Oka manifolds

Franc Forstnerič

Univerza v Ljubljani

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Contents

The theme of these lectures are holomorphic maps from Stein manifolds to complex manifolds having flexible complex structures.

They are called **Oka manifolds**, so named after **Kiyoshi Oka**, a pioneer of complex analysis of 20th century. This term was introduced in a 2009 paper of mine, although they have been implicitly present since the works of Oka (1939) and Hans Grauert (1958).

Here is the contents:

- A brief history and main results of the Oka–Grauert theory
- The work of Mikhail Gromov
- The concept of an Oka manifold and Oka map
- Characterizations of Oka manifolds
- Examples of Oka manifolds
- Applications to minimal surfaces in $\mathbb{R}^n$

The main literature

F. Forstnerič: Stein manifolds and holomorphic mappings. The homotopy principle in complex analysis. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, 56. Second Edition. Springer, Cham, 2017. https://link.springer.com/book/10.1007/978-3-319-61058-0

F. Forstnerič: Recent developments on Oka manifolds. Indag. Math., 34(2) (2023) 367–417. https://doi.org/10.1016/j.indag.2023.01.005

F. Forstnerič and F. Lárusson: Survey of Oka theory. New York J. Math., 17a (2011), 1–28. https://nyjm.albany.edu/j/2011/17a-2.html

A. Alarcón, F. Forstnerič, and F. J. López: Minimal Surfaces from a Complex Analytic Viewpoint. Springer Monographs in Mathematics, Springer, Cham, 2021. https://link.springer.com/book/10.1007/978-3-030-69056-4

A. Alarcón and F. Forstnerič: New complex analytic methods in the theory of minimal surfaces: a survey. J. Aust. Math. Soc., 106:3 (2019) 287–341

Flexibility versus rigidity in complex geometry

A central question of complex geometry is to understand the space of holomorphic maps $X \to Y$ between a pair of complex manifolds. Are there many maps, or few maps? Which properties can such maps have?

There are many maps $\mathbb{C} \to \mathbb{C}$ and $\mathbb{C} \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, but there are no nonconstant holomorphic maps $\mathbb{C} \to \mathbb{C} \setminus \{0,1\}$. Manifolds with the latter are called (Brody) **hyperbolic**.

***** HYPERBOLICITY IS AN OBSTRUCTION THEORY *****

On the opposite side, **Oka theory** studies complex manifolds Y which admit many holomorphic maps $X \to Y$ from complex Euclidean spaces $X = \mathbb{C}^N$, and more generally from any **Stein manifold** X, a closed complex submanifold of some complex Euclidean space. It developed from works of **Oka, Grauert, Gromov**, and many others.

***** OKA THEORY IS AN EXISTENCE THEORY *****

It provides solutions to a wide variety of complex analytic problems in the absence of topological obstructions.

This story started with a theorem of Oka in 1939

Source: Wikipedia

Kiyoshi Oka, 1939:

Complex line bundles over a domain of holomorphy $X \subset \mathbb{C}^n$ have the same classification in the topological and the holomorphic category.

More precisely, every complex line bundle $E \to X$ admits a compatible structure of a holomorphic line bundle, and any two holomorphic line bundles which are topologically equivalent are also holomorphically equivalent.

Cohomological proof of Oka's theorem

Let $\sigma(f) = \mathrm{e}^{2\pi \mathrm{i} f}$. The *exponential sheaf sequence* reads:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{O} & \xrightarrow{\sigma} & \mathcal{O}^* & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{C} & \xrightarrow{\sigma} & \mathcal{C}^* & \longrightarrow & 1
\end{array}$$

A part of the associated long exact sequence on cohomology:

$$\begin{array}{ccccccc}
H^1(X,\mathcal{O}) & \longrightarrow & H^1(X,\mathcal{O}^*) & \xrightarrow{c_1} & H^2(X,\mathbb{Z}) & \longrightarrow & H^2(X,\mathcal{O}) \\
\downarrow & & \downarrow & & \| & & \downarrow \\
0 & \longrightarrow & H^1(X,\mathcal{C}^*) & \xrightarrow{c_1} & H^2(X,\mathbb{Z}) & \longrightarrow & 0
\end{array}$$

