Hyperbolic domains in real Euclidean spaces

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Abstract

Let Ω be a domain in $\mathbb{R}^n$, $n \geq 3$. We introduce an intrinsic Kobayashi-type (Finsler) **minimal pseudometric** $g_\Omega : T\Omega = \Omega \times \mathbb{R}^n \to \mathbb{R}_+$ defined in terms of conformal harmonic discs. Such discs parameterize minimal surfaces in $\mathbb{R}^n$.

Its integrated form is the **minimal pseudodistance** $\rho_\Omega : \Omega \times \Omega \to \mathbb{R}_+$, also defined by chains of conformal harmonic discs.

On the unit ball $\mathbb{B}^n$, $g_{\mathbb{B}^n}$ coincides with the **Cayley–Klein metric**, one of the classical models of hyperbolic geometry.

I shall present several sufficient conditions for a domain $\Omega \subset \mathbb{R}^n$ to be (complete) hyperbolic, meaning that g_Ω is a (complete) metric; equivalently, ρ_Ω is a (complete) distance function.

F. F. & David Kalaj, Hyperbolicity theory for conformal minimal surfaces in $\mathbb{R}^n$. `https://arxiv.org/abs/2102.12403`**, February 2021**

Barbara Drinovec Drnovšek and F. F., Hyperbolic domains in real Euclidean spaces. `https://arxiv.org/abs/2109.06943`**, Sept 2021. To appear in Pure and Appl. Math. Quarterly.**

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disc, and let Ω be a domain in $\mathbb{R}^n$.

Let $\mathrm{CH}(\mathbb{D}, \Omega)$ denote the space of conformal harmonic discs $f : \mathbb{D} \to \Omega$:

$$f_x \cdot f_y = 0, \quad |f_x| = |f_y|; \qquad z = x + \mathrm{i}y \in \mathbb{D}.$$

Fix a pair of points $\mathbf{x}, \mathbf{y} \in \Omega$ and consider finite chains of discs $f_i \in \mathrm{CH}(\mathbb{D}, \Omega)$ and points $a_i \in \mathbb{D}$ $(i = 1, \ldots, k)$ such that

$$f_1(0) = \mathbf{x}, \quad f_{i+1}(0) = f_i(a_i) \text{ for } i = 1, \ldots, k-1, \quad f_k(a_k) = \mathbf{y}.$$

To any such chain we associate the number

$$\sum_{i=1}^{k} \frac{1}{2} \log \frac{1 + |a_i|}{1 - |a_i|} \geq 0.$$

The pseudodistance $\rho_\Omega : \Omega \times \Omega \to \mathbb{R}_+$ is the infimum of the numbers obtained in this way. Clearly it satisfies the triangle inequality.

If $\Omega \subset \mathbb{C}^n$ and we use holomorphic discs, we get the **Kobayashi pseudodistance** $\mathcal{K}_\Omega$ (S. Kobayashi, 1967). Hence, $\rho_\Omega \leq \mathcal{K}_\Omega$. These pseudodistances agree on domains in $\mathbb{C}$, but strict inequality holds if $n > 1$.

The minimal pseudometric

Define a Finsler pseudometric $g_\Omega : \Omega \times \mathbb{R}^n \to \mathbb{R}_+$ on $(\mathbf{x}, \mathbf{v}) \in \Omega \times \mathbb{R}^n$ by

$$g_\Omega(\mathbf{x}, \mathbf{v}) = \inf\{1/r > 0 : \exists f \in \mathrm{CH}(\mathbb{D}, \Omega),\ f(0) = \mathbf{x},\ f_x(0) = r\mathbf{v}\}.$$

Clearly, g_Ω is upper-semicontinuous and absolutely homogeneous:

$$g_\Omega(\mathbf{x}, t\mathbf{v}) = |t|\, g_\Omega(\mathbf{x}, \mathbf{v}) \quad \text{for} \quad t \in \mathbb{R}.$$

If $\Omega \subset \mathbb{C}^n$ and using only holomorphic disc gives the Kobayashi pseudometric.

