Oka manifolds

Franc Forstnerič

Univerza v Ljubljani

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Contents of Lecture 2

In the first lecture, we introduced the convex approximation property, **CAP**, and stated that it characterizes Oka manifolds.

We also introduced the notion of an **elliptic manifold** and stated that

$$\mathbb{C}\text{-homogeneous} \Longrightarrow \text{elliptic} \Longrightarrow \text{CAP} \Longleftrightarrow \text{OKA}.$$

The first two implications cannot be reversed. In this lecture we shall:

- Prove that elliptic $\Longrightarrow$ CAP (Gromov 1989).
- Sketch the proof of CAP $\Longleftrightarrow$ OKA (F. 2005–9).
- Introduce Condition Ell_1 and prove Kusakabe's theorem (2021):

$$\mathrm{Ell}_1 \Longleftrightarrow \text{CAP}.$$

- Indicate the proof of Kusakabe's localization theorem.
- Recent applications of these new results.
- Survey of the known examples of Oka manifolds.

An h-Runge approximation theorem

Theorem (Gromov 1989; Theorem 6.6.1 in my book)

Let $K \subset L$ be Stein compacts in a complex manifold X, and assume that K is $\mathcal{O}(L)$-convex. Let Y be an elliptic or a subelliptic complex manifold.

Given a holomorphic map $f_0 : L \to Y$ and a homotopy of holomorphic maps $f_t : K \to Y$ $(t \in [0,1])$, we can approximate $\{f_t\}$ uniformly on K by a homotopy $\tilde{f}_t : L \to Y$ $(t \in [0,1])$ of holomorphic maps with $\tilde{f}_0 = f_0$.

A parametric version of this result holds as well.

Sketch of proof of the h-Runge theorem

Assume that Y is elliptic. (A similar proof applies to subelliptic manifolds.)
Let $\pi : E \to Y$ be a holomorphic vector bundle and $s : E \to Y$ be a dominating spray:

$$s(0_y) = y \ \text{ and } \ ds_{0_y}(E_y) = T_y Y \ \text{ for every } \ y \in Y.$$

Given a holomorphic map $f_0 : L \to Y$ from a Stein compact $L \subset X$, consider the pullback bundle $\pi_0 : E_0 = f_0^* E \to L$:

$$E_0 = \{(x, e) : x \in L,\ e \in E,\ f_0(x) = \pi(e)\}, \quad \pi_0(x, e) = x.$$

We have a natural map $\iota : E_0 \to E$ given by $\iota(x, e) = e$ such that

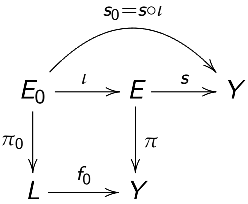
$$f_0 \circ \pi_0 = \pi \circ \iota \quad \text{holds on } E_0,$$

and a holomorphic map $s_0 = s \circ \iota : E_0 \to Y$ given by

$$s_0(x, e) = s(e), \quad (x, e) \in E_0.$$

Note that $s_0(x, 0) = s(0_{f_0(x)}) = f_0(x)$ and $(ds_0)_{(x,0)}(E_{0,x}) = T_{f_0(x)} Y$.

The diagram of maps



$$E_0 = {f_0}^* E, \quad s_0 = s \circ \iota : E_0 \to Y,$$

$$s_0(x, 0) = s(0_{f_0(x)}) = f_0(x), \quad (ds_0)_{(x,0)}(E_{0,x}) = T_{f_0(x)} Y.$$

Sketch of proof, 2

The domination property of s_0 implies that for some $t_0 \in (0,1]$ we can lift the homotopy $f_t : K \to Y$ for $t \in [0, t_0]$ to a homotopy of holomorphic sections $g_t : K \to E_0|_K$ $(t \in [0, t_0])$:

$$(*) \quad s_0(x, g_t(x)) = f_t(x) \ \text{ for } x \in K \text{ and } t \in [0, t_0], \text{ with } g_0 \text{ the zero section.}$$

In fact, splitting $E_0 = E_0' \oplus E_0''$ where $E_0'' = \ker ds_0|_{L \times \{0\}}$, the restriction of ds_0 to the zero section of E_0' gives for every $x \in L$ a $\mathbb{C}$-linear isomorphism

$$(ds_0)_{0_x} : (E_0')_x \xrightarrow{\cong} T_{f_0(x)} Y.$$

Hence, for $t \in [0, t_0]$ there is a unique section g_t of $E_0'|_K$ satisfying $(*)$.

