New developmens on Oka manifolds

Franc Forstnerič

Univerza v Ljubljani

Complex Geometry Workshop
Ruhr University Bochum
5 September 2022

Flexibility versus rigidity in complex geometry

A central question in complex geometry is to understand the space of holomorphic maps $X \to Y$ between a pair of complex manifolds. Are there many maps, or few maps? Which properties can such maps have?

There are many maps $\mathbb{C} \to \mathbb{C}$, but there are no nonconstant holomorphic maps $\mathbb{C} \to \mathbb{C} \setminus \{0, 1\} = Y$. Such manifolds Y are called **(Brody) hyperbolic**.

***** HYPERBOLICITY IS AN OBSTRUCTION THEORY *****

On the opposite side, **Oka theory** is about complex manifolds Y which admit many holomorphic maps $X \to Y$ from any affine complex (**Stein**) manifold X. It developed from works of **Oka, Grauert, Gromov**, and others.

***** OKA THEORY IS AN EXISTENCE THEORY *****

It provides solutions to a wide variety of analytic problems in the absence of topological obstructions.

Oka manifolds

A complex manifold Y is called an **Oka manifold** if maps $X \to Y$ from any Stein manifold (or a reduced Stein space) X satisfy the following conditions:

- every continuous map $f : X \to Y$ can be homotopically deformed to a holomorphic map $\tilde{f} : X \to Y$.
- If the map $f : X \to Y$ is holomorphic on a compact $\mathscr{O}(X)$-convex set K and on a closed complex subvariety X' of X, then there is a homotopy from f to $\tilde{f}$ consisting of maps which are holomorphic near K and close to f on K, and which agree with f on X'.
- The analogous approximation and (jet) interpolation properties hold for families of maps $X \to Y$ depending continuously on a parameter.

Oka manifolds are like bitcoins — precious but hard to discover

- **Oka 1939, Grauert 1958** Complex Lie groups and their homogeneous manifolds are Oka.

- **Gromov 1989** Every elliptic complex manifold is an Oka manifold.

  A complex manifold Y is said to be elliptic if it admits a holomorphic vector bundle $\pi : E \to Y$ and a **dominating holomorphic spray** $F : E \to Y$ such that for every point $y \in Y$ we have that $F(0_y) = y$ and

  $$dF_{0_y} : T_{0_y} E \to T_y Y \text{ maps the fibre } E_y \text{ onto the tangent space } T_y Y.$$

- In particular, if Y admits complete holomorphic vector fields which span the tangent space at every point, then Y is elliptic and hence Oka.

- **F. 2006** A complex manifold Y is Oka iff it satisfies the following

  Convex approximation property: Every holomorphic map $K \to Y$ from a compact convex set in a Euclidean space $\mathbb{C}^n$ is a limit of entire maps $\mathbb{C}^n \to Y$.

Kusakabe's characterization of Oka manifolds

Y. Kusakabe 2021 A complex manifold Y is an Oka manifold if and only if every holomorphic map $f : L \to Y$ from (a neighbourhood of) a compact convex set $L \subset \mathbb{C}^N$ is the core map of a dominating holomorphic spray $F : L \times \mathbb{C}^n \to Y$ for some $n \geq \dim Y$, i.e., such that

$$F(\cdot\,,0) = f \text{ and } \quad \frac{\partial}{\partial z}\Big|_{z=0} F(\zeta, z) : \mathbb{C}^n \to T_{f(\zeta)}Y \text{ is surjective for every } \zeta \in L.$$

This is a restricted version (to compact convex sets) of condition Ell_1 studied by Gromov in 1989. Kusakabe gave a short but ingenious proof that Ell_1 implies CAP; the rest was known before. However, Ell_1 is often easier to verify. As an application of this and an old result of mine, Kusakabe proved

The localization theorem for Oka manifolds:
If a complex manifold $Y = \bigcup_i Y \setminus A_i$ is a union of Zariski-open Oka domains $Y \setminus A_i$, with A_i a closed complex subvariety of Y, then Y is Oka.

These results furnished many new examples of Oka manifolds. In this talk we shall describe some further ones among domains in Euclidean and projective spaces.

Which domains in $\mathbb{C}^n$ are Oka?

- Until Kusakabe's work the only known examples of Oka domains in $\mathbb{C}^n$ were FB-domains, complements of complex hyperplanes, and complements of closed tame (in particular, algebraic) subvarieties of dimension $\leq n-2$.

- **Kusakabe 2021** For every compact polynomially convex set $K \subset \mathbb{C}^n$ for $n > 1$, the complement $\mathbb{C}^n \setminus K$ is Oka.