Theorem

The minimal pseudodistance ρ_Ω is obtained by integrating g_Ω:

$$\rho_\Omega(\mathbf{x}, \mathbf{y}) = \inf_\gamma \int_0^1 g_\Omega(\gamma(t), \dot\gamma(t))\, dt, \qquad \mathbf{x}, \mathbf{y} \in \Omega.$$

The infimum is over piecewise smooth paths $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = \mathbf{x}$ and $\gamma(1) = \mathbf{y}$.

The elementary proof is similar to the one for the Kobayashi pseudometric.

Metric decreasing properties

A conformal surface M is **hyperbolic** if its universal covering space is the disc $\mathbb{D}$. Such a surface carries the **Poincaré metric**, $\mathcal{P}_M$, the unique Riemannian metric such that any conformal covering map $h : \mathbb{D} \to M$ is an isometry from $(\mathbb{D}, \mathcal{P}_{\mathbb{D}})$ onto $(M, \mathcal{P}_M)$. The Poincaré metric on $\mathbb{D}$ is

$$\mathcal{P}_{\mathbb{D}}(z, \xi) = \frac{|\xi|}{1 - |z|^2}, \qquad z \in \mathbb{D},\ \xi \in \mathbb{C}.$$

For every conformal harmonic map $f : \mathbb{D} \to \Omega$ we have that

$$g_\Omega(f(z), df_z(\xi)) \le \mathcal{P}_{\mathbb{D}}(z, \xi), \qquad z \in \mathbb{D},\ \xi \in \mathbb{C},$$

and g_Ω **is the largest pseudometric on Ω with this property.**

For $z = 0$ this is immediate from the definition of g_Ω. For other points, we precompose f by $\phi \in \mathrm{Aut}(\mathbb{D})$ interchanging z and 0.

The same holds for conformal harmonic maps $(M, \mathcal{P}_M) \to (\Omega, g_\Omega)$.

Any rigid map $R : \mathbb{R}^n \to \mathbb{R}^m$ $(n \le m)$ with $R(\Omega) \subset \Omega'$ is metric-decreasing:

$$g_{\Omega'}(R(\mathbf{x}), R(\mathbf{v})) \le g_\Omega(\mathbf{x}, \mathbf{v}), \qquad \mathbf{x} \in \Omega,\ \mathbf{v} \in \mathbb{R}^n.$$

In particular,

$$\Omega \subset \Omega' \implies g_\Omega \geq g_{\Omega'}.$$

We also introduce a Finsler pseudometric on $\Omega \times \mathbb{G}_2(\mathbb{R}^n)$, where $\mathbb{G}_2(\mathbb{R}^n)$ denotes the Grassmann manifold of 2-planes in $\mathbb{R}^n$, by

$$\mathcal{M}_\Omega(\mathbf{x}, \Lambda) = \inf\{1/\|df_0\| : f \in \mathrm{CH}(\mathbb{D}, \Omega),\ f(0) = \mathbf{x},\ df_0(\mathbb{R}^2) = \Lambda\}.$$

Here, $\|df_0\|$ denotes the operator norm of the differential $df_0 : \mathbb{R}^2 \to \mathbb{R}^n$.

It clearly follows that for any vector $\mathbf{v} \in \mathbb{R}^n$ we have

$$g_\Omega(\mathbf{x}, \mathbf{v}) = |\mathbf{v}| \cdot \inf\left\{\mathcal{M}_\Omega(\mathbf{x}, \Lambda) : \Lambda \in \mathbb{G}_2(\mathbb{R}^n), \mathbf{v} \in \Lambda\right\}.$$

Note that the 2-planes Λ containing a given vector $v \neq 0$ form an $(n-2)$-sphere. This is an important difference with respect to the Kobayashi metric — a vector $0 \neq \mathbf{v} \in \mathbb{C}^n$ determines a unique complex line Λ.