The parametric version of the Oka–Weil theorem shows that $\{g_t\}_{t \in [0,t_0]}$ can be approximated uniformly on $K \times [0, t_0]$ by a homotopy of holomorphic sections $\tilde g_t : L \to E_0$ $(t \in [0, t_0])$, with $\tilde g_0$ the zero section. Taking

$$\tilde f_t(x) = s_0(x, \tilde g_t(x)) \in Y \ \text{ for } x \in L \text{ and } t \in [0, t_0]$$

solves the problem on this subinterval. We now apply the same argument to the map $\tilde f_{t_0} : L \to Y$. We are done in finitely many steps.

Ellipticity implies CAP

Corollary

- *A (sub-)elliptic manifold Y satisfies* CAP*, and hence is an Oka manifold.*
- *A Stein Oka manifold is elliptic.*

For the first item, let $K \subset \mathbb{C}^n$ be convex. Then, any map $f_1 : K \to Y$ can be inserted in a homotopy $f_t : K \to Y$ $(t \in [0,1])$ with f_0 a constant map $\mathbb{C}^n \mapsto y_0 \in Y$. (Take $f_t = f_1 \circ \tau_t$ where the homotopy $\tau_t : \mathbb{C}^n \to \mathbb{C}^n$ contracts the identity map τ_1 on $\mathbb{C}^n$ to the constant map $\tau_0 : \mathbb{C}^n \to p \in K$.) Hence, the theorem shows that ellipticity of Y implies CAP.

For the converse part, assume that Y is Stein and Oka. Using the composition of local flows of finitely many (say, N) holomorphic vector fields on Y spanning TY, we can find a neighbourhood $U \subset Y \times \mathbb{C}^N$ of $Y \times \{0\}$ and a holomorphic map $s_0 : U \to Y$ having the properties of a dominating spray.

Since Y satisfies CAP and hence is OKA, we can find a holomorphic map $s : Y \times \mathbb{C}^N \to Y$ which agrees with s_0 to the second order along the zero section $Y \times \{0\}$. Clearly, such s is a dominating spray on Y.

An h-Runge approximation theorem for algebraic maps

There is a version of this result for algebraically (sub-) elliptic manifolds.

Theorem (F., 2006; Theorem 6.15.1 in my book)

Let X be an affine algebraic manifold, Y be an algebraically subelliptic manifold, $K \subset X$ be a compact $\mathcal{O}(X)$-convex set, $f_0 : X \to Y$ be an algebraic map, and $f_t : K \to Y$ $(t \in [0,1])$ be a homotopy of holomorphic maps.

Then, we can approximate $\{f_t\}$ uniformly on K by a homotopy $F_t : X \to Y$ $(t \in [0,1])$ of algebraic maps with $F_0 = f_0$.

If in addition the homotopy f_t is fixed on a closed algebraic subvariety $X' \subset X$ then F can be chosen such that $F(x,t) = f(x)$ for all $x \in X'$ and $t \in \mathbb{C}$.

In particular, a holomorphic map $X \to Y$ that is homotopic to an algebraic map is a limit of algebraic maps uniformly on compacts in X.

In general there exist homotopy classes of maps $X \to Y$ (even for simple manifolds like $Y = \mathbb{CP}^1$) which cannot be represented by algebraic maps. In particular, there exist algebraically nontrivial line bundles on affine algebraic curves. However every holomorphic vector bundle on an open Riemann surface X is holomorphically trivial since $H^2(X;\mathbb{Z}) = 0$.