- **F. 2022** If $K \subset \mathbb{C}^n$ is a compact polynomially convex set then $\mathbb{CP}^n \setminus K$ is Oka. (This follows from the previous theorem and localization.)

  Furthermore, if Γ is a compact union of curves such that the complex curve $\widehat{K \cup \Gamma} \setminus K \cup \Gamma$ has at most finitely many irreducible components, then $\mathbb{C}^n \setminus K \cup \Gamma$ and $\mathbb{CP}^n \setminus K \cup \Gamma$ are Oka manifolds.

- In a recent joint work with E.F. Wold, we found surprisingly small Oka domains in $\mathbb{C}^n$ $(n > 1)$ at the limit of what is possible.

  F. F. and E. F. Wold 2022 **Oka domains in Euclidean spaces.**
  https://arxiv.org/abs/2203.12883

Oka complements of closed convex sets

Theorem (1)

If E is a closed convex set with $\mathscr{C}^1$ boundary in $\mathbb{C}^n$ for $n > 1$ such that $E \cap T_p^{\mathbb{C}} bE$ does not contain an affine real halfline for any $p \in bE$, then $\mathbb{C}^n \setminus E$ is an Oka domain. This holds in particular if bE is strictly convex.

There are many such examples of the form

$$E = \{(z', z_n) \in \mathbb{C}^n : \Im z_n \geq \phi(z', \Re z_n)\},$$

where ϕ is a convex function of class $\mathscr{C}^1$. If $t > 0$, the convex domain

$$\Omega_t^+ = \{\Im z_n > t\phi(z', \Re z_n)\}$$

does not contain any affine complex line, so it is hyperbolic (Barth 1980, Harris 1979, Bracci and Saracco 2009), while the domain

$$\Omega_t^- = \{\Im z_n < t\phi(z', \Re z_n)\}$$

is Oka. For $t < 0$ the picture is reversed, while at $t = 0$ the hyperplane $\{\Im z_n = 0\}$ splits $\mathbb{C}^n$ in a pair of halfspaces which are neither Oka nor hyperbolic.

Oka domains below convex graphs

A convex function $\phi : \mathbb{R}^n \to \mathbb{R}$ is **irreducible** if it is not of the form $\phi = \psi \circ P + l$ where $P : \mathbb{R}^n \to \mathbb{R}^m$ is a linear projection with $m < n$, ψ is a convex function on $\mathbb{R}^m$, and l is a linear function on $\mathbb{R}^n$. (This means that ϕ is not a convex function of a smaller number of variables which is linear in the remaining variables.)

Corollary

If ϕ is an irreducible convex function on $\mathbb{C}^{n-1} \times \mathbb{R}$, then the domain

$$\Omega_\phi = \{(z', z_n) \in \mathbb{C}^n : \Im z_n < \phi(z', \Re z_n)\}$$

is Oka. The same holds for domains $\Omega_\phi = \{(z', z_n) \in \mathbb{C}^n : \Im z_n < \phi(z')\}$.

Proof By Azagra (2013) the condition on ϕ implies that for every $\epsilon > 0$ there is a smooth strictly convex function $\psi : \mathbb{C}^{n-1} \times \mathbb{R} \to \mathbb{R}$ such that $\phi - \epsilon < \psi < \phi$. Hence, the domain $\Omega_\psi = \{\Im z_n < \psi(z', \Re z_n)\}$ is Oka.

This gives an increasing sequence $\phi_1 < \phi_2 < \phi_3 < \cdots$ of smooth strictly convex functions on $\mathbb{C}^{n-1} \times \mathbb{R}$ converging uniformly to ϕ such that the sequence of Oka domains Ω_{ϕ_j} increases to Ω_ϕ as $j \to \infty$. Hence, Ω_ϕ is Oka.

Complements of convex domains containing no lines

Example Every open set in $\mathbb{C}^n$ for $n > 1$ of the form

$$\Im z_n < c|\Re z_n| + \sum_{j=1}^{n-1} \left(a_j|\Re z_j| + b_j|\Im z_j|\right)$$

for $c \geq 0$ and positive numbers $a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1}$ is Oka.

Corollary

If E is a closed convex set in $\mathbb{C}^n$ for $n > 1$ which does not contain any affine real line, then $\mathbb{C}^n \setminus E$ is an Oka domain.

Proof There is a decreasing sequence $E_1 \supset E_2 \supset E_3 \supset \cdots$ of smoothly bounded strongly convex sets in $\mathbb{C}^n$ such that $E = \bigcap_{j=1}^{\infty} E_j$.

