Proper holomorphic embeddings: from classical to recent

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In the beginning, there were Stein manifolds

Source: Wikipedia

Karl Stein, 1951:

A complex manifold X is holomorphically complete (Stein) if

(i) holomorphic functions on X separate points, and

(ii) for every discrete sequence $p_j \in X$ there is a holomorphic function $f \in \mathcal{O}(X)$ with $\lim_{j\to\infty} |f(p_j)| = +\infty$.

Equivalently, for every compact subset $K \subset X$ its holomorphic hull $\widehat{K}_{\mathcal{O}(X)}$ is also compact.

Embedding Stein manifolds in Euclidean spaces

Like many interesting theorems, this one has a complex genesis.

Theorem (Remmert, 1956; Narasimhan, 1960; Bishop, 1961)

Let X be an n-dimensional Stein manifold. In the Frechet space $\mathscr{O}(X, \mathbb{C}^N)$ of holomorphic maps $X \to \mathbb{C}^N$, endowed with compact-open topology,

1. *almost proper mappings form a residual set if $N \geq n$,*
2. *proper maps are dense if $N \geq n+1$,*
3. *proper immersions are dense if $N \geq 2n$,*
4. *proper embeddings are dense if $N \geq 2n+1$.*

The following result adds interpolation on a subvariety.

Theorem (Acquistapace, Broglia, and Tognoli, 1975)

Assume that X is a reduced Stein space of dimension n, X' is a closed complex subspace of X, and $\varphi : X' \hookrightarrow \mathbb{C}^N$ is a proper holomorphic embedding for some $N \geq 2n+1$. Then the set of all holomorphic maps $f : X \to \mathbb{C}^N$ that extend φ and are proper, injective, and regular on X_{reg}, is dense in the space of all holomorphic maps $X \to \mathbb{C}^N$ extending φ.

Minimal embedding dimension

The stated result is dimensionwise optimal if we insist on density.
What if we only ask about the existence of a map of a given kind?

Example (Forster, 1970)

For each $n \geq 2$ there is a Stein manifold X^n which does not admit a proper holomorphic embedding in $\mathbb{C}^{[3n/2]}$ or a holomorphic immersion in $\mathbb{C}^{[3n/2]-1}$.

Forster's example gave rise to the conjecture that these numbers, increased by one, are optimal embedding/immersion dimensions. This was confirmed only after a suitable development of Oka theory by Gromov in 1989.

Theorem (Eliashberg and Gromov, 1992; Schürmann, 1997)

Every Stein manifold X of dimension n immerses properly holomorphically in $\mathbb{C}^{\left[\frac{3n+1}{2}\right]}$, and if $n > 1$ then X embeds properly holomorphically into $\mathbb{C}^{\left[\frac{3n}{2}\right]+1}$.

A Stein manifold of dimension $n = 1$ is an open Riemann surface. It is not known whether every such embeds in $\mathbb{C}^2$. Much work was done on this question **(Kasahara and Nishino, Laufer, Alexander, Globevnik and Stensønes, Wold, F^2, Alarcón and López, Ritter, Kutzschebauch, Di Salvo,...)**

Embeddings in strongly (pseudo-) convex domains

If Ω is a bounded strongly pseudoconvex domain in $\mathbb{C}^N$ and $X^n \subset \mathbb{C}^N$ is a complex submanifold intersecting $b\Omega$ transversely, then $D = X \cap \Omega$ is a strongly pseudoconvex domain in X.

In the early 1970s, questions were asked which strongly pseudoconvex domains arise in this way, especially if Ω is some model domain such as the ball $\mathbb{B}^N$.

Fornæss 1974 Every relatively compact strongly pseudoconvex domain in a Stein manifold arises in this way for some strongly convex domain $\Omega \subset \mathbb{C}^N$.

Fornæss's proof is based on his lemma on convexifying a boundary point of a strongly pseudoconvex domain by a holomorphic map defined on a neighbourhood of the closure of the domain. This embeds it into a product of convex domains, and then one smoothens the corners.