Failure of the basic algebraic basic Oka principle

Example

(Loday) Let Σ^n denote the complex n-sphere, i.e., the affine variety

$$\Sigma^n = \left\{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} : z_0^2 + \cdots + z_n^2 = 1\right\}.$$

Then Σ^n retracts onto the real n-sphere S^n. Also, Σ^n is algebraically elliptic for $n \geq 2$. This can either be seen directly (in my book in minimal surfaces with Alarcón and López, Section 1.15), and it also follows from the fact that Σ^n is homogeneous for the complex Lie group $SO(n+1, \mathbb{C})$, and $\mathrm{Hom}(SO(n+1, \mathbb{C}), \mathbb{C}^*) = 1$ for $n \geq 2$.

Loday showed that every algebraic map $\Sigma^p \times \Sigma^q \to \Sigma^{p+q}$ is null-homotopic when p and q are odd, but there always exists a homotopically nontrivial continuous (hence also holomorphic) map $\Sigma^p \times \Sigma^q \to \Sigma^{p+q}$. □

Failure of the algebraic Oka principle

Theorem (Lárusson and Truong, 2019)

If Y is an algebraic manifold which contains a rational curve $\mathbb{CP}^1$ or is compact, then Y does not have any of the algebraic Oka properties.

We explain the basic idea for the case when Y is a projective manifold.

It is easily seen that the algebraic interpolation property (aIP), or the basic algebraic Oka property (aBOP), implies the existence of a nontrivial rational curve $g\colon \mathbb{CP}^1 \to Y$. Assuming now Y that admits such a curve, we will show that Y does not satisfy aIP; a similar argument excludes the other properties.

The basic case to consider is $Y = \mathbb{CP}^1$. Let $X \subset \mathbb{C}^2$ be an affine algebraic curve whose projective closure is not rational. Then, X admits an algebraic line bundle $L \to X$ all of whose nonzero tensor powers are algebraically nontrivial, and every such bundle is the pullback of the universal bundle $U \to \mathbb{CP}^1$ by an algebraic map $f\colon X \to \mathbb{CP}^1$.

Since X is an open Riemann surface, f is null-homotopic, and hence it extends to a continuous map $\mathbb{C}^2 \to \mathbb{CP}^1$.

Failure of the algebraic Oka principle, 2

If $\mathbb{CP}^1$ satisfies the algebraic interpolation property, aIP, then f also extends to a regular map $\mathbb{C}^2 \to \mathbb{CP}^1$. By the Quillen–Suslin theorem, it follows that the line bundle $f^*U \to \mathbb{C}^2$ is algebraically trivial.

This contradicts the assumption that the restriction $L = f^*U|_X \to X$ is algebraically nontrivial, so $\mathbb{CP}^1$ does not satisfy the algebraic interpolation property.

In the general case when Y is a projective manifold and $g : \mathbb{CP}^1 \to Y$ is a nontrivial rational curve, taking an ample line bundle $E \to Y$, the pullback $g^*E \to \mathbb{CP}^1$ is algebraically nontrivial. With $f : X \to \mathbb{CP}^1$ as above, we see as before that the map $g \circ f : X \to Y$ does not extend to an algebraic map $\mathbb{C}^2 \to Y$. Hence, Y does not satisfy aIP.

For a general compact algebraic manifold Y, one uses finitely many blowups in order to obtain a projective manifold.

CAP $\implies$ OKA (Theorem 5.4.4 in my book)

We shall now outline the proof of the basic Oka principle with approximation for maps $X \to Y$, where X is a Stein manifold and Y satisfies CAP.

We are given a continuous map $f = f_0 : X \to Y$ which is holomorphic on a neighbourhood of a compact $\mathcal{O}(X)$-convex set $K \subset X$. The goal is to construct a sequence of continuous maps $f_j : X \to Y$ $(j = 1, 2, \ldots)$ and a normal exhaustion of X by compact $\mathcal{O}(X)$-convex sets

$$K = K_0 \subset K_1 \subset \cdots \subset \bigcup_{j=0}^{\infty} K_j = X$$

such that the following hold for every $j = 1, 2, \ldots$:

- f_j is holomorphic on a neighborhood of K_j,
- f_j approximates f_{j-1} on K_{j-1}, and
- there is a homotopy of maps $f_{j-1,t} : X \to Y$ $(t \in [0,1])$, with $f_{j-1} = f_{j-1,0}$ and $f_j = f_{j-1,1}$, which are holomorphic and close to f_{j-1} on K_{j-1}.