The Cayley–Klein metric on the ball $\mathbb{B}^n$ of $\mathbb{R}^n$ for $n \geq 3$

Theorem (F.–Kalaj 2021)

The minimal metric $g_{\mathbb{B}^n}$ on the unit ball $\mathbb{B}^n$ equals the **Cayley–Klein metric***:*

$$g_{\mathbb{B}^n}(\mathbf{x}, \mathbf{v})^2 = \frac{(1 - |\mathbf{x}|^2)|\mathbf{v}|^2 + |\mathbf{x} \cdot \mathbf{v}|^2}{(1 - |\mathbf{x}|^2)^2} = \frac{|\mathbf{v}|^2}{1 - |\mathbf{x}|^2} + \frac{|\mathbf{x} \cdot \mathbf{v}|^2}{(1 - |\mathbf{x}|^2)^2}.$$

We also have that

$$g_{\mathbb{B}^n}(\mathbf{x}, \mathbf{v}) = \frac{\sqrt{1 - |\mathbf{x}|^2 \sin^2 \phi}}{1 - |\mathbf{x}|^2} |\mathbf{v}|, \qquad \mathbf{x} \in \mathbb{B}^n, \ \mathbf{v} \in \mathbb{R}^n,$$

where $\phi \in [0, \pi/2]$ is the angle between the vector $\mathbf{v}$ and the line $\mathbb{R}\mathbf{x} \subset \mathbb{R}^n$, and

$$\mathcal{M}_{\mathbb{B}^n}(\mathbf{x}, \Lambda) = \frac{\sqrt{1 - |\mathbf{x}|^2 \sin^2 \theta}}{1 - |\mathbf{x}|^2} |\mathbf{v}|, \qquad \mathbf{x} \in \mathbb{B}^n, \ \Lambda \in \mathbb{G}_2(\mathbb{R}^n),$$

where $\theta \in [0, \pi/2]$ is the angle between the plane Λ and the line $\mathbb{R}\mathbf{x} \subset \mathbb{R}^n$.

The **Beltrami–Cayley–Klein model of hyperbolic geometry** was introduced and studied by **Arthur Cayley (1859)**, **Eugenio Beltrami (1868)**, and **Felix Klein (1871–73)**.

The underlying space is the unit ball, geodesics are straight line segments with endpoints on the boundary sphere, and the distance between points on a geodesic is given by the cross ratio.

This metric is the restriction of the *Kobayashi metric* (or, up to a scalar multiple, of the **Bergman metric**) on the complex ball $\mathbb{B}^n_{\mathbb{C}} \subset \mathbb{C}^n$ to points in $\mathbb{B}^n = \mathbb{B}^n_{\mathbb{C}} \cap \mathbb{R}^n$ and vectors in $\mathbb{R}^n$.

It is a special case of the metric on convex domains in $\mathbb{R}^n$ which was introduced and studied by **David Hilbert** in 1885.

Hyperbolic domains

Definition

A domain $\Omega \subset \mathbb{R}^n$ for $n \geq 3$ is **hyperbolic** if ρ_Ω is a distance function, and is **complete hyperbolic** if (Ω, ρ_Ω) is a complete metric space.

Example

(A) The ball $\mathbb{B}^n \subset \mathbb{R}^n$, $n \geq 3$, is complete hyperbolic.

(B) Every bounded domain $\Omega \subset \mathbb{R}^n$ is hyperbolic since it is contained in a ball. However, it need not be complete hyperbolic. For example, if $b\Omega$ is smooth and contains a strongly concave boundary point $\mathbf{p} \in b\Omega$, there is a conformal linear disc $\Sigma \subset \Omega \cup \{\mathbf{p}\}$ containing $\mathbf{p}$. Then, $\mathbf{p}$ is at finite ρ_Ω-distance from Ω.

(C) The half-space $\mathbb{H}^n = \{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 > 0\}$ is not hyperbolic since the pseudodistance $\rho_{\mathbb{H}^n}$ vanishes on planes $x_1 = const$. However, we will show that the minimal distance to the hyperplane $b\mathbb{H} = \{x_1 = 0\}$ is infinite.

Basic properties of hyperbolic domains

Theorem

The following conditions are equivalent for a domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$.