If X is Stein then $H^1(X,\mathcal{O}) = 0 = H^2(X,\mathcal{O})$, and hence

$$\mathrm{Pic}(X) = H^1(X,\mathcal{O}^*) \cong H^1(X,\mathcal{C}^*) \cong H^2(X,\mathbb{Z}).$$

Stein manifolds

Source: Wikipedia

Karl Stein, 1951:

A complex manifold X is holomorphically complete (Stein) if

1. holomorphic functions on X separate points, and
2. for every discrete sequence $p_j \in X$ there is a holomorphic function $f \in \mathcal{O}(X)$ with $\lim_{j\to\infty} |f(p_j)| = +\infty$.

   Equivalently, for every compact subset $K \subset X$ its holomorphic hull $\widehat{K}_{\mathcal{O}(X)}$ is also compact.

Examples and characterizations of Stein manifolds

- Open Riemann surfaces (Behnke and Stein, 1949).
- Domains of holomorphy in $\mathbb{C}^n$ (Cartan and Thullen, 1932).
- Closed complex submanifolds of $\mathbb{C}^N$ (and of Stein manifolds). Conversely, every Stein n-manifold is a closed complex submanifold of $\mathbb{C}^{2n+1}$ (Remmert, Bishop, Narasimhan, 1956–61).
- Holomorphic coverings of Stein manifolds.
- Cartesian products of Stein manifolds.
- The total space of a holomorphic vector bundle over a Stein base.
- (Grauert, 1958) A complex manifold X is Stein iff it admits a strictly plurisubharmonic exhaustion function $\rho : X \to \mathbb{R}$, $dd^c\rho = i\partial\bar\partial\rho > 0$.
- (Siu, 1976) Every Stein subvariety of a complex space admits an open Stein neighbourhood.

A **Stein space** is a complex space with singularities having similar function theoretic properties. Examples are closed complex subvarieties of complex Euclidean spaces.

The Oka–Grauert Principle

Source: Wikipedia

Hans Grauert, 1958: Let G be a complex Lie group. Then, for principal G-bundles on a Stein space, the holomorphic classification coincides with the topological classification. This holds in particular for complex vector bundles on Stein spaces.

Equivalently, we have an isomorphism

$$H^1(X, \mathcal{O}_X^G) \xrightarrow{\cong} H^1(X, \mathcal{C}_X^G)$$

induced by the inclusion $\mathcal{O}_X^G \hookrightarrow \mathcal{C}_X^G$ of the sheaf of holomorphic maps $X \to G$ into the sheaf of continuous maps.

Reduction to a mapping problem

Every complex vector bundle $E \to X$ of rank k embeds into a trivial bundle $X \times \mathbb{C}^N$ for some $N \geq k$. This gives a map

$$f\colon X \to Gr_k(\mathbb{C}^N), \quad f(x) = E_x \subset \mathbb{C}^N \ \ (x \in X)$$

into the Grassman manifold of k planes in $\mathbb{C}^N$. If $U_{k,N} \to Gr_k(\mathbb{C}^N)$ denotes the universal rank k vector bundle, we have that

$$E \cong f^*(U_{k,N}).$$

Thus, every complex vector bundle $E \to X$ of rank k is the pullback of the universal bundle by some map $f\colon X \to Gr_k(\mathbb{C}^N)$ and $N \in \mathbb{N}$:

$$\begin{array}{ccc} E = f^*U_{k,N} & \longrightarrow & U_{k,N} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\ f\ } & Gr_k(\mathbb{C}^N) \end{array}$$

Holomorphic maps $X \to Gr_k(\mathbb{C}^N)$ induce holomorphic vector bundles $E \to X$. Homotopic maps induce isomorphic vector bundles.

Holomorphic structures on vector bundles

Conclusions:

- If every continuous map $f\colon X \to Gr_k(\mathbb{C}^N)$ is homotopic to a holomorphic map, then every topological complex vector bundle on X admits an equivalent holomorphic structure.
- If every homotopy $f_t : X \to Gr_k(\mathbb{C}^N)$ $(t \in [0,1])$ of continuous maps, connecting a pair of holomorphic maps f_0 and f_1, is homotopic with fixed ends to a homotopy consisting of holomorphic maps, then every pair of complex vector bundles on X which are topologically equivalent are also holomorphically equivalent.

