# Oka manifolds

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European Research Council
Executive Agency
Established by the European Commission

**Masaryk University, Brno**
**Lecture 3, April 6, 2023**

# In this lecture. . .

. . . we present some recent developments of Oka theory after 2017. They are presented in more detail in the open access survey paper

F. Forstnerič: Recent developments on Oka manifolds.
Indag. Math., 34(2) (2023) 367–417.
https://doi.org/10.1016/j.indag.2023.01.005

We begin with Kusakabe's characterization of the class of Oka manifolds by condition $\mathrm{Ell}_1$, whose main new application is the Zariski localization theorem for Oka manifolds. This led to many new examples of Oka manifolds, especially among domains in Euclidean and projective spaces, which are discussed in the sequel.

We also present a recent application of these newly developed techniques to proper holomorphic embeddings of Stein manifolds in Euclidean spaces avoiding certain unbounded closed convex subsets.

# Kusakabe's characterization of Oka manifolds

**Gromov, 1986 Ergebnisse book, p. 72**
A complex manifold $Y$ enjoys condition $\mathrm{Ell}_1$ if every holomorphic map $X \to Y$ from a Stein manifold is the core of a dominating spray $X \times \mathbb{C}^N \to Y$.
It is easily seen that $\mathrm{OKA} \Longrightarrow \mathrm{Ell}_1$.

**Kusakabe, 2021** A complex manifold $Y$ enjoys condition $\mathrm{C\text{-}Ell}_1$ if for every compact convex set $K \subset \mathbb{C}^n$ $(n \in \mathbb{N})$, open set $U \subset \mathbb{C}^n$ containing $K$, and holomorphic map $f: U \to Y$ there are an open set $V$ with $K \subset V \subset U$ and a dominating spray $F: V \times \mathbb{C}^N \to Y$ over $f|_V$.

**Theorem (Kusakabe 2021)**

*A complex manifold which satisfies condition $\mathrm{C\text{-}Ell}_1$ is an Oka manifold. Hence, the following conditions on a complex manifold are equivalent:*

$$\mathrm{Oka} \iff \mathrm{Ell}_1 \iff \mathrm{C\text{-}Ell}_1.$$

It suffices to prove that $\mathrm{C\text{-}Ell}_1 \Longrightarrow$ CAP; the rest was known before. Furthermore, it suffices to verify this condition on special pairs of compact convex polyhedra.

# Proof of Kusakabe's theorem

Let $K \subset L$ be special polyhedral pair in $\mathbb{C}^n$ with $K = \{z \in L : \lambda(z) \leq 0\}$ for some linear function $\lambda : \mathbb{C}^n \to \mathbb{R}$.

Since $K$ is convex and $Y$ is connected, the space $\mathcal{O}(K, Y)$ is connected. Denote by $\mathcal{A}$ the set of all $f \in \mathcal{O}(K, Y)$ which can be approximated uniformly on $K$ by maps in $\mathcal{O}(L, Y)$. Clearly $\mathcal{A}$ is nonempty and closed in $\mathcal{O}(K, Y)$.

It remains to show that $\mathcal{A}$ is also open in $\mathcal{O}(K, Y)$, so $\mathcal{A} = \mathcal{O}(K, Y)$.

Fix $f \in \mathcal{A}$ and represent it by a map $f \in \mathcal{O}(U, Y)$ from an open set $U \subset \mathbb{C}^n$ containing $K$. Condition $\mathrm{C\text{-}Ell}_1$ gives a convex open set $V$, with $K \subset V \subset U$, and a dominating holomorphic spray $F : V \times \mathbb{C}^N \to Y$ with $F(\cdot\,, 0) = f|_V$. By factoring out the kernel of the derivative

$$\partial F(z, w)/\partial w|_{w=0} : \mathbb{C}^N \to T_{f(z)} Y, \quad z \in V$$

(which is a trivial holomorphic subbundle of $V \times \mathbb{C}^N$ with trivial quotient) we may assume that $N = \dim Y$ and the above map is an isomorphism for every $z \in V$. Hence, up to shrinking $V$ around $K$, there is an open ball $0 \in W \subset \mathbb{C}^N$ such that the map $\widetilde{F} = (\mathrm{Id}, F) : V \times \mathbb{C}^N \to V \times Y$ given by

$$\widetilde{F}(z, w) = (z, F(z, w)), \quad z \in V, \ w \in \mathbb{C}^N$$

maps $V \times W$ biholomorphically onto its image in $V \times Y$.