1. (i) *The family $\mathrm{CH}(\mathbb{D}, \Omega)$ of conformal harmonic discs $\mathbb{D} \to \Omega$ is pointwise equicontinuous for some metric ρ on Ω inducing its natural topology.*
2. (ii) *Every point $\mathbf{p} \in \Omega$ has a neighbourhood $U \subset \Omega$ and $c > 0$ such that*
$$g_\Omega(\mathbf{x}, \mathbf{u}) \geq c|\mathbf{u}|, \qquad \mathbf{x} \in U, \ \mathbf{u} \in \mathbb{R}^n.$$
3. (iii) *Ω is hyperbolic.*
4. (iv) *The minimal distance ρ_Ω induces the standard topology of Ω.*

A domain $\Omega \subset \mathbb{R}^n$ is called **taut** if $\mathrm{CH}(\mathbb{D}, \Omega)$ is a normal family.

Theorem

The following hold for any domain Ω in $\mathbb{R}^n$, $n \geq 3$:

$$\textit{complete hyperbolic} \implies \textit{taut} \implies \textit{hyperbolic}$$

Theorem

The following are equivalent for a convex domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$.

1. Ω *is complete hyperbolic.*
2. Ω *is hyperbolic.*
3. Ω *does not contain any 2-dimensional affine subspaces.*
4. Ω *is contained in the intersection of $n-1$ halfspaces determined by linearly independent linear functionals.*

For comparison: A convex domain in $\mathbb{C}^n$ is Kobayashi hyperbolic if and only if it does not contain any affine complex line (Barth (1980), Harris (1979)).

The main implication is $(iv) \Rightarrow (i)$. We first show that **the minimal distance to an affine hyperplane is infinite.** This follows from the Schwarz lemma for positive harmonic functions $f : \mathbb{D} \to (0, +\infty) : \; |\nabla f(0)| \leq 2f(0)$. For $\mathbb{H}^n = \{x_1 > 0\}$ this gives

$$g_{\mathbb{H}^n}((x_1, \ldots, x_n), (v_1, \ldots, v_n)) \geq \frac{|v_1|}{2x_1}.$$

For any path $\gamma(t) = (\gamma_1(t), \ldots) \in \mathbb{H}^n$, $t \in [0, 1)$, it follows that

$$\int_0^1 g_{\mathbb{H}^n}(\gamma(t), \dot\gamma(t))\, dt \geq \int_0^1 \frac{|\dot\gamma_1(t)|}{2\gamma_1(t)}\, dt.$$

If $\gamma(t) \to 0$ or $\gamma(t) \to +\infty$ as $t \to 1$ then the integral is $+\infty$.

Hyperbolicity of convex domains, 2

Hence, **a convex domain is locally complete hyperbolic at every boundary point.**

Assume that Ω satisfies condition (iv). Up to a translation and rotation, there are linearly independent unit vectors $\mathbf{y}_1, \ldots, \mathbf{y}_{n-1} \in \mathbb{R}^{n-1} \times \{0\}$ such that

$$\Omega \subset \bigcap_{i=1}^{n-1} \mathbb{H}_i \quad \text{where} \quad \mathbb{H}_i = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \cdot \mathbf{y}_i > 0\} \quad \text{for} \quad i = 1, \ldots, n-1.$$

Let $\mathbf{x}(t) = (\mathbf{x}'(t), x_n(t)) \in \Omega$ $(t \in [0,1))$ be a divergent path. Set

$$x_i(t) := \mathbf{x}(t) \cdot \mathbf{y}_i = \mathbf{x}'(t) \cdot \mathbf{y}_i > 0 \quad \text{for} \quad i = 1, \ldots, n-1, \ t \in [0,1).$$

If $\mathbf{x}(t)$ clusters at some point $\mathbf{p} \in b\Omega$ as $t \to 1$, then $\mathbf{x}(t)$ has infinite g_Ω-length. Likewise, if one of the functions $x_i(t)$ for $i = 1, \ldots, n-1$ clusters at $+\infty$, then the path $\mathbf{x}(t)$ has infinite minimal length in $\mathbb{H}_i$, and hence also in $\Omega \subset \mathbb{H}_i$.

