Recent developments on Oka manifolds

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Flexibility versus rigidity in complex geometry

A central question of complex geometry is to understand the space of holomorphic maps $X \to Y$ between a pair of complex manifolds. Are there many maps, or few maps? Which properties can such maps have?

There are many holomorphic maps $\mathbb{C} \to \mathbb{C}$ and $\mathbb{C} \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, but there are no nonconstant algebraic maps $\mathbb{C} \to \mathbb{C}^*$ or holomorphic maps $\mathbb{C} \to \mathbb{C} \setminus \{0, 1\}$. Manifolds with the latter property are called **hyperbolic**.

***** HYPERBOLICITY IS AN OBSTRUCTION THEORY *****

On the opposite side, **Oka theory** studies complex manifolds Y which admit many holomorphic maps $X \to Y$ from complex Euclidean spaces $X = \mathbb{C}^N$, and more generally from any **Stein manifold** X. It developed from works of **Oka, Grauert, Gromov**, and others.

***** OKA THEORY IS AN EXISTENCE THEORY *****

It provides solutions to a wide variety of complex analytic problems in the absence of topological obstructions.

The Oka principle for line bundles

Source: Wikipedia

Kiyoshi Oka, 1939:

Every complex line bundle $E \to X$ over a domain of holomorphy $X \subset \mathbb{C}^n$ admits a compatible structure of a holomorphic line bundle, and any two holomorphic line bundles which are topologically equivalent are also holomorphically equivalent.

In cohomological language:

$$\mathrm{Pic}(X) = H^1(X, \mathscr{O}^*) \cong H^2(X, \mathbb{Z}).$$

Stein manifolds were introduced in 1951

Source: Wikipedia

Karl Stein, 1951:

A complex manifold X is said to be holomorphically complete (Stein) if

1. holomorphic functions on X separate points, and
2. for every discrete sequence $p_j \in X$ there is a holomorphic function $f \in \mathcal{O}(X)$ with $\lim_{j\to\infty} |f(p_j)| = +\infty$.

   Equivalently, for every compact subset $K \subset X$ its holomorphic hull $\widehat{K}_{\mathcal{O}(X)}$ is also compact.

Examples and characterizations of Stein manifolds

- **Cartan and Thullen, 1932:** A domain in $\mathbb{C}^n$ is Stein iff it is a domain of holomorphy.
- **Behnke and Stein, 1949:** Every open Riemann surface is Stein.
- **Remmert, Bishop, Narasimhan, 1956–61:** A complex manifold X is Stein iff it embeds as a closed complex submanifolds of some $\mathbb{C}^N$. Can take $N = 2\dim X + 1$.
- **Grauert, 1958:** A complex manifold X is Stein iff it admits a strictly plurisubharmonic exhaustion function $\rho : X \to \mathbb{R}$, $dd^c\rho = i\partial\bar{\partial}\rho > 0$.
- **Siu, 1976:** Every Stein subvariety of a complex space admits an open Stein neighbourhood

A **Stein space** is a complex space with singularities having similar function theoretic properties.

The Oka–Grauert Principle, 1958

Source: Wikipedia

Hans Grauert, 1958: Let G be a complex Lie group. For principal G-bundles on a Stein space, the holomorphic classification coincides with the topological classification. This holds in particular for complex vector bundles on Stein spaces.

Equivalently, we have an isomorphism

$$H^1(X, \mathcal{O}_X^G) \xrightarrow{\cong} H^1(X, \mathcal{C}_X^G)$$

induced by the inclusion $\mathcal{O}_X^G \hookrightarrow \mathcal{C}_X^G$ of the sheaf of holomorphic maps $X \to G$ into the sheaf of continuous maps.

Oka manifolds

Grauert's theorem follows from the following result of his.