If the approximations are close enough then the sequence f_j converges uniformly on compact in X to a holomorphic map $F = \lim_{j\to\infty} f_j : X \to Y$, and the homotopies $f_{j-1,t}$ also converge to a homotopy from f to F.

Strongly plurisubharmonic (SPSH) exhaustion functions

Given a neighbourhood U_0 of $K = K_0$, we choose a SPSH Morse exhaustion function $\rho : X \to \mathbb{R}$ such that $K_0 \subset \{\rho \le 0\} \subset U_0$. The exhaustion K_j is built by using ρ. Set $X_t = \{\rho \le t\}$ for $t \in \mathbb{R}$. The proof combines two basic cases:

1. **The noncritical case:** Let $a < b$ be such that ρ has no critical values on $[a, b]$. There is no change of topology from X_a to X_b.

2. **The critical case:** Let $c \in \mathbb{R}$ be a critical value of ρ. These values are isolated since ρ is Morse. Choose $a < c < b$ such that c is the only critical value on $[a, b]$. The topology of X_t changes when t crosses the critical value c. We must explain how to approximately extend f from X_a to X_b.

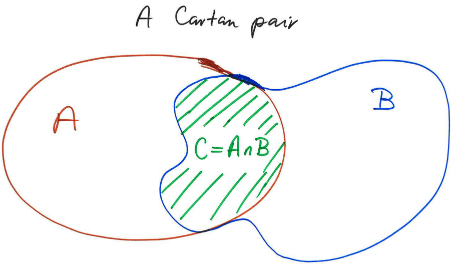
The CAP condition on the manifold Y is only used in the noncritical case (to fatten the region of holomorphicity of the map). This gives rise to

F. & Slapar (2007), The soft Oka principle: Every continuous map $f : (X, J_0) \to Y$ from a Stein manifold (X, J_0) to an arbitrary complex manifold Y is homotopic to a holomorphic map $F : (X, J_1) \to Y$ with respect to a Stein structure J_1 on X which is homotopic to J_0.

The noncritical case: Cartan pairs

A pair (A, B) of compact subsets in a complex manifold X is a **Cartan pair** if it satisfies the following two conditions:

1. The sets $D = A \cup B$ and $C = A \cap B$ are Stein compacts (i.e., they have bases of open Stein neighbourhoods), and
2. A and B are *separated* in the sense that $\overline{A \setminus B} \cap \overline{B \setminus A} = \varnothing$.
3. A Cartan pair (A, B) is **special** if the sets $C \subset B$ are convex in some holomorphic coordinates on a neighbourhood of B. In such case, B is called a **convex bump on A**.



Extending the map across a convex bump

Fact: If ρ has no critical points on $[a, b]$ then $X_b = \{\rho \leq b\}$ is obtained from $X_a = \{\rho \leq a\}$ by successively attaching convex bumps finitely many times. The reason is that every strongly pseudoconvex domain is locally at any boundary point strongly convex in suitable local coordinates.

This reduces our problem to the following:

Lemma

Assume that the complex manifold Y enjoys CAP. Let (A, B) be a special Cartan pair in X. Given a holomorphic map $f_0 : A \to Y$, we can approximate it on A by a holomorphic map $F : A \cup B \to Y$.

The existence of a homotopy is immediate from the construction. A finite application of this lemma extends the map from X_a to X_b, with approximation on X_a.

Extending the map across a convex bump, 2

We may assume that A and B are compact strongly pseudoconvex domains. Here are the steps in the proof of the extension lemma:

- The graph of f_0 over A in $X \times Y = Z$ is a Stein compact, and hence it has a Stein neighbourhood. This allows us to construct a holomorphic spray $f\colon A \times U \to Y$, where $0 \in U \subset \mathbb{C}^N$ is a ball, such that $f(\cdot\,, 0) = f_0$ and
$$\frac{\partial}{\partial z}\Big|_{z=0} f(x,z) : \mathbb{C}^N \to T_{f_0(x)} Y \quad \text{is surjective for every } x \in A.$$
- Let $C = A \cap B$. Since $C \subset B$ are convex sets in some local holomorphic coordinates on X and the set $C \times U$ is also convex, we can apply CAP to approximate f (shrinkig U slightly) by a holomorphic map $g : B \times U \to Y$.
- If the approximation of f by g is close enough on $C \times U$, we can find a smaller ball $0 \in U' \subset U$ and a **holomorphic transition map**
$$\gamma : C \times U' \to C \times U, \quad \gamma(x,z) = (x, c(x,z))$$
close to the identity map $\gamma_0(x,z) = (x,z)$ such that
$$f = g \circ \gamma \quad \text{holds on } C \times U'.$$