It remains to consider the case when the functions $x_i(t)$ are bounded,

$$0 < \mathbf{x}(t) \cdot \mathbf{y}_i \le c_1 \quad \text{for} \quad i = 1, \ldots, n-1, \ t \in [0,1) \tag{1}$$

and the path $\mathbf{x}(t) \in \Omega$ does not cluster anywhere on $b\Omega$. In this case, the last component $x_n(t) \in \mathbb{R}$ of $\mathbf{x}(t)$ clusters at $\pm\infty$ as $t \to 1$, and hence $\int_0^1 |\dot{x}_n(t)| dt = +\infty$. To see that the path $\mathbf{x}(t)$ has infinite g_Ω-length, it suffices to show that

$$g_\Omega(\mathbf{x}(t), \dot{\mathbf{x}}(t)) \ge c_2 |\dot{x}_n(t)|,$$

where $c_2 > 0$ only depends on $c_1 > 0$ and the vectors $\mathbf{y}_1, \ldots, \mathbf{y}_{n-1}$.

Hyperbolicity of convex domains, 3

Fix a point $\mathbf{x} \in \Omega$ satisfying (1) and a unit vector $\mathbf{v} = (\mathbf{v}', v_n) \in \mathbb{R}^n$, and consider a conformal harmonic map $f = (f_1, f_2, \ldots, f_n) : \mathbb{D} \to \Omega$ such that $f(0) = \mathbf{x}$ and $f_x(0) = r\mathbf{v}$ for some $r > 0$. Then, $f_y(0) = r\mathbf{w} = r(\mathbf{w}', w_n)$ where $(\mathbf{v}, \mathbf{w})$ is an orthonormal frame:

$$0 = \mathbf{v} \cdot \mathbf{w} = \mathbf{v}' \cdot \mathbf{w}' + v_n w_n, \qquad |\mathbf{v}| = |\mathbf{w}| = 1.$$

From this and the Cauchy–Schwarz inequality it follows that

$$v_n^2(1 - |\mathbf{w}'|^2) = v_n^2 w_n^2 = |\mathbf{v}' \cdot \mathbf{w}'|^2 \le |\mathbf{v}'|^2 |\mathbf{w}'|^2 = (1 - v_n^2)|\mathbf{w}'|^2,$$

and hence

$$|v_n| \le |\mathbf{w}'| \le c_3 \max_{i=1,\ldots,n-1} |\mathbf{w} \cdot \mathbf{y}_i|$$

where $c_3 > 0$ depends on the vectors $\mathbf{y}_1, \ldots, \mathbf{y}_{n-1} \in \mathbb{R}^{n-1} \times \{0\}$. Therefore,

$$r|v_n| \le c_3 \max_{i=1,\ldots,n-1} r|\mathbf{w} \cdot \mathbf{y}_i| \le 2c_3 \max_{i=1,\ldots,n-1} \mathbf{x} \cdot \mathbf{y}_i,$$

where the second estimate follows from the Schwarz lemma applied to the conformal harmonic disc $z \mapsto \tilde{f}(z) = f(\mathrm{i}z)$ in each of the half-spaces $\mathbb{H}_i$. (Note that $\tilde{f}(0) = \mathbf{x}$ and $\tilde{f}_x(0) = f_y(0) = r\mathbf{w}$.) Together with the assumption (1) this gives

$$g_\Omega(\mathbf{x}, \mathbf{v}) \ge \frac{1}{r} \ge \frac{|v_n|}{2c_3 \max_{i=1,\ldots,n-1} \mathbf{x} \cdot \mathbf{y}_i} \ge \frac{|v_n|}{2c_1 c_3} = c_2 |v_n|,$$

Applying this with $\mathbf{x} = \mathbf{x}(t)$ and $\mathbf{v} = \dot{\mathbf{x}}(t)$ yields $g_\Omega(\mathbf{x}(t), \dot{\mathbf{x}}(t)) \ge c_2 |\dot{x}_n(t)|$, proving that Ω is complete hyperbolic.

Minimal plurisubharmonic functions ...

Let Ω be a domain in $\mathbb{R}^n$. An upper-semicontinuous function $u : \Omega \to [-\infty, +\infty)$ is said to be **minimal plurisubharmonic, MPSH**, if for every affine 2-plane $L \subset \mathbb{R}^n$ the restriction $u : L \cap \Omega \to [-\infty, +\infty)$ is subharmonic (in conformal affine coordinates on L).