Theorem (Grauert 1958)

Let X be a Stein space and Y be a complex homogeneous manifold. Then:

- *Every continuous map $f_0 : X \to Y$ is homotopic to a holomorphic map $f_1 : X \to Y$.*
- *If in addition f_0 is holomorphic on a compact $\mathscr{O}(X)$-convex subset $K \subset X$ and on a closed complex subvariety $X' \subset X$, then the homotopy $f_t : X \to Y$ $(t \in [0,1])$ from f_0 to f_1 can be chosen to be holomorphic and uniformly close to f_0 on K, and to agree with f_0 on X'.*
- *The analogous result holds for continuous families of maps $X \to Y$.*
- *These results also hold for sections of any holomorphic fibre bundle $h : Z \to X$ whose fibre is a complex homogeneous manifold.*

In the special case $Y = \mathbb{C}$, this combines the **Oka–Weil approximation theorem** and the **Oka–Cartan extension theorem**.

A complex manifold Y satisfying this theorem is called an **OKA MANIFOLD**.

Basic properties of Oka manifolds

- Oka manifolds are in a precise sense dual to Stein manifolds: the former are the most natural sources of holomorphic maps, while Oka manifolds are the most natural targets.
- On any Oka manifold Y, the Kobayashi infinitesimal metric and the Eisenmann volume forms vanish identically, and every negative plurisubharmonic function on Y is constant.

  Thus, Oka manifolds are completely anti-hyperbolic.
- **F., 2017:** Every Oka manifold Y admits a strongly dominating holomorphic map $f\colon \mathbb{C}^n \to Y$ with $n = \dim Y$; i.e., $f(\mathbb{C}^n \setminus \mathrm{br}\, f) = Y$.
- **Kobayashi and Ochiai, 1975:** A compact complex manifold of general Kodaira type is not dominable by $\mathbb{C}^n$, so it is not Oka.

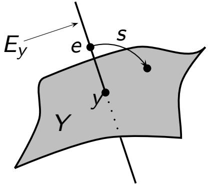
Gromov's ellipticity

The goal: Find verifiable necessary and sufficient conditions characterizing Oka manifolds.

Observation: If Y is an Oka manifold, then for every point $y \in Y$ there is a dominating holomorphic map $s_y : \mathbb{C}^n \to Y$ such that

$$s_y(0) = y \quad \text{and} \quad \mathrm{d}s_y(0) : T_0\mathbb{C}^n = \mathbb{C}^n \to T_yY \ \text{ is surjective.}$$

Gromov, 1989: A complex manifold Y is called **elliptic** if it admits a **dominating spray**: A holomorphic map $s : E \to Y$, defined on the total space of a holomorphic vector bundle E over Y, such that $s(0_y) = y$ and $s : E_y \to Y$ is a submersion at 0_y for all $y \in Y$.



First examples of dominating sprays

(a) Let G be a complex Lie group and $\mathfrak{g} = T_1 G \cong \mathbb{C}^p$ ($p = \dim G$) be its Lie algebra. Then the map

$$s: G \times \mathbb{C}^p \to G, \quad s(g, v) = \exp(v) g$$

is a dominating spray on G.

(b) More generally, if Y is a G-homogeneous manifold, then the map

$$s: Y \times \mathbb{C}^p \to Y, \quad s(y, v) = \exp(v) y$$

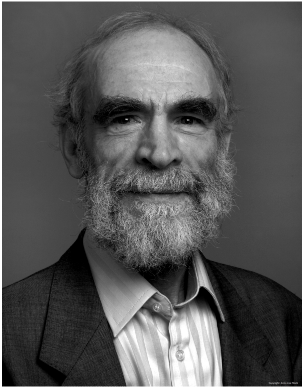
is a dominating spray on Y.

(c) Assume that a complex manifold Y is **holomorphically flexible**, in the sense that it admits finitely many $\mathbb{C}$-complete holomorphic vector fields $V_1, \ldots, V_m$ which span the tangent space of Y at every point. Denote by ϕ_j^t the flow of V_j for time $t \in \mathbb{C}$. Then, the map $s: Y \times \mathbb{C}^m \to Y$ given by

$$s(y, t_1, \ldots, t_m) = \phi_1^{t_1} \circ \phi_2^{t_2} \circ \cdots \circ \phi_m^{t_m}(y),$$

is a dominating spray on Y.