By Theorem 1, the domain $\Omega_j = \mathbb{C}^n \setminus E_j$ is Oka for every $j \in \mathbb{N}$.

Hence, $\mathbb{C}^n \setminus E = \bigcup_{j=1}^{\infty} \Omega_j$ is an increasing union of Oka domains, so it is Oka.

The main theorem

Theorem 1 follows from our main result, Theorem 2. Given a closed set $E \subset \mathbb{C}^n$, we denote by $\overline{E} \subset \mathbb{CP}^n$ its projective closure.

Theorem (2)

If E is a closed subset of $\mathbb{C}^n$ for $n > 1$ and $\Lambda \subset \mathbb{CP}^n$ is a hyperplane such that $\overline{E} \cap \Lambda = \varnothing$ and $\overline{E}$ is polynomially convex in $\mathbb{CP}^n \setminus \Lambda \cong \mathbb{C}^n$, then $\mathbb{C}^n \setminus E$ is Oka.

In particular, if $\mathbb{CP}^n \setminus \overline{E}$ is a union of projective hyperplanes and the set of such hyperplanes is connected, then $\Omega = \mathbb{C}^n \setminus E$ is Oka.
(We call such $\overline{E}$ **projectively convex***.)*

The second part follows from the first one by observing that if K is a compact set in $\mathbb{CP}^n$ and $\Lambda_t \subset \mathbb{CP}^n$ $(t \in [0,1])$ is a path of hyperplanes not intersecting K, then $\bigcup_{t\in(0,1]} \Lambda_t$ does not belong to the polynomial hull of K in $\mathbb{CP}^n \setminus \Lambda_0$.

We showed that every convex set $E \subset \mathbb{C}^n$ in Theorem 1 has projectively convex closure. In fact, $\mathbb{CP}^n \setminus \overline{E}$ is a union of hyperplanes parallel to the complex tangent spaces $T_p^{\mathbb{C}} bE$ for $p \in bE$. Hence, Theorem 2 implies Theorem 1.

Example: an Oka tube

Another example of a small Oka domain is a tube in $\mathbb{C}^n$ of the form

$$\Omega = \{z = (z', z_n) \in \mathbb{C}^n : |z_n| \le f(|z'|)\},$$

where $f \ge 1$ is an increasing, strongly convex function on $\mathbb{R}_+$ satisfying $f(0) = 1$ and $f(t) \approx ct$ for some $c > 0$ as $t \to +\infty$ such that for every $a > 0$ the linear function

$$x \mapsto g(x) = f(a) + f'(a)(x - a)$$

(the tangent line to the graph of f at the point $(a, f(a))$) satisfies

$$g(0) = f(a) - af'(a) > 0.$$

This implies that Ω is a union of a connected family of affine complex hyperplanes whose closures in $\mathbb{CP}^n$ do not intersect $\overline{\mathbb{C}^n \setminus \Omega}$, so Theorem 2 shows that Ω is Oka.

Such tubes are small neighbourhoods of the tube $\mathbb{C}^{n-1} \times \{|z_n| < 1\}$, and their complement is not convex.

Proof of Theorem 2

Let $H = \mathbb{CP}^n \setminus \mathbb{C}^n$ denote the hyperplane at infinity. Set $K = \overline{E}$. Choose a projective hyperplane $\Lambda \subset \mathbb{CP}^n$ with $K \cap \Lambda = \varnothing$.

Let $z = (z_1, \ldots, z_n)$ be affine coordinates on $\mathbb{CP}^n \setminus \Lambda \cong \mathbb{C}^n$ in which $H \setminus \Lambda = \{z_n = 0\}$. By the hypothesis, K is polynomially convex in these coordinates. It now suffices to prove the following result.

Theorem (3)

If K is a compact polynomially convex set in $\mathbb{C}^n$ then $(\mathbb{C}^{n-1} \times \mathbb{C}^*) \setminus K$ is Oka.

Assume that Theorem 3 holds. Then,

$$\mathbb{C}^n \setminus (E \cup \Lambda) = \mathbb{CP}^n \setminus (H \cup E \cup \Lambda) = (\mathbb{CP}^n \setminus \Lambda) \setminus (H \cup E)$$

is Oka. Choose hyperplanes $\Lambda_0 = \Lambda, \Lambda_1, \ldots, \Lambda_n$ in $\mathbb{CP}^n \setminus K$ close to Λ such that $\bigcap_{i=0}^n \Lambda_i = \varnothing$. Then, $\mathbb{C}^n \setminus (E \cup \Lambda_i)$ is Oka for every i and

$$\Omega = \mathbb{C}^n \setminus E = \bigcup_{i=0}^n (\mathbb{C}^n \setminus E) \setminus \Lambda_i$$

is a union of Zariski open Oka domains, so it is Oka.