A gluing lemma

Lemma (Proposition 5.8.1 in my book)

Let $(A, B, C = A \cap B)$ *be a Cartan pair in* X*. Given a holomorphic map*

$$\gamma : C \times U' \to C \times U$$

as above, close to the identity map, and a slightly smaller ball $0 \subset V \subset U'$*, there are holomorphic maps*

$$\begin{aligned} \alpha(x,z) &= (x, a(x,z)), \quad x \in A, \ z \in V, \\ \beta(x,z) &= (x, b(x,z)), \quad x \in B, \ z \in V \end{aligned}$$

close to the identity on their respective domains (depending on how close γ *is to the identity) such that*

$$\gamma \circ \alpha = \beta \quad \text{holds on } C \times V.$$

This is a nonlinear version of Cousin-I problem, generalizing Cartan's lemma. It is proved by using the solution to the $\bar\partial$-equation with bounds on strongly pseudoconvex domains and the implicit function theorem in Banach spaces.

Conclusion of the noncritical case

Recall that

$$f = g \circ \gamma \quad \text{holds on } C \times U'$$

and

$$\gamma \circ \alpha = \beta \quad \text{holds on } C \times V.$$

It follows that

$$f \circ \alpha = g \circ \gamma \circ \alpha = g \circ \beta \quad \text{holds on } C \times V.$$

Hence, $f \circ \alpha$ and $g \circ \beta$ amalgamate into a holomorphic map

$$F : (A \cup B) \times V \to Y$$

approximating f on $A \times V$. The map

$$F_0 = F(\cdot\,, 0) : A \cup B \to Y$$

then provides the desired holomorphic approximation of $f_0 : A \to Y$.

This lemma has been used in numerous constructions of holomorphic maps, both in Oka theory and wider. It was used in the construction of proper holomorphic maps of Stein manifolds into more general complex manifolds.

The critical case

Suppose that $p \in X$ is a critical point of a Morse SPSH function $\rho : X \to \mathbb{R}$ with Morse index k. Then, $k \in \{0, 1, \ldots, n = \dim_{\mathbb{C}} X\}$.

By a small change of ρ at p, we have in local holomorphic coordinates $z = (z', z'') = (x' + \mathrm{i}y', x'' + \mathrm{i}y'') \in \mathbb{C}^k \oplus \mathbb{C}^{n-k}$ on X at p, with $z(p) = 0$, that

$$\rho(z) = \rho(0) - |x'|^2 + |x''|^2 + \sum_{j=1}^{n} \lambda_j y_j^2$$

where $\lambda_j > 1$ for $j \in \{1, \ldots, k\}$ and $\lambda_j \geq 1$ for $j \in \{k+1, \ldots, n\}$.

Assume $\rho(0) = 0$. Fix $c_0 > 0$ such that ρ has no critical values in $[-c_0, c_0] \setminus \{0\}$. When $t \in \mathbb{R}$ passes the value 0, the change of topology of $X_t = \{\rho \leq t\}$ at $p = 0$ is described by attaching to X_{-c_0} the totally real disc handle

$$E = \{(x' + \mathrm{i}0', 0'') : |x'|^2 \leq c_0\}$$

and thickening the union $X_{-c_0} \cup E$ to a strongly pseudoconvex handlebody.