A function $u \in \mathscr{C}^2(\Omega)$ is MPSH if and only if

$$\Delta(u|_{\mathbf{x}+\Lambda})(\mathbf{x}) = \mathrm{tr}_\Lambda \mathrm{Hess}_u(\mathbf{x}) \geq 0 \quad \text{for every} \quad (\mathbf{x}, \Lambda) \in \Omega \times \mathbb{G}_2(\mathbb{R}^n),$$

and this holds if and only if

$$(*) \qquad \lambda_1(\mathbf{x}) + \lambda_2(\mathbf{x}) \geq 0 \quad \text{for all } \mathbf{x} \in \Omega,$$

where $\lambda_1(\mathbf{x}), \lambda_2(\mathbf{x})$ denote the smallest eigenvalues of $\mathrm{Hess}_u(\mathbf{x})$.

We say that $u \in \mathscr{C}^2(\Omega)$ is **strongly minimal plurisubharmonic** if strong inequality holds in (*).

This class of functions was studied by **Harvey and Lawson** in a series of papers. Their use in the theory of minimal surfaces is summarized in my monograph with **Alarcón** and **López** (Minimal surfaces from a complex analytic viewpoint, Springer, 2021).

Proposition

An upper-semicontinuous function $u : \Omega \to [-\infty, +\infty)$ is MPSH if and only if for each conformal harmonic map $f : M \to \Omega$ from a conformal surface the function $u \circ f : M \to \mathbb{R}$ is subharmonic. If $u \in \mathscr{C}^2(\Omega)$ is strongly MPSH and f is an immersion, then $u \circ f$ is strongly subharmonic on M.

For functions $u \in \mathscr{C}^2(\Omega)$ this follows from the following formula, which holds for every conformal harmonic map $f : \mathbb{D} \to \Omega$:

$$\Delta(u \circ f)(z) = \mathrm{tr}_{df_z(\mathbb{R}^2)} \mathrm{Hess}_u(f(z)) \cdot \|df_z\|^2, \quad z \in \mathbb{D}.$$

Lemma

Let $\mathbf{x}$ be the Euclidean coordinate on $\mathbb{R}^n$, $n \geq 3$.

(a) *The function $\log|\mathbf{x}|$ is MPSH on $\mathbb{R}^n$.*

(b) *If u is MPSH on $\Omega \subset \mathbb{R}^n$ then for any $\mathbf{p} \in \Omega$ the function $\mathbf{x} \mapsto |\mathbf{x} - \mathbf{p}|^2 e^{u(\mathbf{x})}$ and its logarithm are MPSH on Ω.*

Minimally convex domains

A domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, with smooth boundary is **minimally convex** if admits a defining function ρ such that

$$(*) \quad \mathrm{tr}_\Lambda \mathrm{Hess}_\rho(\mathbf{p}) \geq 0 \ \text{ for every } \mathbf{p} \in b\Omega \text{ and 2-plane } \Lambda \subset T_\mathbf{p} b\Omega.$$

The domain Ω is **strongly minimally convex** if strict inequality holds.

Condition (*) says that $b\Omega$ has nonnegative (resp. positive) mean sectional curvature on every tangent 2-plane. This holds if and only if the principal normal curvatures $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_{n-1}$ of $b\Omega$ at $\mathbf{p} \in b\Omega$ satisfy

$$\nu_1 + \nu_2 \geq 0 \quad (\text{resp. } \nu_1 + \nu_2 > 0).$$

A domain in $\mathbb{R}^3$ bounded by a minimal surface is minimally convex.

Alarcón, Drinovec Drnovšek, F., López 2019: Every bordered Riemann surface admits many proper conformal harmonic immersions into an arbitrary minimally convex domain.

F. 2022: A bounded (strongly) minimally convex domain $\Omega \subset \mathbb{R}^n$ admits a defining function u which is (strongly) MPSH on $\overline{\Omega} = \{u \leq 0\}$.

Strongly minimally convex domains are complete hyperbolic

Theorem

Every bounded strongly minimally convex domain is complete hyperbolic.

This is an analogue of **Graham's theorem** (1975) that bounded strongly pseudoconvex domains in $\mathbb{C}^n$ are complete Kobayashi hyperbolic.