F^2, 1984 A generic strongly pseudoconvex domain with $\mathscr{C}^\infty$ boundary cannot be mapped properly holomorphically into any ball $\mathbb{B}^N$ by a map that extends smoothly to a boundary point. The obstruction appears at the level of formal power series of the defining function, and the proof is similar to Poincaré's argument (1906) that most pairs of such hypersurfaces in $\mathbb{C}^n$ for $n > 1$ are not locally biholomorphically equivalent.

Embeddings in balls and polydiscs

Løw, December 1984 Every bounded strongly pseudoconvex domain embeds properly holomorphically in a high dimensional polydisc Δ^N.

Løw, F^2, January 1985 Every bounded strongly pseudoconvex domain embeds properly holomorphically in a high dimensional ball $\mathbb{B}^N$.

These construction use a new idea – to push the boundary of the image $f(\overline{D}) \subset \mathbb{B}^N$ closer to the sphere $b\mathbb{B}^N$ in a controlled way by using holomorphic peak functions on the strongly pseudoconvex domain D.

Holomorphic peak functions were studied extensively at the time. They were used in the **proof of the inner function conjecture by Erik Løw (1982), based on the previous work by Monique Hakim and Nessim Sibony.** The result says that on the unit ball $\mathbb{B}^n$ there are holomorphic functions $f : \mathbb{B} \to \Delta$ whose a.e. boundary values on $b\mathbb{B}^n$ have modulus one. The same kind of functions exist on every strongly pseudoconvex domain.

In the works of Løw and myself from 1985, peak function were used for the first time to construct proper holomorphic embeddings.

Embeddings in balls and polydiscs

The main idea is to cover the boundary of D by finitely many families of open sets, each family consisting of connected caps with positive distances between them. For each family, choose a function in $\mathscr{A}(\overline{D})$ which has modulus close to 1 near the centre of each cap in the family and is very small outside their union. Using these functions, we push $f(bD)$ in direction roughly orthogonal to the radius vector, where the direction vectors corresponding to different families are nearly orthonormal. This pushes bD in a controlled way closer to $b\mathbb{B}^N$, and then one applies induction. The following result uses a similar technique.

Globevnik 1987 Given $n \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that for any closed interpolation set $K \subset b\mathbb{B}^n$ for $\mathscr{A}(\overline{\mathbb{B}}^n)$ and continuous map $f : K \to b\mathbb{B}^N$ there is a map $F \in \mathscr{A}(\overline{\mathbb{B}}^n, \overline{\mathbb{B}}^N)$ with extends f and satisfies $F(b\mathbb{B}^n) \subset b\mathbb{B}^N$.

In subsequent works, more carefully shaped peak function were used to reduce the codimension to the minimal possible one.

Noell and Stensønes, 1989–1990 Proper holomorphic maps from strongly or weakly pseudoconvex domains in $\mathbb{C}^2$ to the polydisc $\Delta^3 \subset \mathbb{C}^3$.

Hakim 1990 For every smoothly bounded strongly pseudoconvex domain $D \subset \mathbb{C}^n$ there is a proper holomorphic map $f : D \to \mathbb{B}^{n+1}$ extending to a continuous map $f : \overline{D} \to \overline{\mathbb{B}}^{n+1}$ with $f(bD) \subset b\mathbb{B}^{n+1}$.

How smooth can such maps be?

F^2, 1989 If $f : \mathbb{B}^n \to \mathbb{B}^N$ for $2 \leq n < N$ is a proper holomorphic map which extends to a map of class $\mathscr{C}^{N-n+1}$ to a neighborhood of some point $p \in b\mathbb{B}^n$, then f is a rational map.

Cima and Suffridge, 1990 Such a map has no singularities on $b\mathbb{B}^n$.

Faran, Webster, D'Angelo, and many others Existence and classification theory of rational proper maps between balls. Study of the fixed-point-free subgroups G of the unitary group $\mathbb{U}(n)$ such that $\mathbb{B}^n/G$ embeds properly holomorphically in a ball. (Quotients $b\mathbb{B}^n/G$ are spherical space forms.)

The technique of using peak functions in constructions of proper holomorphic maps was optimized by Avner Dor in 1990s. His main result is:

Dor 1995 Every smoothly bounded strongly pseudoconvex domain $D \subset \mathbb{C}^n$ admits a proper holomorphic embedding to any given pseudoconvex domain $\Omega \subset \mathbb{C}^N$ if $N > 2n$ and immersion if $N \geq 2n$.