Properties of this type became known as **Oka properties**. Both properties above have affirmative answer if X is Stein.

Theorem (Grauert 1958)

Every complex Lie group and, more generally, every complex homogeneous manifold enjoys all natural Oka properties for holomorphic maps from any Stein space.

What precisely are the Oka properties?

By definition, a Stein manifolds X admits many holomorphic maps $X \to \mathbb{C}$. Now replace $\mathbb{C}$ by a complex manifold Y and ask the following

Main question: For which complex manifolds Y do there exist many holomorphic maps $X \to Y$ from any Stein manifold X?

What is a good way to interpret the notion 'many maps' ?

Start with two classical 19th century theorems:

Weierstrass Theorem: On a discrete subset of a domain X in $\mathbb{C}$ we can prescribe the values of a holomorphic function on X.

Runge Theorem. If $K \subset \mathbb{C}$ is a compact set without holes, then every holomorphic function $K \to \mathbb{C}$ can be approximated uniformly on K by entire functions.

Higher dimensional analogues

Cartan Extension Theorem: If T is a (closed complex) subvariety of a Stein manifold X, then every holomorphic function $T \to \mathbb{C}$ extends to a holomorphic function $X \to \mathbb{C}$.

Oka-Weil Approximation Theorem. Let K be a *holomorphically convex* compact subset of a Stein manifold X (i.e., for every point $p \in X \setminus K$ there exists $f \in \mathcal{O}(X)$ such that $|f(p)| > \sup_K |f|$). Then, every holomorphic function $K \to \mathbb{C}$ can be approximated uniformly on K by holomorphic functions $X \to \mathbb{C}$.

These two properties can be merged: one can at the same time approximate and interpolate by holomorphic functions on Stein manifolds.

These are fundamental properties of Stein manifolds.

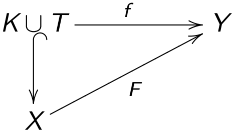
We can also view them as properties of the target manifold, the complex number field $\mathbb{C}$.

We now formulate them as properties of an arbitrary target manifold Y, with source any Stein manifold (or Stein space).

The Basic Oka Property

A complex manifold Y enjoys **BOPAI — the basic Oka property with approximation and interpolation** — if the following hold:

For every Stein inclusion $T \hookrightarrow X$ and every compact $\mathcal{O}(X)$-convex set $K \subset X$, a continuous map $f\colon X \to Y$ that is holomorphic on $K \cup T$ can be deformed to a holomorphic map $F\colon X \to Y$ by a homotopy of maps which are fixed on T, holomorphic on K, and approximate f on K.

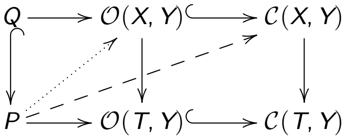


By Oka–Weil and Oka–Cartan, $\mathbb{C}$, and hence $\mathbb{C}^n$, satisfy BOPAI.

By Grauert, every complex homogeneous manifold satisfies BOPAI.

The Parametric Oka Property (POPAI)

Let $Q \subset P$ be a pair of compact Hausdorff spaces. Consider maps $f\colon P \times X \to Y$.



A complex manifold Y enjoys POPI (the parametric Oka property with interpolation) if every lifting in the big square can be deformed to a lifting in the left-hand square. For POPAI, add approximation on a compact $\mathcal{O}(X)$-convex set $K \subset X$ into the picture.

Applying this with parameter pairs $Q = \emptyset \subset P = S^k =$ k-sphere and $Q = S^k \subset P = B^{k+1} = (k+1)$-ball gives

Corollary

If POPAI holds for maps $X \to Y$ then the inclusion $\mathcal{O}(X, Y) \hookrightarrow \mathcal{C}(X, Y)$ is a **weak homotopy equivalence***:*

$$\pi_k(\mathcal{O}(X, Y)) \cong \pi_k(\mathcal{C}(X, Y)), \quad \forall k = 0, 1, 2, \ldots$$

The Oka-Grauert Principle

Theorem (Grauert, 1958)

Every complex homogeneous manifold Y enjoys POPAI for maps $X \to Y$ from any Stein space X and all pairs of finite polyhedra $Q \subset P$.