# Proof of Kusakabe's theorem, 2

Since $f \in \mathcal{A}$ is approximable, there are a neighbourhood $\Omega \subset \mathbb{C}^n$ of $L$ and a map $g \in \mathcal{O}(\Omega, Y)$ such that

$$\{(z, g(z)) : z \in K\} \subset \widetilde{F}(V \times W).$$

Up to shrinking $\Omega$ around $L$, there is a dominating holomorphic spray

$$G : \Omega \times W \to Y, \quad G(\cdot\,, 0) = g.$$

Replacing $G(z, w)$ by $G(z, tw)$ for some $t > 0$, there is an open convex set $U_1 \subset \mathbb{C}^n$ with $K \subset U_1 \Subset V \cap \Omega$ such that the map

$$\widetilde{G}(z, w) = (z, G(z, w))$$

satisfies

$$\widetilde{G}(U_1 \times W) \Subset \widetilde{F}(V \times W) \subset V \times Y.$$

Since the map $\widetilde{F}$ is biholomorphic on $V \times W$, there is a unique holomorphic map $H : U_1 \times W \to W$ such that

$$F(z, H(z, w)) = G(z, w) \quad \text{for all } (z, w) \in U_1 \times W.$$

# Proof of Kusakabe's theorem, 3

Pick a small $\epsilon > 0$ and set

$$A = \{z \in L : \lambda(z) \le 2\epsilon\} \subset U_1, \qquad B = \{z \in L : \lambda(z) \ge \epsilon\} \subset \Omega.$$

The polyhedra $A$ and $B$ form a Cartan pair with

$$A \cup B = L \quad \text{and} \quad C := A \cap B = \{z \in L : \epsilon \le \lambda(z) \le 2\epsilon\}.$$

Pick a convex open set $U_0 \subset \mathbb{C}^n$ such that $K \subset U_0 \subset U_1$ and $\overline{U}_0 \cap C = \varnothing$.

Choose any holomorphic map $\phi : U_0 \to \mathbb{C}^N$. Since $K$ and $C$ are disjoint compact convex sets in $\mathbb{C}^n$, their union is polynomially convex.

The Oka–Weil theorem furnishes a holomorphic map $\tilde{\phi} : A \times W \to \mathbb{C}^N$ which approximates $H$ on $C \times W$, and it approximates $\phi(z)$ on $(z, w) \in K \times W$.

Hence, the holomorphic map $\Phi : A \times W \to Y$ defined by

$$\Phi(z, w) = F(z, \tilde{\phi}(z, w)) \quad \text{for } z \in A \text{ and } w \in W$$

approximates $G$ on $C \times W$, while on $K \times W$ it approximates the map

$$(z, w) \mapsto f_\phi(z) := F(z, \phi(z)) \quad \text{for } z \in K \text{ and } w \in W.$$

# Proof of Kusakabe's theorem, 4

Recall that the spray $G$ is dominating over $C$. Hence, if the approximations are close enough, we can glue $\Phi$ and $G$ into a holomorphic spray

$$\Theta : L \times W' \to Y, \quad 0 \in W' \Subset W.$$

By the construction, its core map

$$\tilde{f} := \Theta(\cdot\,, 0) : L \to Y$$

approximates the map $f_\phi$ on $K$, which shows that $f_\phi \in \mathcal{A}$.

Since the map $\widetilde{F}$ is injective holomorphic on $V \times W$, every holomorphic map $K \to Y$ sufficiently uniformly close to $f$ is of the form $f_\phi$ for a suitable choice of $\phi$, so it belongs to the set $\mathcal{A}$ of approximable maps.

This shows that the set $\mathcal{A}$ is open as claimed, and therefore $\mathcal{A} = \mathcal{O}(K, Y)$, completing the proof of Kusakabe's theorem.

# Kusakabe's localization theorem for Oka manifolds

**Theorem (Kusakabe, 2021)**

*If $Y$ is a complex manifold which is a union of Zariski open Oka domains, then $Y$ is an Oka manifold.*

This is one of the most important new results in Oka theory and a wonderful tool for constructing new examples of Oka manifolds. Previously, a localization theorem was known only for algebraically subelliptic manifolds.

The proof uses the following result, which follows easily from Theorems 7.2.1 and 8.6.1 in my book.