Gromov's Oka principle



Source: Wikipedia

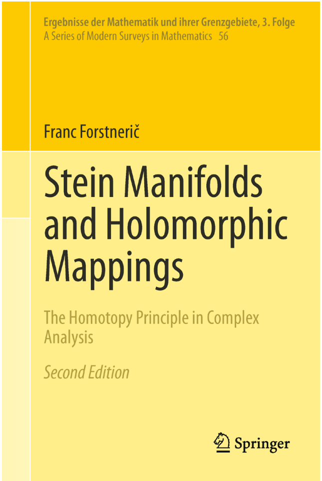
Gromov 1989: Every elliptic manifold is an Oka manifold.

Furthermore, the Oka principle holds for sections of elliptic submersions $h : Z \to X$ over a Stein base X.

F. 2002: A complex manifold Y is **subelliptic** if there exist finitely many sprays $s_j : E_j \to Y$ such that $\sum_j ds_j(E_{j,y}) = T_y Y$ for all $y \in Y$.

Every subelliptic manifold is Oka.

Gromov's theorem generalizes the Oka–Grauert principle



A detailed proof of Gromov's Oka principle was given by **Jasna Prezelj** and myself during 2000–2002. It is presented in my book (2011, 2017).

Lárusson, 2004: Construction of a model category for Oka theory.

F., 2010 Oka principle for sections of stratified subelliptic submersions $Z \to X$ over a Stein space.

Equivariant Oka theory: **Heinzner and Kutzschebauch, 1995; Kutzschebauch, Lárusson & Schwarz, 2017–**

Algebraically (sub-) elliptic manifolds

An algebraic manifold Y is called **algebraically (sub-) elliptic** if it admits a dominating algebraic spray (resp. a finite dominating family of algebraic sprays).

- **Gromov, 1989:** If Y is covered by finitely many Zariski open charts which are algebraically subelliptic, then Y is algebraically subelliptic.
- The complement of an algebraic subvariety A of codimension at least two in $\mathbb{C}^n$, $\mathbb{CP}^n$, or in a complex Grassmanian is algebraically subelliptic.
- **Kaliman and Zaidenberg, 2022**: Every algebraically subelliptic manifold is algebraically elliptic.
- **Arzhantsev, Kaliman, and Zaidenberg, 2023**: If Y is compact of dimension $n > 1$ covered by Zariski open sets isomorphic to domains in $\mathbb{C}^n$, then Y is algebraically elliptic.

The relative algebraic Oka principle

Let X be an affine algebraic variety and Y be an algebraically subelliptic manifold.

F., 2006: Every holomorphic map $X \to Y$ that is homotopic to an algebraic map is a limit of algebraic maps, uniformly on compacts in X.

F., 2017 If Y is compact, it admits a surjective strongly dominating morphism $\mathbb{C}^n \to Y$ with $n = \dim Y$.

Kusakabe, 2022 The same holds if Y is not compact and $n = \dim Y + 1$.

Lárusson and Truong, 2019:
The Oka principle fails for maps from affine algebraic manifolds X to any algebraic manifold Y which is compact or contains a rational curve:

for some such X there is a continuous map $X \to Y$ which is not homotopic to an algebraic map.

Characterization of Oka manifolds by CAP

Gromov's ellipticity conditions do not seem to satisfy interesting functorial properties. They are sufficient but not necessary for the Oka principle.

In his 1989 paper, Gromov asked whether Oka manifolds can be characterized by the Runge approximation property from a class of simple domains in Euclidean spaces. This question was answered affirmatively during 2005–2009, which led to the first unification of Oka theory.

F. 2005–2009:

- A complex manifold Y is an **Oka manifold** if and only if it enjoys the **Convex approximation property (CAP)**:

  Every holomorphic map $K \to Y$ from a compact convex set K in $\mathbb{C}^n$ is a limit of entire maps $\mathbb{C}^n \to Y$.
- **All natural Oka properties are pairwise equivalent.**
- **Theorem "up-down":** If $Y \to Z$ is a holomorphic fibre bundle with Oka fibre, then Y is an Oka manifold iff Z is an Oka manifold.