Proof of Theorem 3

In view of Kusakabe's characterisation of Oka manifolds by Condition Ell_1, it suffices to prove the following.

Theorem (4)

Assume that

- *K is a compact polynomially convex set in $\mathbb{C}^n$ for some $n > 1$,*
- *L is a compact (polynomially) convex set in $\mathbb{C}^N$ for some $N \in \mathbb{N}$, and*
- *$f : L \to \mathbb{C}^n$ is a holomorphic map such that*

$$f(\zeta) \in (\mathbb{C}^{n-1} \times \mathbb{C}^*) \setminus K \quad \textit{holds for all } \zeta \in L.$$

Then there is a holomorphic map $F : L \times \mathbb{C}^n \to \mathbb{C}^n$ such that for every $\zeta \in L$,

$$F(\zeta, 0) = f(\zeta) \textit{ and the map } F(\zeta, \cdot\,) : \mathbb{C}^n \to (\mathbb{C}^{n-1} \times \mathbb{C}^*) \setminus K \textit{ is injective.}$$

It follows that

$$\Omega_\zeta = \{F(\zeta, z) : z \in \mathbb{C}^n\} \subset (\mathbb{C}^{n-1} \times \mathbb{C}^*) \setminus K$$

is a family of Fatou–Bieberbach domains depending holomorphically on $\zeta \in L$.

Proof of Theorem 4

Since the set $L \subset \mathbb{C}^N$ is polynomially convex, we may assume that the map f is defined on a Stein neighbourhood U of L which is a Runge in $\mathbb{C}^N$.

The graph $\Gamma = \{(\zeta, f(\zeta)) : \zeta \in U\}$ is a closed Stein submanifold of the Stein domain $X = U \times \mathbb{C}^n$ which is a Runge in $\mathbb{C}^{N+n}$. The restricted graph

$$\Gamma_L = \{(\zeta, f(\zeta)) \in X : \zeta \in L\} \subset L \times \Big((\mathbb{C}^{n-1} \times \mathbb{C}^*) \setminus K\Big) \tag{1}$$

is clearly $\mathscr{O}(\Gamma)$-convex, hence also $\mathscr{O}(X)$-convex and polynomially convex in $\mathbb{C}^N \times \mathbb{C}^n$ (since X is Runge in $\mathbb{C}^N \times \mathbb{C}^n$).

It follows that $(L \times K) \cup \Gamma_L$ is $\mathscr{O}(X)$-convex and hence polynomially convex, so it has a basis of Runge Stein neighbourhoods.

Proof of Theorem 4, part 2

Let $\pi : \mathbb{C}^N \times \mathbb{C}^n \to \mathbb{C}^N$ denote the projection. Consider the injective π-fibre preserving holomorphic map $\Phi = (\mathrm{Id}, \phi)$ of the form

$$\Phi(\zeta, z) = (\zeta, \phi(\zeta, z)) \quad \text{for} \quad (\zeta, z) \in \Omega$$

on a small Runge Stein neighbourhood $\Omega = \Omega' \cup \Omega''$ of $(L \times K) \cup \Gamma_L$ in $\mathbb{C}^N \times \mathbb{C}^n$ which equals the identity map on a neighbourhood Ω' of $L \times K$ and

$$\phi(\zeta, z) = f(\zeta) + \frac{1}{2}(z - f(\zeta)) = \frac{1}{2} f(\zeta) + \frac{1}{2} z$$

for (ζ, z) in a neighbourhood Ω'' of the graph Γ_L in (1). Thus, $\phi(\zeta, \cdot\,)$ is a contraction by the factor $1/2$ around the point $f(\zeta) \in \mathbb{C}^n$ for every $\zeta \in L$.

For a suitable choice of the neighbourhood Ω'' of Γ_L the map $\phi = \phi_{1/2}$ is connected to $\phi_0(\zeta, z) = z$ by the isotopy

$$\phi_t(\zeta, z) = t f(\zeta) + (1 - t) z \quad \text{for } 0 \leq t \leq \frac{1}{2}.$$

On $\Omega' \supset L \times K$ we take the constant isotopy $\phi_t(\zeta, z) = \phi_0(\zeta, z) = z$ for all t. Clearly, the trace of the isotopy $\Phi_t = (\mathrm{Id}, \phi_t)$ for $t \in [0, 1/2]$ consists of Runge domains $\Phi_t(\Omega) \subset \Omega$.