The analogous result holds for sections $X \to Z$ of a holomorphic G-bundle $h : Z \to X$ (with G a complex Lie group) over a Stein space X.

For Stein manifolds X, a different proof was given in 1986 by Henkin and Leiterer (published in 1998).

Since every isomorphism between G-bundles is a section of an associated G-bundle, it follows that

The holomorphic and the topological classifications of principal G-bundles over any Stein space coincide. This holds in particular for complex vector bundles.

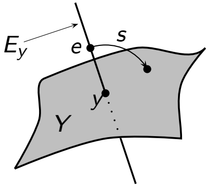
How to characterize Oka properties?

The goal: Find simple (verifiable) necessary and sufficient conditions for Oka property of Y by testing on maps from simple domains in $\mathbb{C}^n$.

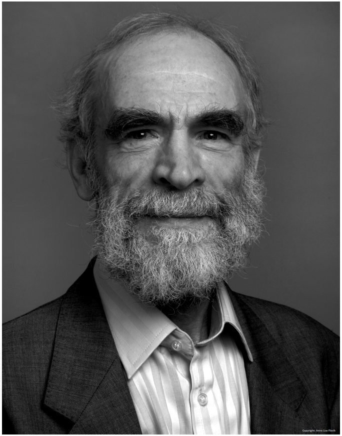
Observation: If Y satisfies BOPAI, then for every point $y \in Y$ there is a dominating holomorphic map $s_y : \mathbb{C}^n \to Y$ such that

$$s_y(0) = y \quad \text{and} \quad \mathrm{d}s_y(0) : T_0\mathbb{C}^n = \mathbb{C}^n \to T_yY \ \text{ is surjective.}$$

Gromov, 1989: A complex manifold Y is called **elliptic** if it admits a **dominating spray**: A holomorphic map $s : E \to Y$, defined on the total space of a holomorphic vector bundle E over Y, such that $s(0_y) = y$ and $s : E_y \to Y$ is a submersion at 0_y for all $y \in Y$.



Gromov's Oka principle



Source: Wikipedia

Gromov 1989: If Y is an elliptic manifold, then maps $X \to Y$ from any Stein manifold satisfy the parametric Oka property.

The analogous result holds for sections of elliptic submersions $h : Z \to X$ over a Stein base.

F. 2002: A complex manifold Y is **subelliptic** if there exist finitely many sprays $s_j : E_j \to Y$ such that $\sum_j ds_j(E_{j,y}) = T_y Y$ for all $y \in Y$.

Subellipticity also implies POPAI.

Examples of dominating sprays

(a) Let G be a complex Lie group and $\mathfrak{g} = T_1 G \cong \mathbb{C}^p$ ($p = \dim G$) be its Lie algebra. Then the map

$$s : G \times \mathbb{C}^p \to G, \quad s(g, v) = \exp(v)g$$

is a dominating spray on G. Note that $\exp(v)$ is the time-1 map of the left invariant vector field V on G determined by $v \in \mathfrak{g}$.

(b) More generally, if Y is a G-homogeneous manifold, then the map

$$s : Y \times \mathbb{C}^p \to Y, \quad s(y, v) = \exp(v)y$$

is a dominating spray on Y.

Hence, **Gromov's theorem implies the Oka–Grauert principle.**

(c) Assume that a complex manifold Y is **holomorphically flexible**, in the sense that it admits finitely many $\mathbb{C}$-complete holomorphic vector fields $V_1, \ldots, V_m$ which span the tangent space of Y at every point. Denote by ϕ_j^t the flow of V_j for time $t \in \mathbb{C}$. Then, the map $s : Y \times \mathbb{C}^m \to Y$ given by

$$s(y, t_1, \ldots, t_m) = \phi_1^{t_1} \circ \phi_2^{t_2} \circ \cdots \circ \phi_m^{t_m}(y),$$

is a dominating spray on Y.

Further examples of (sub-) elliptic manifold

An algebraic manifold Y is called **algebraically elliptic** if it admits a dominating algebraic spray, and **algebraically subelliptic** if it admits a finite dominating family of algebraic sprays.

If an algebraic manifold Y is covered by finitely many Zariski open charts which are algebraically subelliptic, then Y is also algebraically subelliptic.