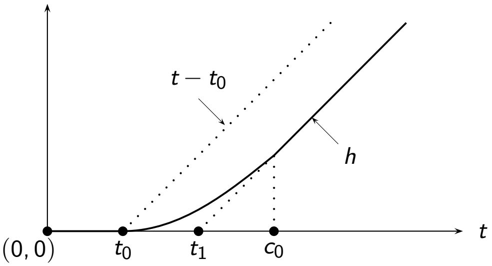
The SPSH function τ

More precisely, we choose a SPSH function τ of the form

$$\tau(z) = -h(|x'|^2) + |x''|^2 + \sum_{j=1}^{n} \lambda_j y_j^2 \tag{1}$$

where $h\colon \mathbb{R} \to [0, +\infty)$ is a smooth convex increasing function as in the following illustration, with $h(t) = t - t_1$ for $t \geq c_0$. Here,

$$1 < \mu < \min\{\lambda_1, \ldots, \lambda_k\} \quad \text{and} \quad t_0 = c_0\big(1 - \frac{1}{\mu}\big)^2 \in (0, c_0).$$

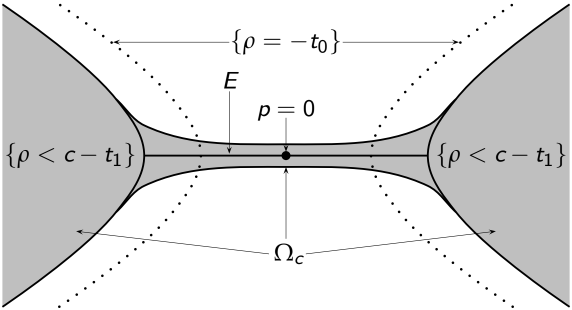


A strongly pseudoconvex handlebody

The function τ satisfies the following conditions:

- (a) $\{\rho \le -c_0\} \cup E \subset \{\tau \le 0\} \subset \{\rho \le -t_0\} \cup E$,
- (b) $\{\rho \le c_0\} \subset \{\tau \le 2c_0\} \subset \{\rho < 3c_0\}$,
- (c) $\tau = \rho + t_1$ on $\{|x'|^2 \ge c_0\}$ for some $t_1 \in (t_0, c_0)$, and
- (d) τ has no critical values in $(0, +\infty)$.

The details can be found in Section 3.11 of my book. The illustration shows the strongly pseudoconvex handlebody $\Omega_c = \{\tau < c\}$ for $c \in (0, c_0)$.



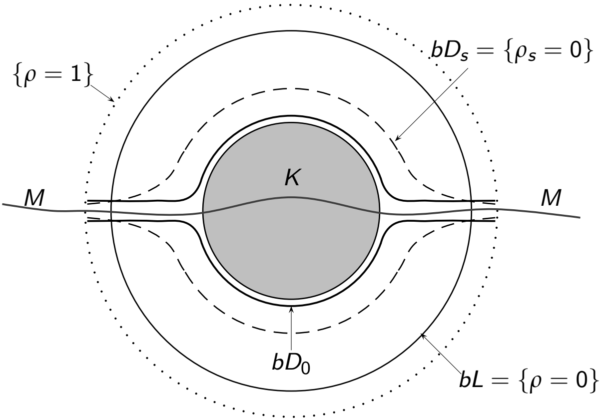
Extending the map across the handle

- By the noncritical case, we may assume that our map $f\colon X \to Y$ is holomorphic on $X_{-t_0} = \{\rho \le -t_0\}$, where $0 < t_0 < c_0$ was defined above. We make f smooth on the totally real disc E.
- The graph of f over the handlebody $H_{t_0} := X_{-t_0} \cup E$ has a basis of Stein neighbourhoods in $X \times Y$. Hence, we can use the Mergelyan theorem to approximate f on H_{t_0} by a holomorphic map on a neighbourhood $U \supset H_{t_0}$. (See Theorem 3.8.1 in my book.)
- For small enough $c > 0$ we have that $\{\tau \le c\} \subset U$ (see condition (a) in the list of properties of τ).
- By the noncritical case of the proof, applied with the function τ, we can approximate f on H_{t_0} by a map $\tilde{f}\colon X \to Y$ that is holomorphic on $\{\tau \le 2c_0\}$. By Condition (b), this set contains $X_{c_0} = \{\rho \le c_0\}$, so we have crossed the critical level of ρ at p. We now revert back to ρ and apply the noncritical case up to the next critical level of ρ.

This completes the induction and hence the proof of CAP $\Longrightarrow$ OKA.