*** **MSC 2020:** New subfield **32Q56 Oka principle and Oka manifolds** ***

Kusakabe's characterization of Oka manifolds

Gromov, 1986: A complex manifold Y enjoys condition Ell_1 if every holomorphic map $X \to Y$ from a Stein manifold is the core of a dominating spray $X \times \mathbb{C}^N \to Y$.

It is easily seen that $\mathrm{OKA} \implies \mathrm{Ell}_1$. Does the converse hold?

Kusakabe, 2021 A complex manifold Y enjoys condition $\mathrm{C\text{-}Ell}_1$ if the above holds for every (bounded) convex domain $X \subset \mathbb{C}^n$.

Theorem (Kusakabe 2021)

A complex manifold which satisfies condition $\mathrm{C\text{-}Ell}_1$ is an Oka manifold. Hence, the following conditions on a complex manifold are equivalent:

$$\mathrm{Oka} \iff \mathrm{Ell}_1 \iff \mathrm{C\text{-}Ell}_1.$$

The new implication is $\mathrm{C\text{-}Ell}_1 \implies \mathrm{CAP}$.

A localization theorem for Oka manifolds

The following is an interesting application.

Theorem (Kusakabe, 2021)

If Y is a complex manifold which is a union of Zariski open Oka domains, then Y is an Oka manifold.

This is a wonderful tool for constructing new examples of Oka manifolds. Previously, a localization theorem was known only for algebraically subelliptic manifolds.

The proof uses characterization of Oka manifolds by $\mathrm{C\text{-}Ell}_1$ and the following result, which follows easily from Theorems 7.2.1 and 8.6.1 in my book.

Lemma

Let Ω be a Zariski open Oka domain in a complex manifold Y. Given a Stein manifold X and a holomorphic map $f\colon X \to Y$, there is a holomorphic spray $F\colon X \times \mathbb{C}^N \to Y$ over f which is dominating on $f^{-1}(\Omega)$.

Stein manifolds with Varolin's density property

Definition (Varolin 2000)

A complex manifold X has the density property if every holomorphic vector field on X can be approximated, uniformly on compacts in X, by Lie combinations (sums and Lie backets) of complete holomorphic vector fields on X.

Andersén and Lempert, 1992: $\mathbb{C}^n$ for $n > 1$ has the density property.

Remark: Every Stein manifold with the density property is an Oka manifold.

F. and Rosay, 1993: Let X be a Stein manifold with the density property. If $\Omega_0 \subset X$ is a pseudoconvex Runge domain and $F_t : \Omega_0 \to \Omega_t \subset X$ $(t \in [0, 1])$ is a smooth isotopy of biholomorphic maps such that $F_0 = \mathrm{Id}_{\Omega_0}$ and the domain $\Omega_t = F_t(\Omega_0)$ is Runge in X for all t, then F_1 can be approximated uniformly on compacts in Ω_0 by holomorphic automorphisms of X.

After the initial work of Varolin, the theory of such manifolds was mainly developed by **Kaliman & Kutzschebauch** and their collaborators.

Complements of polynomially convex sets are Oka

Kusakabe, preprint 2020; F. and Wold, 2020

If K is a compact polynomially convex set in $\mathbb{C}^n$ $(n > 1)$ then $\mathbb{C}^n \setminus K$ is Oka. The same holds in any Stein manifold with the density property.

To see this, we verify condition $\mathrm{C\text{-}Ell}_1$. Let $L \subset \mathbb{C}^N$ be a compact convex set and $f\colon U \to \mathbb{C}^n \setminus K$ be a holomorphic map from a Runge open neighbourhood $U \subset \mathbb{C}^N$ of L. Let $\Gamma = \{(\zeta, f(\zeta)) : \zeta \in L\}$. The set

$$(L \times K) \cup \Gamma$$

is then polynomialy convex in $\mathbb{C}^N \times \mathbb{C}^n$.

Let $G(\zeta, z) = (\zeta, \psi(\zeta, z))$ be the identity on a neighborhood of $U \times K$, and the contraction

$$\psi(\zeta, z) = \frac{1}{2} z + \frac{1}{2} f(\zeta)$$

to the point $f(\zeta)$ for each (ζ, z) in a neighbourhood of Γ.