Arzhantsev et al., 2013 An algebraic manifold Y is **algebraically flexible** if finitely many complete algebraic vector fields on Y with algebraic flows (locally nilpotent derivations, LNDs) pointwise span the tangent bundle of Y.

- (a) Every algebraically flexible manifold is algebraically elliptic.
- (b) If $A \subset \mathbb{C}^n$ is a tame analytic subvariety with $\dim A \leq n-2$ then $\mathbb{C}^n \setminus A$ is holomorphically flexible and hence elliptic.
- (c) If in addition A is algebraic then $Y = \mathbb{C}^n \setminus A$ is algebraically flexible and hence algebraically elliptic.
- (d) The complement of an algebraic subvariety A of codimension at least two in $\mathbb{CP}^n$ or in a complex Grassmanian is algebraically subelliptic.
- (e) **Kaliman and Zaidenberg, 2022**: Every algebraically subelliptic manifold is algebraically elliptic.

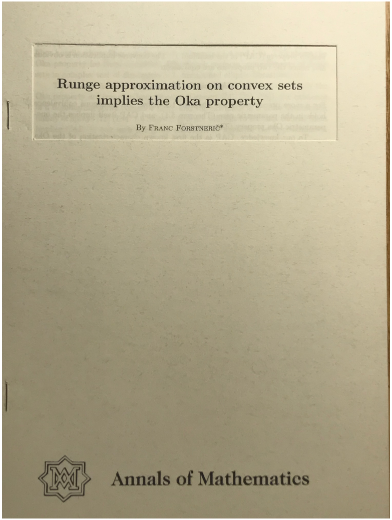
Are there simpler sufficient conditions for Oka properties?

The (sub) ellipticity conditions introduced by Gromov in 1989 do not satisfy any reasonable functorial properties, so it remained unclear which properties hold for the class of manifolds satisfying Oka properties. As a consequence, very few new examples appeared in the next 15 years.

In his 1989 paper, Gromov asked many questions regarding simpler sufficient conditions implying the Oka properties. In particular, he asked whether they are implied by the Runge approximation property from a class of simple domains in Euclidean spaces.

This question was answered affirmatively during 2005–2009, and it led to the first unification of Oka theory.

Convex approximation property implies Oka properties



F., 2005–2009 Assume that a complex manifold Y enjoys the **convex approximation property (CAP)**:

- every holomorphic map $K \to Y$ from a compact convex set in a Euclidean space $\mathbb{C}^n$ is a limit of entire maps $\mathbb{C}^n \to Y$.

Then it also enjoys POPAI:

- every continuous map $X \to Y$ from a Stein manifold X can be deformed to a holomorphic map
- approximation and interpolation of holomorphic maps
- analogous properties hold for families of maps $X \to Y$

Enter Oka manifolds — 2009

A complex manifold satisfying these equivalent properties is called an

***** OKA MANIFOLD *****

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COMPTES RENDUS MATHEMATIQUE

Complex Analysis

Oka manifolds

Franc Forstnerič

Faculty of Mathematics and Physics, University of Ljubljana, and Institute of Mathematics, Physics and Mechanics, Jadranska 19, 1000 Ljubljana, Slovenia

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To Mikhaël Gromov on the occasion of his receiving the Abel Prize

MSC 2020: New subfield **32Q56 Oka principle and Oka manifolds**

The main theorem on Oka manifolds

F., 2010 (Theorem 5.4.4 in the book)

Let X be a reduced Stein space, Z be a complex space, and $h : Z \to X$ be a stratified holomorphic fibre bundle all of whose fibres enjoy CAP. Then, sections $X \to Z$ satisfy POPAI (the parametric Oka property with approximation and interpolation):

Let K be compact $\mathcal{O}(X)$-convex subset of X, X' be a closed complex subvariety of X, $Q \subset P$ be compact Hausdorff spaces, and $f \colon P \times X \to Z$ be a continuous map such that

- (a) for every $p \in P$, $f(p, \cdot\,) : X \to Z$ is a section of $Z \to X$ that is holomorphic on a neighborhood of K and $f(p, \cdot\,)|_{X'}$ is holomorphic on X', and
- (b) $f(p, \cdot\,)$ is holomorphic on X for every $p \in Q$.