Proof of Theorem 4, part 3

Varolin 2001 The Lie algebra of holomorphic (algebraic) vector fields on $\mathbb{C}^n$ vanishing on a hyperplane $\mathbb{C}^{n-1} \times \{0\}$ has the (algebraic) density property.

Hence, by the parametric version of the main result of Andersén–Lempert theory, we can approximate Φ on $(L \times K) \cup \Gamma_L$ by a holomorphic map

$$\Psi : V \times \mathbb{C}^n \to V \times \mathbb{C}^n, \quad \Psi(\zeta, z) = (\zeta, \psi(\zeta, z)),$$

where $V \subset U$ is a neighbourhood of L, such that for every $\zeta \in V$ we have that

- $\psi(\zeta, \cdot\,) \in \mathrm{Aut}(\mathbb{C}^n)$,
- $\psi(\zeta, z) = z$ for every $z = (z', 0) \in \mathbb{C}^{n-1} \times \{0\}$, and
- $\psi(\zeta, f(\zeta)) = f(\zeta)$.

Choose $a, b \in \mathbb{R}$ such that

$$0 < a < 1/2 < b < 1 \quad \text{and} \quad b^2 < a.$$

If the approximation of ϕ by ψ is close enough then the estimate

$$a|z - f(\zeta)| \le |\psi(\zeta, z) - f(\zeta)| \le b|z - f(\zeta)| \tag{2}$$

holds in a neighbourhood of the graph Γ_L of f, and ψ is arbitrarily close to the map $(\zeta, z) \mapsto z$ on a neighbourhood of $L \times K$.

Proof of Theorem 4, part 4

This gives a sequence of holomorphic maps ψ_k of the same kind as ψ such that the estimate (2) holds for all of them on the same neighbourhood of Γ_L, and ψ_k converges to the map $(\zeta, z) \mapsto z$ on a neighbourhood of $L \times K$ as $k \to \infty$. Consider the sequence of random iterations

$$\theta_k(\zeta, \cdot\,) = \psi_k(\zeta, \cdot\,) \circ \psi_{k-1}(\zeta, \cdot\,) \circ \cdots \circ \psi_1(\zeta, \cdot\,) \in \mathrm{Aut}(\mathbb{C}^n).$$

Due to the condition $b^2 < a$ in the estimate (2) the attracting basin $B_\zeta \subset \mathbb{C}^n$ of the sequence θ_k at the fixed point $f(\zeta)$ is biholomorphic to $\mathbb{C}^n$ (Wold 2005).

If the convergence of the sequence ψ_k to the map $(\zeta, z) \mapsto z$ is fast enough on a neighbourhood of $L \times K$, then none of the basins B_ζ intersect K.

Furthermore, the condition $\psi_k(\zeta, (z', 0)) = (z', 0)$ for all $\zeta \in L$, $z' \in \mathbb{C}^{n-1}$, and $k \in \mathbb{N}$ ensures that the basin B_ζ does not intersect $\mathbb{C}^{n-1} \times \{0\}$.

This gives a holomorphic map $F : V \times \mathbb{C}^n \to \mathbb{C}^n$ such that the image B_ζ of $F(\zeta, \cdot\,)$ is a Fatou–Bieberbach domain in $(\mathbb{C}^{n-1} \times \mathbb{C}^*) \setminus K$ centred at $f(\zeta)$ for every $\zeta \in V$, so Theorem 4 is proved.

Open problems

We have seen that complements of most closed convex sets in $\mathbb{C}^n$ $(n > 1)$ are Oka. One expects that Oka property is naturally related to pseudoconcavity.

Problem

(a) *Is every domain with a connected strongly Levi pseudoconcave boundary in $\mathbb{C}^n$ for $n > 1$ an Oka domain?*

(b) *Is every smoothly bounded Oka domain in $\mathbb{C}^n$ Levi pseudoconcave?*

(c) *Is there a smooth real hypersurface Σ in $\mathbb{C}^n$ for $n > 1$ such that the connected components of $\mathbb{C}^n \setminus \Sigma$ are Oka? The same question for $\mathbb{CP}^n$.*

Note that an Oka manifold does not admit any bounded plurisubharmonic functions. In particular, an Oka domain has no strongly pseudoconvex boundary points, so (b) has an affirmative answer for $n = 2$.

Parts (a) and (b) may be called the **dual Levi problem**.

In dimension $n = 2$, part (c) is equivalent to the well-known open problem on the (non-)existence of a Levi-flat hypersurface in $\mathbb{CP}^2$.