More general target manifolds

After Dor's result, the main problem was to extend these results to more general target manifolds. The first step in this direction was:

Globevnik, 2000 The disc Δ admits a proper holomorphic immersion $f : \Delta \to Y$ to any Stein surface, and a proper holomorphic embedding to any Stein manifold of dimension > 2. Furthermore, given a point $y_0 \in Y$ and a tangent vector $v \in T_{y_0}Y$, one can choose f such that $f(0) = p$ and $f'(0) = \lambda v$ for some $\lambda > 0$.

Drinovec Drnovšek, 2004 There are proper holomorphic discs in Stein manifolds avoiding a given closed pluripolar set.

Further progress required a technique for gluing manifold-valued holomorphic maps, to replace the solution of the Cousin-I problem in the linear case.

Such techniques were first developed in the context of **Oka theory**. Precise up-to-the-boundary versions were obtained by Drinovec Drnovšek and myself (Duke Math. J. 2007), and a more succinct proof using the implicit function theorem in Banach spaces was given in my paper in Asian J. Math. 2007.

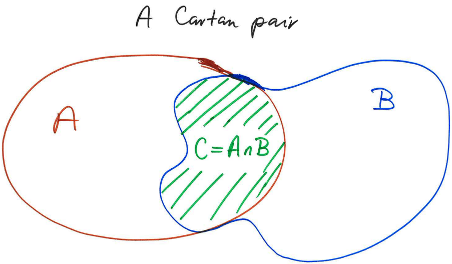
Cartan pairs

A pair (A, B) of compact subsets in a complex manifold X is a **Cartan pair** if it satisfies the following two conditions:

(i) the sets $D = A \cup B$ and $C = A \cap B$ are Stein compacts (i.e., they have bases of open Stein neighbourhoods), and

(ii) A and B are *separated* in the sense that $\overline{A \setminus B} \cap \overline{B \setminus A} = \varnothing$.

(iii) A Cartan pair (A, B) is **strongly pseudoconvex** if the sets A, B, C, and $D = A \cup B$ are strongly pseudoconvex domains. Every Cartan pair can be approximated from the outside by SPSC Cartan pairs.



Gluing sprays of maps

Assume that (A, B) is a SPSC Cartan pair in a Stein manifold X. Let Y be a complex manifold and $f_0 : A \to Y$ be a map of class $\mathscr{A}(A, Y)$.

- The graph of f_0 over A in $X \times Y = Z$ is a Stein compact, and hence it has a Stein neighbourhood. This allows us to construct a holomorphic spray $f : A \times U \to Y$, where $0 \in U \subset \mathbb{C}^N$ is a ball, such that $f(\cdot\,, 0) = f_0$ and

$$\frac{\partial}{\partial z}\Big|_{z=0} f(x, z) : \mathbb{C}^N \to T_{f_0(x)} Y \quad \text{is surjective for every } x \in A.$$

- Assume that we also have holomorphic map $g : B \times U \to Y$ of class $\mathscr{A}(B)$ which approximates f sufficiently closely on $C \times U$.
- If f and g are close enough on $C \times U$, we can find a smaller ball $0 \in U' \subset U$ and a **holomorphic transition map**

$$\gamma : C \times U' \to C \times U, \quad \gamma(x, z) = (x, c(x, z))$$

close to the identity map $\gamma_0(x, z) = (x, z)$ such that

$$f = g \circ \gamma \quad \text{holds on } C \times U'.$$

A splitting lemma

Lemma (Splitting lemma; Proposition 5.8.1 in my book)

Let $(A, B, C = A \cap B)$ *be a SPSC Cartan pair in a Stein manifold* X. *Given a holomorphic map* $\gamma : C \times U' \to C \times U$ *as above, close to the identity map, and a slightly smaller ball* $0 \subset V \subset U'$, *there are holomorphic maps*

$$\begin{aligned} \alpha(x,z) &= (x, a(x,z)), \quad x \in A,\ z \in V, \\ \beta(x,z) &= (x, b(x,z)), \quad x \in B,\ z \in V \end{aligned}$$

close to the identity on their respective domains such that

$$\gamma \circ \alpha = \beta \quad \textit{holds on } C \times V.$$

The maps α *and* β *may be chosen to depend smoothly on* γ.