Then there is a homotopy $f_t : P \times X \to Z$ $(t \in [0, 1])$, with $f_0 = f$, such that f_t enjoys properties (a) and (b) for all $t \in [0, 1]$ and also the following:

- (i) $f_1(p, \cdot\,)$ is holomorphic on X for all $p \in P$,
- (ii) f_t is uniformly close to f on $P \times K$ for all $t \in [0, 1]$, and
- (iii) $f_t = f$ on $(Q \times X) \cup (P \times X')$ for all $t \in [0, 1]$.

Oka manifolds are strongly dominable

Corollary (F., 2017)

Every Oka manifold Y is the image of a strongly dominating holomorphic map $f\colon \mathbb{C}^n \to Y$ with $n = \dim Y$; i.e., $f(\mathbb{C}^n \setminus \mathrm{br}\, f) = Y$.

Problem

Does the converse hold?

Corollary

On any Oka manifold Y, the Kobayashi infinitesimal metric and the Eisenmann volume form vanish identically. Furthermore, every negative plurisubharmonic function on Y is constant (i.e., Y is Liouville).

Thus, Oka manifolds are completely anti-hyperbolic.

Kobayashi and Ochiai 1975: A compact complex manifold of general Kodaira type is not dominable by $\mathbb{C}^n$. Hence, no such manifold is Oka.

The main theorem revisited

The proof of the main theorem on Oka manifolds also gives the following result which is useful in applications. For simplicity we only state the basic case.

Theorem

Assume that X is a reduced Stein space, K is a compact $\mathcal{O}(X)$-convex set in X, X' is closed complex subvariety of X, Ω is an Oka domain in a complex manifold Y, and $f_0 : X \to Y$ is a continuous map which is holomorphic on a neighbourhood of K and on X' such that

$$f_0(\overline{X \setminus K}) \subset \Omega.$$

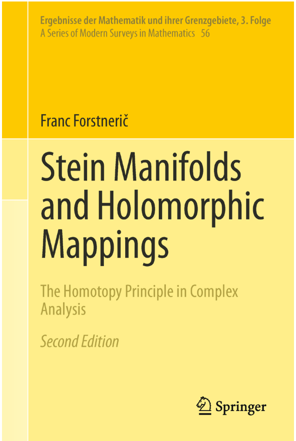
Then there is a homotopy $f_t : X \to Y$ $(t \in [0,1])$ connecting f_0 to a holomorphic map $f_1 : X \to Y$ satisfying the conclusion of the main theorem and

$$f_t(\overline{X \setminus K}) \subset \Omega \text{ for all } t \in [0,1].$$

This is especially useful in light of a recent result of **Kusakabe (2020)** saying that, for a compact holomorphically convex set K in $\mathbb{C}^n$ $(n > 1)$ and more generally in any Stein manifold Y with the holomorphic (Varolin) density property, the complement $\Omega = Y \setminus K$ is Oka.

How to find Oka manifolds?

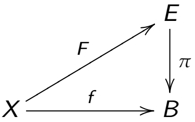
Oka manifolds are like bitcoins — precious but hard to discover.



- Oka theory is a major topic of my monograph (2011, 2017).
- The main theorem on Oka manifolds (Theorem 5.4.4) takes a large part of Chapter 5 to prove. Gromov's Oka principle is presented in detail in Chapter 6.
- Major developments were made by Yuta Kusakabe after 2017, and the main ones will be discussed later.

The Oka principle for liftings

Given a map $\pi : E \to B$, we say that a map $F : X \to E$ is a **lifting** of a map $f : X \to B$ if $\pi \circ F = f$. Similarly one defines homotopies of liftings.



Corollary (The Oka principle for liftings)

Assume that $\pi : E \to B$ is a stratified holomorphic fibre bundle all of whose fibres are Oka manifolds.

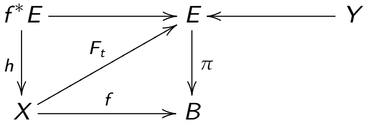
If X is a reduced Stein space and $f : X \to B$ is a holomorphic map, then any continuous lifting $F_0 : X \to E$ of f admits a homotopy of liftings $F_t : X \to E$ $(t \in [0, 1])$ such that F_1 is a holomorphic lifting.