Conversely: if $\nu_1 + \nu_2 < 0$ at some point $\mathbf{p} \in b\Omega$ then $\mathbf{p}$ is at finite minimal distance from the interior. In this case there exists an embedded conformal harmonic disc $f : \mathbb{D} \to \Omega \cup \{\mathbf{p}\}$ with $f(0) = \mathbf{p}$ and $f(\mathbb{D}^*) \subset \Omega$.

Corollary

If M is an embedded surface in $\mathbb{R}^3$ such that the minimal distance to any point $\mathbf{p} \in M$ is infinite, then M is a minimal surface.

Problem

Is the minimal distance to an embedded minimal surface $M \subset \mathbb{R}^3$ infinite?

A pseudometric defined by MPSH functions

Our proof uses the existence of a strongly minimally plurisubharmonic defining function and an **analogue of the Sibony metric** in this category.

We define the pseudometric $\mathcal{F}_\Omega : \Omega \times \mathbb{G}_2(\mathbb{R}^n) \to \mathbb{R}_+$ by

$$\mathcal{F}_\Omega(\mathbf{x}, \Lambda) = \frac{1}{2} \sup_u \sqrt{\mathrm{tr}_\Lambda \mathrm{Hess}_u(\mathbf{x})}, \quad \mathbf{x} \in \Omega, \ \Lambda \in \mathbb{G}_2(\mathbb{R}^n),$$

where the supremum is over all MPSH functions $u : \Omega \to [0, 1]$ such that u is of class $\mathscr{C}^2$ near $\mathbf{x}$, $u(\mathbf{x}) = 0$, and $\log u$ is MPSH on Ω.

The **Sibony metric** is defined in the same way, using log-plurisubharmonic functions on domains in $\mathbb{C}^n$ and complex lines $\Lambda \subset \mathbb{C}^n$.

The main point is that $\mathcal{F}_\Omega$ gives a lower bound for the minimal pseudometric:

Proposition

For any domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, we have that $\mathcal{F}_\Omega \leq \mathcal{M}_\Omega$.

Proof of the proposition

Fix $(\mathbf{x}, \Lambda) \in \Omega \times \mathbb{G}_2(\mathbb{R}^n)$. Let $f \in \mathrm{CH}(\mathbb{D}, \Omega)$ be such that $f(0) = \mathbf{x}$ and $df_0(\mathbb{R}^2) = \Lambda$. Let $u : \Omega \to [0,1]$ be as in the definition of $\mathcal{F}_\Omega$.

The function $v := u \circ f : \mathbb{D} \to [0,1]$ is then subharmonic, of class $\mathscr{C}^2$ near the origin, $v(0) = 0$, and $\log v = \log u \circ f : \mathbb{D} \to [-\infty, 0)$ is also subharmonic.

By **Sibony (1981)** we have that

$$\Delta v(0) \le 4.$$

(The unique extremal function with $\Delta v(0) = 4$ is $v(x + \imath y) = x^2 + y^2$.) Hence,

$$\mathrm{tr}_\Lambda \mathrm{Hess}_u(\mathbf{x}) \cdot \|df_0\|^2 = \Delta v(0) \le 4.$$

Equivalently,

$$\frac{1}{2}\sqrt{\mathrm{tr}_\Lambda \mathrm{Hess}_u(\mathbf{x})} \le \frac{1}{\|df_0\|}.$$

The supremum of the left hand side over all admissible functions u equals $\mathcal{F}_\Omega(\mathbf{x}, \Lambda)$, while the infimum of the right hand side over all conformal harmonic discs f as above equals $\mathcal{M}_\Omega(\mathbf{x}, \Lambda)$. Hence, $\mathcal{F}_\Omega \le \mathcal{M}_\Omega$.