Complements of polynomially convex sets are Oka, 2

By the parametric version of the Forstnerič–Rosay theorem, we can approximate G uniformly on a neighbourhood of $(L \times K) \cup \Gamma$ by a holomorphic automorphism $\Phi \in \mathrm{Aut}(U \times \mathbb{C}^n)$ of the form

$$\Phi(\zeta, z) = (\zeta, \phi(\zeta, z)), \quad \zeta \in U, \ z \in \mathbb{C}^n.$$

Iteration of this procedure leads to a holomorphic maps $F \colon U \times \mathbb{C}^n \to \mathbb{C}^n$ such that for all $\zeta \in U$ we have $F(\zeta, 0) = f(\zeta)$ and

$$F(\zeta, \cdot\,) : \mathbb{C}^n \to \mathbb{C}^n \setminus K \text{ is a Fatou–Bieberbach map.}$$

Hence, F is a dominating holomorphic spray with the core f and taking values in $\mathbb{C}^n \setminus K$.

Thus, $\mathbb{C}^n \setminus K$ satisfies condition $\mathrm{C\text{-}Ell}_1$, so it is Oka by Kusakabe's theorem.

Furthermore, $\mathbb{CP}^n \setminus K$ is Oka for every such K.

Oka complements of unbounded convex sets in $\mathbb{C}^n$

F. & Wold, 2022 Complements of most closed convex sets $E \subset \mathbb{C}^n$ for $n > 1$ are Oka. In particular:

- (a) If E has $\mathcal{C}^1$ boundary and $E \cap T_p^{\mathbb{C}} bE$ for $p \in bE$ does not contain any real halfline, then $\mathbb{C}^n \setminus E$ is Oka.
- (b) If E is a closed convex set in $\mathbb{C}^n$ which does not contain any affine real line, then $\mathbb{C}^n \setminus E$ is Oka.

Let $K = \overline{E} \subset \mathbb{CP}^n$ and $H = \mathbb{CP}^n \setminus \mathbb{C}^n$. Condition (a) implies that $\mathbb{CP}^n \setminus K$ is the union of a connected family of complex hyperplanes. For any such hyperplane Λ, K is polynomially convex in $\mathbb{CP}^n \setminus \Lambda \cong \mathbb{C}^n$. Choose affine coordinates $z = (z_1, \ldots, z_n)$ on $\mathbb{CP}^n \setminus \Lambda$ such that $H \setminus \Lambda = \{z_n = 0\}$. The conclusion now follows from the following result and the localization theorem.

Theorem (F. and Wold, 2022)

If K is a compact polynomially convex set in $\mathbb{C}^n$ $(n > 1)$ then

$$\Omega = (\mathbb{C}^{n-1} \times \mathbb{C}^*) \setminus K \quad \textit{is an Oka domain.}$$

Selected applications of the Oka principle

- **Sections avoiding analytic subvarieties:** Let $E \to X$ be a holomorphic vector bundle, and let $\Sigma \subset E$ be a tame complex subvariety with fibres $\Sigma_x \subset E_x$ of codimension ≥ 2. Then, $E \setminus \Sigma \to X$ is an elliptic submersion. Hence, sections $X \to E$ avoiding Σ satisfy the Oka principle.
  We also have the Oka principle for removal of intersections with Σ.

- **Eliashberg and Gromov, 1992; Schürmann, 1997:** Existence of proper holomorphic embeddings $X^n \hookrightarrow \mathbb{C}^{\left[\frac{3n}{2}\right]+1}$ and of proper holomorphic immersions $X^n \hookrightarrow \mathbb{C}^{\left[\frac{3n+1}{2}\right]}$ when X^n is Stein (with $n > 1$ for embeddings).

- **Eliashberg and Gromov, 1986**: h-principle for holomorphic immersions $X^n \to \mathbb{C}^N$, $N > n$.

- **F, 2003:** h-principle for holomorphic submersions $X^n \to \mathbb{C}^q$, $n > q$.