Furthermore, if F_0 is holomorphic on a subset $K \cup X' \subset X$ as in the main theorem, then the homotopy $\{F_t\}_{t \in [0,1]}$ can be chosen to satisfy properties (i) and (ii) in that theorem.

Proof

Assume first that $\pi : E \to B$ is a holomorphic fibre bundle with Oka fibre Y.

Let $h : f^*E \to X$ denote the pullback bundle whose fibre over a point $x \in X$ is $E_{f(x)} \cong Y$.



Sections $X \to f^*E$ of h are in one-to-one correspondence with liftings $F : X \to E$ of the map $f : X \to B$. Since the fibre Y of $h : f^*E \to X$ is Oka, the conclusion follows from the main theorem.

In the general case, we stratify X by a decreasing sequence of closed complex subvarieties so that the strata are mapped by f to the strata of B; then f^*E is also a stratified fibre bundle over X and we conclude the proof as before.

The parametric Oka property for liftings also holds, but the proof is more involved. (F. F., Invariance of the parametric Oka property, Complex Analysis, pp. 125–143. Trends in Math., Birkhäuser, Basel, 2010.)

Oka maps

A holomorphic map $h : Y \to Z$ is said to be an **Oka map** if it satisfies the parametric Oka principle for liftings and it is a topological fibration (i.e., it enjoys the homotopy lifting property).

Example

Every holomorphic fibre bundle $Y \to Z$ with Oka fibre is an Oka map.
In particular, every holomorphic covering projection is an Oka map.

It is easily seen that an Oka map $h : Y \to Z$ over a connected base is a surjective submersion with Oka fibres $Y_z = h^{-1}(z)$, $z \in Z$. In particular:

A complex manifold Y is Oka iff the projection $Y \mapsto \mathrm{point}$ is an Oka map.

The converse fails. Let $g : \mathbb{D} = \{|z| < 1\} \to \mathbb{C}$ be a continuous function and

$$Y = \{(z, y) : z \in \mathbb{D},\ y \in \mathbb{C},\ y \neq g(z)\} \xrightarrow{h} \mathbb{D}, \quad h(z, y) = z.$$

Then the fibres of h are $\mathbb{C}^*$ and hence Oka, but h is an Oka map iff g is holomorphic, in which case Y is fibrewise isomorphic to $\mathbb{D} \times \mathbb{C}^*$. This follows from a result of Eremenko on the holomorphic variation of the missing value of entire maps $\mathbb{C} \to \mathbb{C}$.

Up–down

Characterization of Oka manifolds by CAP gives new examples and constructions of Oka manifolds. Here is an important example.

Theorem (Up–down)

If $h : Y \to Z$ is an Oka map, then Y is an Oka manifold if and only if Z is an Oka manifold.

This holds in particular if $Y \to Z$ is a holomorphic fibre bundle with Oka fibre, and in particular if it is a holomorphic covering space.

Corollary

The following are equivalent for a Riemann surface Y:

- (a) *Y is an Oka manifold.*
- (b) *Y is not Kobayashi hyperbolic.*
- (c) *Y is one of the surfaces $\mathbb{CP}^1$, $\mathbb{C}$, $\mathbb{C}^*$, or a torus.*

Proof of theorem up–down

Assume first that Y is an Oka manifold and let us prove that so is Z. We shall use the characterization of Oka manifolds by CAP.

Let K be a compact convex in $\mathbb{C}^n$ and $f\colon K \to Z$ be a holomorphic map.

Since $h : Y \to Z$ is an Oka map and K is contractible, f lifts to a holomorphic map $g : K \to Y$ with $h \circ g = f$.

Since Y is Oka, we can approximate g as closely as desired uniformly on K by a holomorphic map $G : \mathbb{C}^n \to Y$.

The map $F = h \circ G : \mathbb{C}^n \to Z$ then approximates f on K. This shows that Z enjoys CAP, and hence it is Oka.

A holomorphic retraction lemma

To proves the converse part, we shall need the following lemma (F., Lemma 3.4 in Ann. Inst. Fourier, 54(6):1913–1942, 2004).

It generalizes the Docquier–Grauert holomorphic retraction lemma.