Sketch of proof of the theorem on complete hyperbolicity

We use the above proposition with MPSH function of the form

$$\Psi(\mathbf{y}) = \theta\left(r^{-2}|\mathbf{y}-\mathbf{x}|^2\right) e^{\lambda u(\mathbf{y})}, \qquad \mathbf{y} \in \Omega,$$

where $\theta : [0, \infty) \to [0, 1]$ is a smooth increasing function such that

$$\theta(t) = t \ \text{ for } 0 \le t \le \frac{1}{2}, \qquad \theta(t) = 1 \ \text{ for } t \ge 1,$$

u is a strongly MPSH defining functions for Ω, $\mathbf{x} \in \Omega$, and $r > 0$ and $\lambda > 0$ are suitably chosen constants. In this way, we show that

$$g_\Omega(\mathbf{x}, \mathbf{v}) \ge C \frac{|\mathbf{v}|}{\sqrt{\operatorname{dist}(\mathbf{x}, b\Omega)}}, \qquad \mathbf{x} \in \Omega, \ \mathbf{v} \in \mathbb{R}^n.$$

To show completeness of g_Ω we need a stronger estimate

$$g_\Omega(\mathbf{x}, \mathbf{v}) \ge C \frac{|\mathbf{v}|}{\operatorname{dist}(\mathbf{x}, b\Omega)} \tag{2}$$

for vectors $\mathbf{v}$ which are normal to $b\Omega$ at the closest point $\mathbf{p} \in b\Omega$ to $\mathbf{x}$.

Sketch of proof, 2

We follow Ivashkovich and Rosay (2004). The existence of a local negative strongly MPSH peak function, and also of the MPSH anti-peak functions $\mathbf{z} \mapsto \log|\mathbf{x} - \mathbf{p}|$ at points $\mathbf{p} \in b\Omega$, implies that for some $c > 0$ we have

$$|\nabla f(z)| \leq c\sqrt{|u(f(0))|} \approx \sqrt{\mathrm{dist}(f(0), b\Omega)}, \quad |z| \leq \frac{1}{2} \tag{3}$$

for every $f \in \mathrm{CH}(\mathbb{D}, \Omega)$ whose centre $f(0)$ is close enough to $b\Omega$. (This amounts to a **localization argument**, showing that most of the disc is mapped by f close to $f(0)$, and then applying the Schwarz lemma for bounded harmonic functions.) This gives

$$\begin{aligned} |\Delta(u \circ f)(z)| &= |\mathrm{tr}_{df_z(\mathbb{R}^2)} \mathrm{Hess}_u(f(z))| \cdot \|df_z\|^2 \\ &\leq c_1 |\nabla f(z)|^2 \leq C_1 |u(f(0))|, \quad |z| \leq \frac{1}{2} \end{aligned}$$

for some constant $c_1 > 0$ and $C_1 = c^2 c_1 > 0$. We claim that this gives

$$|\nabla(u \circ f)(0)| \leq C_2 |u(f(0))|, \quad f \in \mathrm{CH}(\mathbb{D}, \Omega), \tag{4}$$

which implies (2) and hence establishes complete hyperbolicity of Ω. (Note that $u \circ f$ is essentially the normal component of f.)

Proof of (4)

By rescaling we may assume that (3) holds for all $z \in \mathbb{D}$.
Set $v = u \circ f : \mathbb{D} \to (-\infty, 0)$, so we have that

$$|\Delta v(z)| \leq C_1 |v(0)|, \qquad z \in \mathbb{D}.$$

We extend Δv to $\mathbb{C}$ by setting it equal to 0 on $\mathbb{C} \setminus \overline{\mathbb{D}}$. The function

$$g(z) = v(z) - \Big(\frac{1}{2\pi} \log|\cdot| * \Delta v\Big)(z) - C_1|v(0)|, \quad z \in \mathbb{D}$$

is then harmonic on $\mathbb{D}$. Note that

$$\left|\frac{1}{2\pi} \log|\cdot| * \Delta v\right| \leq C_1|v(0)|.$$

Hence, $g \leq v < 0$ on $\mathbb{D}$ and $|g(0)| < (2C_1 + 1)|v(0)|$. Schwarz lemma for negative harmonic functions gives $|\nabla g(0)| \leq 2|g(0)|$, and hence

$$|\nabla v(0)| \leq |\nabla g(0)| + \sup_{\mathbb{D}} |\Delta v| \leq 2|g(0)| + C_1|v(0)| \leq (5C_1 + 2)|v(0)|.$$

This is the estimate (3) with $C = 5C_1 + 2$.

$\sim$ **Thank you for your attention** $\sim$

The castle of Ljubljana and the mountains of Kamnik
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