This is a nonlinear version of Cousin-I problem. It is proved by using the solution to the $\bar{\partial}$-equation with bounds on strongly pseudoconvex domains and the implicit function theorem in Banach spaces.

Gluing f_0 and g_0

Recall that

$$f = g \circ \gamma \quad \text{holds on } C \times U'$$

and

$$\gamma \circ \alpha = \beta \quad \text{holds on } C \times V.$$

It follows that

$$f \circ \alpha = g \circ \gamma \circ \alpha = g \circ \beta \quad \text{holds on } C \times V.$$

Hence, $f \circ \alpha$ and $g \circ \beta$ amalgamate into a holomorphic map

$$F : (A \cup B) \times V \to Y.$$

The holomorphic map

$$F_0 = F(\cdot\,, 0) : D = A \cup B \to Y$$

is such that $F_0|_A$ approximates f_0 and $F_0|_B$ approximates $g_0 = g(\cdot\,, 0)$.

Combining this with the local lifting technique by peak functions, adjusted to use for sprays of holomorphic maps, we obtained the following lemma.

Lifting boundaries of images of SPSC domains

Lemma (Drinovec Drnovšek & F^2, Amer. Math. J. 2010)

Assume that

1. ρ *is a strongly PSH exhaustion function on a Stein manifold* Y*,*
2. X *is a Stein manifold with* $\dim Y \geq 2 \dim X$*,*
3. D *is a smoothly bounded strongly pseudoconvex domain in* X*, and*
4. $f : \overline{D} \to Y$ *is a holomorphic map satisfying* $a < \rho(f(x)) < b$ *for some* $a < b$ *and for all* $x \in bD$*.*

Given $\epsilon > 0$ *and a compact set* $K \subset D$*, there is a holomorphic immersion* $F : \overline{D} \to Y$ *satisfying*

(a) $\rho(F(x)) > b$ *for all* $x \in bD$*,*

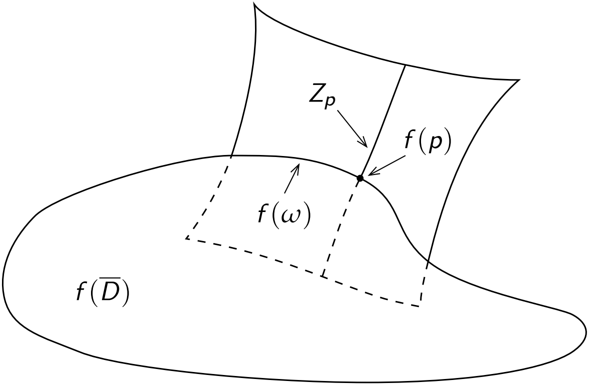
(b) $\rho(F(x)) > \rho(f(x)) - \epsilon$ *for all* $x \in \overline{D}$*, and*

(c) $\mathrm{dist}_Y(F(x), f(x)) < \epsilon$ *for all* $x \in K$*.*

If $\dim Y > 2 \dim X$ *then* F *can be chosen an embedding.*

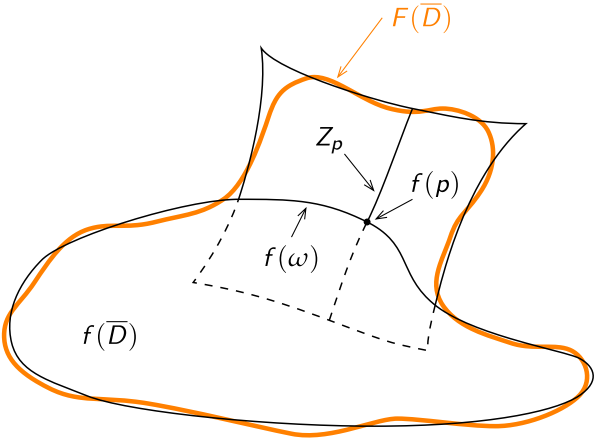
The conclusion also holds (without demanding that F *be an immersion) if* $\dim Y > \dim X$ *and* ρ *has no critical values in* $[a, b]$*.*

The boundary deformation



On this illustration, we see a family of Levi hypersurfaces $Z_p \subset Y$ attached to $f(\overline{D})$ along the image $f(\omega)$ of a boundary cap $\omega \subset bD$. The function ρ grows quadratically on each Z_p.