How to deal with interpolation on a subvariety of X

A possible way is illustrated in the following figure. This is also used in the case of stratified fibre bundles with Oka fibres. Another one is to follow the standard approach but using the lemma for gluing sections of coherent analytic sheaves, obtained by **Luca Studer (2021)**.



Examples of Oka manifolds known up to 2017

- $\mathbb{C}^n, \mathbb{CP}^n$, complex Lie groups and their homogeneous spaces
- $\mathbb{C}^n \setminus A$ where A is a tame analytic subvariety of codimension > 1
- $\mathbb{CP}^n \setminus A$ where A is a subvariety of codimension > 1
- Hirzebruch surfaces ($\mathbb{CP}^1$ bundles over $\mathbb{CP}^1$)
- Hopf manifolds (quotients of $\mathbb{C}^n \setminus \{0\}$ by cyclic groups)
- Algebraic manifolds that are locally Zariski affine ($\cong \mathbb{C}^n$);
- certain modifications of such (blowing up points, removing subvarieties of codimension ≥ 2)
- $\mathbb{C}^n$ blown up at all points of a tame discrete sequence
- complex tori of dimension > 1 with finitely many points removed, or blown up at finitely many points
- toric varieties $X = (\mathbb{C}^m \setminus Z)/G$, where Z is a union of coordinate subspaces of $\mathbb{C}^m$ and G is a subgroup of $(\mathbb{C}^*)^m$ acting on $\mathbb{C}^m \setminus Z$ by diagonal matrices.

Compact complex surfaces

Surfaces of general type ($\kappa = 2$) are not dominable, and hence not Oka. A complete list of compact complex surfaces, classified according to the value of their Kodaira dimension $\kappa < 2$, can be found in the book by **Barth et al.** (Table 10 on p. 244). The following information is based on the article **F. and Lárusson, IMRN 2014**.

- $\kappa = -\infty$:
  1. **Rational surfaces are Oka.** Every nonsingular rational surface is obtained by repeatedly blowing up a minimal rational surface. The minimal rational surfaces are $\mathbb{CP}^2$ and the Hirzebruch surfaces Σ_r for $r \in \mathbb{Z}_+$; these are holomorphic $\mathbb{CP}^1$-bundles over $\mathbb{CP}^1$. Repeated blowups preserve the Oka property for surfaces in this class, so non-minimal rational surfaces are also Oka.
  2. **A ruled surface is Oka if and only if its base is Oka.** In fact, a ruled surface is the total space X of a holomorphic fibre bundle with fibre $\mathbb{CP}^1$ over a compact curve C. Such X is Oka if and only if the base C is Oka, which is so if and only if C is either $\mathbb{CP}^1$ or a torus.
  3. **Surfaces of class VII:** Minimal Hopf surfaces and minimal Enoki surfaces are Oka. Inoue surfaces, Inoue-Hirzebruch surfaces, and intermediate surfaces, minimal or blown up, are not strongly Liouville, and hence not Oka.

Compact complex surfaces, 2

- $\kappa = 0$: **Bielliptic surfaces, Kodaira surfaces, and tori are Oka.** It is unknown whether any or all K3 surfaces or Enriques surfaces are Oka.
  1. Tori are complex homogeneous and hence Oka.
  2. Every bielliptic surface, and also every primary Kodaira surface, is the total space of a holomorphic fibre bundle with torus fibre over a torus, so it is Oka by the Up-Down Theorem.
  3. Secondary Kodaira surfaces are proper unramified holomorphic quotients of primary Kodaira surfaces, so they are Oka. They are elliptic fibrations over $\mathbb{CP}^1$ with $b_1(X) = 1$ and with nontrivial canonical bundle.
  4. A K3 surface is a surface X with trivial canonical bundle and $b_1(X) = 0$. Examples include Kummer surfaces and most elliptic surfaces, i.e., surfaces admitting a fibration onto a torus with a torus as generic fibre. All elliptic fibrations in the K3 class are ramified, and we do not know whether any or all of them are Oka.
- $\kappa = 1$: These are **properly elliptic surfaces. Buzzard and Lu (2000)** determined which of them are dominable by $\mathbb{C}^2$. Nothing further is known about the Oka property for these surfaces.