X$ of $K \cup M$ and a biholomorphism $\Phi : \Omega \rightarrow \Phi(\Omega) \subset Y$ such that

$$\Phi|_M = f = F|_M \quad \text{and} \quad \sup_{x \in K} \text{dist}_Y(\Phi(x), F(x)) < \epsilon.$$

More generally, if $f = F|_M : M \rightarrow Y$ is a holomorphic immersion and we replace $\nu_{N,Y}$ by the normal bundle of f , then the map Φ in the theorem can be chosen a holomorphic immersion.

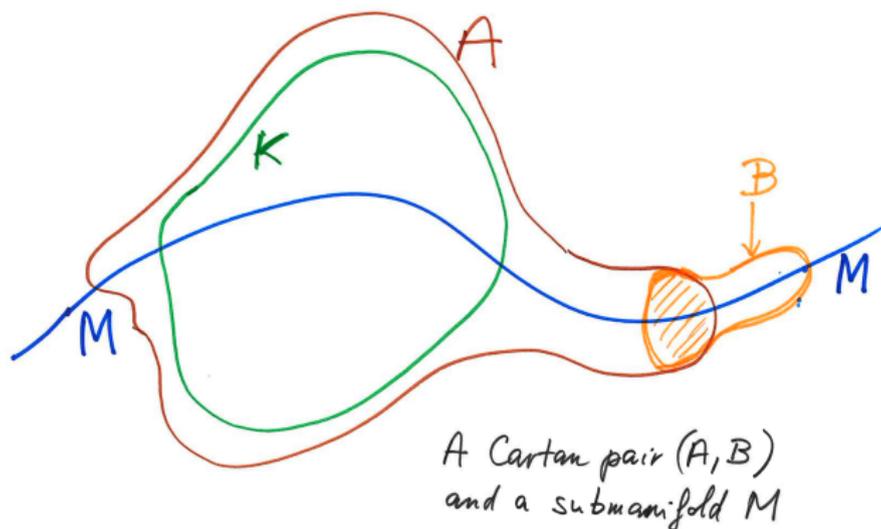
$$\begin{array}{ccc}
 \nu_{M,X} & \xrightarrow{\Theta} & \nu_{N,Y} \\
 \downarrow & & \downarrow \\
 M & \xrightarrow{f} & N
 \end{array}$$

The special case with $K = \emptyset$ of this result coincides with the **Docquier–Grauert tubular neighbourhood theorem** (1960), which asserts that a neighbourhood of a Stein submanifold M in a complex manifold X is biholomorphic to a neighbourhood of the zero section in the normal bundle $\nu_{M,X}$. (The ambient manifold X may be arbitrary due to Siu's theorem.)

The main theorem now follows by applying Theorem C to the map $F : K_1 \cup M \rightarrow \mathbb{C}^n$ constructed above.

Proof of generalized D-G theorem

We prove the generalized Docquier–Grauert theorem by stepwise extending the map $F : K_1 \cup M \rightarrow F(K_1) \cup N \subset \mathbb{C}^n$ to a biholomorphic map on a neighbourhood of $K \cup M$. Every step uses the gluing lemma with interpolation on a Cartan pair as in the following illustration.



Gluing biholomorphic maps with interpolation

The basic form of the following result was proved by myself in 2002.
The interpolation condition is a nontrivial addition.

Theorem (Splitting lemma with interpolation)

Let X be a complex manifold with a distance function dist , M be a closed complex submanifold of X , and (A, B) be a Cartan pair in X .

Given an open set $\tilde{C} \subset X$ containing C , there are open sets $A' \supset A$, $B' \supset B$ with $C' = A' \cap B' \subset \tilde{C}$ and constants $\epsilon_0 > 0$, $c_0 > 0$ such that the following holds.

For every injective holomorphic map $\gamma : \tilde{C} \rightarrow X$ with

$$\epsilon = \text{dist}_{\tilde{C}}(\gamma, \text{Id}) < \epsilon_0 \quad \text{and} \quad \gamma|_{\tilde{C} \cap M} = \text{Id}$$

there exist injective holomorphic maps $\alpha : A' \rightarrow X$, $\beta : B' \rightarrow X$ which depend continuously on γ , they agree with the identity map on $A' \cap M$ and $B' \cap M$, respectively, and satisfy

$$\gamma \circ \alpha = \beta \text{ on } C', \quad \text{dist}_{A'}(\alpha, \text{Id}) < c_0 \epsilon, \quad \text{dist}_{B'}(\beta, \text{Id}) < c_0 \epsilon.$$

THANK YOU

FOR YOUR ATTENTION