Lemma

Let $h\colon Y \to Z$ be a holomorphic submersion of a Stein manifold Y onto a complex manifold Z. Then there are an open Stein domain $W \subset Z \times Y$ containing the Stein submanifold (the graph of h)

$$S = \{(z, y) \in Z \times Y : z = h(y)\}$$

and a holomorphic retraction $\widetilde{\rho}\colon W \to S$ of the form

$$\widetilde{\rho}(z, y) = (z, \rho(z, y)) \quad \text{for } (z, y) \in W.$$

Note that $\rho(z, \cdot\,)$ is a holomorphic retraction onto the fibre over z, depending holomorphically on z.

Proof of theorem up–down, 2

The proof of the theorem can now be completed as follows. Assuming that Z is an Oka manifold, we shall verify that Y enjoys CAP.

Consider the manifolds $\widetilde{Y} = \mathbb{C}^n \times Y$, $\widetilde{Z} = \mathbb{C}^n \times Z$ and the projection

$$\tilde{h} : \widetilde{Y} \to \widetilde{Z}, \quad \tilde{h}(\zeta, y) = (\zeta, h(y)) \text{ for } \zeta \in \mathbb{C}^n \text{ and } y \in Y.$$

Let $f \colon K \to Y$ be a holomorphic map. Set $g = h \circ f \colon K \to Z$. The graphs

$$\Gamma_f = \{(\zeta, f(\zeta)) : \zeta \in K\} \subset \widetilde{Y} \quad \text{and} \quad \Gamma_g = \{(\zeta, g(\zeta)) : \zeta \in K\} \subset \widetilde{Z}$$

are Stein submanifolds of $\widetilde{Y}$ and $\widetilde{Z}$, which admit open Stein neighbourhoods $Y' \subset \widetilde{Y}$ and $Z' \subset \widetilde{Z}$ by Siu's theorem. They can be chosen such that $\tilde{h}|_{Y'} : Y' \to Z'$ is a surjective holomorphic submersion.

For every point $p = (\zeta, z) \in Z'$, the lemma furnishes a holomorphic retraction ρ_p from a neighbourhood of the fibre $Y'_p = Y' \cap \tilde{h}^{-1}(p)$ onto Y'_p, depending holomorphically on $p \in Z'$.

Proof of theorem up–down, 3

- Since Z is Oka, we can approximate the map $g : K \to Z$ uniformly on K by a holomorphic map $G : \mathbb{C}^n \to Z$.
- If the approximation is close enough, then for every $\zeta \in K$ the point $(\zeta, f(\zeta)) \in \Gamma_f \subset Y'$ lies in the domain of the retraction $\rho_{(\zeta, G(\zeta))}$.
- Let $f_1(\zeta) \in Y$ denote the projection of the point $\rho_{(\zeta, G(\zeta))}(\zeta, f(\zeta)) \in Y' \subset \mathbb{C}^n \times Y$ to Y.
- The map $f_1 : K \to Y$ is holomorphic, uniformly close to f, and it satisfies $h \circ f_1 = G$; i.e., f_1 is an h-lifting of G over K.
- Since $h : Y \to Z$ is an Oka map, $G : \mathbb{C}^n \to Z$ is a holomorphic map, and f_1 is a holomorphic lifting of G over K, we can approximate f_1 uniformly on K by a holomorphic map $F : \mathbb{C}^n \to Y$ satisfying $h \circ F = G$. Hence, Y enjoys CAP and so is an Oka manifold.

A couple of corollaries

Corollary

If $h : Y \to Z$ is a holomorphic vector bundle, or more generally a holomorphic fibre bundle whose fibre is a complex homogeneous manifold (such as $\mathbb{CP}^n$), then Y is Oka if and only if Z is Oka.

Corollary

If $h : Y \to Z$ is a holomorphic fibre bundle whose base Z and fibre are one of the Riemann surfaces $\mathbb{CP}^1$, $\mathbb{C}$, $\mathbb{C}^*$, or a torus $\mathbb{C}/\Gamma$, then Y is an Oka manifold. In particular, all Hirzebruch surfaces are Oka manifolds.

Note that Hirzebruch surfaces are $\mathbb{CP}^1$-bundles over $\mathbb{CP}^1$.

Problem

Are any or all K3 surfaces Oka manifolds?
Are rationally connected manifolds Oka?