- **Open problem:** Does every Stein manifold X of dimension $n > 1$ with trivial tangent bundle TX admits a holomorphic immersion $X^n \to \mathbb{C}^n$?

Holomorphic factorization problems

- **Ivarsson and Kutzschebauch, Annals of Math. 2013 (Solution of the Gromov–Vaserstein Problem)**:

  Let X be a Stein space and $f\colon X \to SL_m(\mathbb{C})$ be a null-homotopic holomorphic map. There exist finitely many holomorphic maps $G_1, \ldots, G_k : X \to \mathbb{C}^{m(m-1)/2}$ such that

  $$f(x) = \begin{pmatrix} 1 & 0 \\ G_1(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2(x) \\ 0 & 1 \end{pmatrix} \ldots \begin{pmatrix} 1 & G_k(x) \\ 0 & 1 \end{pmatrix}.$$

  Their proof uses a theorem of **Vaserstein (1988)** on factorization of continuous maps, together with the Oka principle for sections of stratified elliptic submersions over Stein spaces.

- **Ivarsson, Kutzschebauch & Løw, 2019; Schott, 2022**

  Factorization of holomorphic symplectic matrices into elementary factors.

- **Ionita and Kutzschebauch, 2023:** Decomposition of null-homotopic holomorphic vector bundle automorphisms of a rank 2 vector bundle over a Stein space into products of unipotent automorphisms.

Proper holomorphic maps avoiding closed convex sets

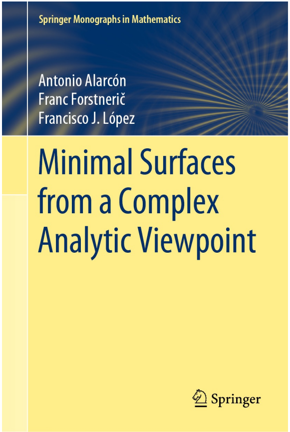
The following result shows that proper holomorphic maps from Stein manifolds to Euclidean spaces can omit suprisingly big sets.

Drinovec Drnovšek & F., 2023

Let E be a closed convex set in $\mathbb{C}^n$ $(n > 1)$ contained in a closed halfspace H such that $E \cap bH$ is nonempty and bounded. Then:

- Every Stein manifold X of dimension $< n$ admits a proper holomorphic map $f \colon X \to \mathbb{C}^n$ with $f(X) \subset \Omega = \mathbb{C}^n \setminus E$.
- If in addition $2 \dim X + 1 \leq n$ then f can be chosen an embedding, and if $2 \dim X = n$ then it can be chosen an immersion.
- If in addition E is strictly convex, we also obtain the interpolation property for such maps on closed complex subvarieties.

Applications to minimal surfaces (2021)



A minimal surface in $\mathbb{R}^n$, $n \geq 3$, is given by a conformal harmonic immersion $F\colon R \to \mathbb{R}^n$ from an open Riemann surface R.

Let θ be a nonvanishing holomorphic 1-form on R.
The map $f = 2\partial F/\theta : R \to \mathbb{C}^n$ is holomorphic and takes values in

$$A_* = \{z_1^2 + z_2^2 + \cdots + z_n^2 = 0\} \setminus \{0\}.$$

Conversely, every holomorphic $f\colon R \to A_*$ having vanishing real periods integrates to a conformal minimal surface $F = \Re \int f\theta : R \to \mathbb{R}^n$.

A_* is an algebraically elliptic manifold. Applications of the Oka principle yield a variety of new results on minimal surfaces in Euclidean spaces.

Additional surveys

- F. Lárusson: What is an Oka manifold?
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- F. Forstnerič and F. Lárusson: Survey of Oka theory.
  New York J. Math., 17a (2011), 1–28.
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- F. Forstnerič: Recent developments on Oka manifolds.
  Indag. Math., 34(2):367-417, 2023.
- F. Kutzschebauch, F. Lárusson & G. W. Schwarz: Equivariant Oka theory: survey of recent progress.
  Complex Anal. Synerg. 8 (2022), no. 3, Paper No. 15.