The boundary deformation



The modification F has been made. It approximates f closely except near the cap ω, where it turns in the direction of the hypersurfaces Z_p $(p \in \omega)$ and goes for a certain distance, thereby increasing $\rho \circ F$ on ω.

Proper holomorphic maps to q-convex manifolds

Our techniques also work in q-convex manifolds for certain values of q. For lifting, we need a smooth exhaustion function ρ on Y whose Levi form at any point of Y has at least $2\dim X$ positive eigenvalues.

Corollary (Drinovec Drnovšek & F[2], Amer. J. Math. 2010)

Let X be a Stein manifold of dimension n, $D \Subset X$ be a smoothly bounded strongly pseudoconvex domain, and Y be a complex manifold of dimension $\dim Y = N \geq 2n$. *Let $q \in \{1, \ldots, N-2n+1\}$. Then the following hold:*

(a) *If Y is q-convex (i.e., there is an exhaustion function on Y whose Levi form at any point has at least $N - q + 1 \geq 2n$ positive eigenvalues) then there exists a proper holomorphic immersion $D \to Y$.*

(b) *If Y is q-complete then every continuous map $f : \overline{D} \to Y$ that is holomorphic in D can be approximated, uniformly on compacts in D, by proper holomorphic immersions $D \to Y$ (embeddings if $N > 2n$).*

Question: Under which condition on a manifold Y can we properly embed or immerse any Stein manifold X of suitable dimension in Y?

Then, there were Oka manifolds...

A complex manifold Y is an **Oka manifold**[1] if every holomorphic map $K \to Y$ from a compact convex set $K \subset \mathbb{C}^n$ is a limit of entire maps $\mathbb{C}^n \to Y$.

The main theorem: Maps $X \to Y$ from any Stein space X to an Oka manifold Y satisfy the parametric Oka principle with approximation and interpolation. In particular:

Theorem

Let Ω be an Oka domain in a complex manifold Y. Given a Stein manifold X, a compact $\mathscr{O}(X)$-convex set $K \subset X$ and a holomorphic map $f : K \to Y$ such that $f(bK) \subset \Omega$, we can approximate f uniformly on K by holomorphic maps $F : X \to Y$ with $F(X \setminus \mathring{K}) \subset \Omega$.

There exist compact and also noncompact Oka manifolds without any closed complex curves (e.g. certain tori and punctured tori of dimension > 1). To get suitable target manifolds for proper holomorphic maps from Stein manifolds, we need a stronger condition.

[1] F. Forstnerič, Oka manifolds, C. R. Acad. Sci. Paris **347:17-18** (2009)
MSC 2020: 32Q56 Oka principle and Oka manifolds

Enter the hero: manifolds with the density property

Varolin, 2000 A complex manifold Y is said to have the **density property** if every holomorphic vector field on Y can be approximated uniformly on compacts by Lie combinations of complete holomorphic vector fields.

On such manifolds, every isotopy of biholomorphic maps between Stein Runge domains can be approximated by holomorphic automorphisms of Y.

A Stein manifold with the density property is an Oka manifold.
It has recently been discovered that **such manifolds are also Oka at infinity.**

Theorem

Let Y be a Stein manifold with the density property. Then:

(a) *Y is an Oka manifold.*

(b) **Kusakabe, 2020; Wold & F^2, 2020**
If $L \subset Y$ is a compact $\mathscr{O}(Y)$-convex subset then $Y \setminus L$ is Oka.

The reason for (b) is that $Y \setminus L$ contains holomorphic families of Fatou–Bieberbach domains with given holomorphically varying family of centres, and hence Kusakabe's new characterization of Oka manifolds by Condition Ell_1 holds (Indiana Univ Math. J., 2021).

Embeddings in Stein manifolds with the density property

Theorem (Andrist, F^2, Ritter, Wold, 3 papers, 2014–2019)

Let Y be a Stein manifold with the density property or the volume density property, and let D be a smoothly bounded strongly pseudoconvex domain in a Stein manifold X such that $\dim Y \geq 2\dim X$.

Then, every continuous map $f : X \to Y$ is homotopic to a proper holomorphic immersion $F : X \to Y$ (embedding if $\dim Y > 2\dim X$), with approximation on compact $\mathcal{O}(X)$-convex sets.

Using part (b) in the previous theorem and the lemma on lifting boundaries, the proof is very simple if Y has the density property. Let ρ be a SPSH exhaustion function on Y.

In an inductive step, we first push $f_k(bD_k)$ into $\{\rho > b\} \subset Y$ for a given $b \in \mathbb{R}$. Since $\{\rho > b\}$ is an Oka domain in Y, f_k can be approximated on $\overline{D}_k$ by a holomorphic map $f_{k+1} : X \to Y$ sending $X \setminus D_k$ to $\{\rho > b\}$. Pick a bigger SPSC domain $D_{k+1} \subset X$ containing $\overline{D}_k$ to get the next map $f_{k+1} : \overline{D}_{k+1} \to Y$ sending $\overline{D_{k+1} \setminus D_k}$ to $\{\rho > b\}$.

An inductive application gives proper holomorphic maps $X \to Y$.

Small Oka domains in $\mathbb{C}^n$

With E.F. Wold we recently found surprisingly small Oka domains in $\mathbb{C}^n$ $(n > 1)$ at the limit of what is possible. In particular, we proved the following.

Theorem (Wold & F[2], 2022, to appear in IMRN)

If E is a closed convex set in $\mathbb{C}^n$ $(n > 1)$ which does not contain any affine real line, then $\Omega = \mathbb{C}^n \setminus E$ is an Oka domain.

The idea is to consider $\mathbb{C}^n$ as an affine chart in $\mathbb{CP}^n = \mathbb{C}^n \cup H$ with $H = \mathbb{CP}^{n-1}$. The projective closure $K = \overline{E}$ has the property that $\mathbb{CP}^n \setminus K$ is a union of a connected family of complex hyperplanes. Fix such a hyperplane Λ. Then, K is polynomially convex in the affine chart $\mathbb{CP}^n \setminus \Lambda \cong \mathbb{C}^n$, and we can choose affine coordinates $z = (z', z_n)$ on it such that $H \setminus \Lambda = \{z_n = 0\}$.

Then, we prove that $\mathbb{C}^n \setminus (H \cup K) = (\mathbb{C}^{n-1} \times \mathbb{C}^*) \setminus K$ is an Oka domain.

In the original chart, this says that $\mathbb{C}^n \setminus (E \cup \Lambda)$ is Oka.

By moving Λ and using Kusakabe's localization theorem for Oka manifold, we get that $\mathbb{C}^n \setminus E$ is Oka as well.

Proper embeddings in $\mathbb{C}^n$ avoiding large convex sets

Here is a brand new application of these techniques.

Definition

A closed convex set E in $\mathbb{R}^n$ has **bounded convex exhaustion hulls (BCEH)** if for every compact convex set $K \subset \mathbb{R}^n$,

$$\text{the set} \quad h(E,K) = \mathrm{Conv}(E \cup K) \setminus E \quad \text{is bounded.}$$

Theorem (Drinovec Drnovšek & F[2], November 2022)

Let E be an unbounded closed convex set in $\mathbb{C}^n$ $(n > 1)$ having BCEH.

Given a Stein manifold X with $\dim X < n$, a compact $\mathcal{O}(X)$-convex set K in X, and a holomorphic map $f_0 : K \to \mathbb{C}^n$ with $f(bK) \subset \Omega = \mathbb{C}^n \setminus E$, we can approximate f_0 uniformly on K by proper holomorphic maps $f : X \to \mathbb{C}^n$ satisfying $f(X \setminus \mathring{K}) \subset \Omega$.

The map f can be chosen an embedding if $2\dim X < n$ and an immersion if $2\dim X \leq n$.

∼ Thank you for your attention ∼

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