**Lemma**

*Let $\Omega$ be a Zariski open Oka domain in a complex manifold $Y$. Given a Stein manifold $X$ and a holomorphic map $f\colon X \to Y$, there is a holomorphic spray $F\colon X \times \mathbb{C}^N \to Y$ over $f$ which is dominating on $f^{-1}(\Omega)$.*

# Proof of the localization theorem

It suffices to show that $Y$ enjoys condition $\mathrm{C\text{-}Ell}_1$.

Let $K$ be a compact convex set in $\mathbb{C}^n$ and $f \in \mathscr{O}(U, Y)$ be a holomorphic map on an open neighbourhood $U \subset \mathbb{C}^n$ of $K$.

Let $\Omega_i$ be a collection of Zariski open Oka domains in $Y$ with $\bigcup_i \Omega_i = Y$. Since $K$ is compact, $f(K)$ is contained in the union of finitely many $\Omega_i$'s; call them $\Omega_1, \ldots, \Omega_m$.

The lemma furnishes a spray $F_1 : U \times \mathbb{C}^{N_1} \to Y$ with the core $f$ which is dominating on $f^{-1}(\Omega_1)$.

Applying the same lemma to $F_1$ furnishes a spray

$$F_2 : (U \times \mathbb{C}^{N_1}) \times \mathbb{C}^{N_2} \to Y$$

with the core $F_1$ which is dominating on $F_1^{-1}(\Omega_2)$. Considering $F_2$ as a spray over $f \colon U \to Y$, it is dominating on $f^{-1}(\Omega_1 \cup \Omega_2)$.

After $m$ steps of this kind we obtain a spray $F \colon U \times \mathbb{C}^N \to Y$ over $f$ which is dominating on a neighbourhood of $K$.

# Stein manifolds with Varolin's density property

In the sequel, we shall need the following notion which developed from the seminal work of Andersén and Lempert (1992).

## Definition (Varolin 2000)

A complex manifold $X$ has the density property if every holomorphic vector field on $X$ can be approximated, uniformly on compacts in $X$, by Lie combinations (sums and Lie backets) of complete holomorphic vector fields on $X$.

**Andersén and Lempert, 1992:** $\mathbb{C}^n$ for $n > 1$ has the density property.

## Theorem (Andersén and Lempert, 1992; Forstnerič and Rosay, 1993)

*Let $X$ be a Stein manifold with the density property. If $\Omega_0 \subset X$ is a pseudoconvex Runge domain and $F_t : \Omega_0 \to \Omega_t \subset X$ $(t \in [0,1])$ is a smooth isotopy of biholomorphic maps such that $F_0 = \mathrm{Id}_{\Omega_0}$ and the domain $\Omega_t = F_t(\Omega_0)$ is Runge in $X$ for all $t$, then $F_1$ can be approximated uniformly on compacts in $\Omega_0$ by holomorphic automorphisms of $X$.*

# Complements of polynomially convex sets are Oka

**Theorem (Kusakabe, preprint 2020; F. and Wold, 2020)**

*If $K$ is a compact polynomially convex set in $\mathbb{C}^n$ $(n > 1)$ then $\mathbb{C}^n \setminus K$ is Oka. The same holds if we replace $\mathbb{C}^n$ by any Stein manifold with the density property.*

To see this, we verify condition $\mathrm{C}\text{-}\mathrm{Ell}_1$. Let $L \subset \mathbb{C}^N$ be a compact convex set and $f\colon U \to \mathbb{C}^n \setminus K$ be a holomorphic map from a Runge open neighbourhood $U \subset \mathbb{C}^N$ of $L$. Let $\Gamma = \{(\zeta, f(\zeta)) : \zeta \in L\}$. The set

$$(L \times K) \cup \Gamma$$

is then polynomialy convex in $\mathbb{C}^N \times \mathbb{C}^n$.

Let $G(\zeta, z) = (\zeta, \psi(\zeta, z))$ be the identity on a neighborhood of $U \times K$, and the contraction

$$\psi(\zeta, z) = \frac{1}{2} z + \frac{1}{2} f(\zeta)$$

to the point $f(\zeta)$ for each $(\zeta, z)$ in a neighbourhood of $\Gamma$.

# Complements of polynomially convex sets are Oka, 2

By the parametric version of the Forstnerič–Rosay theorem, we can approximate $G$ uniformly on a neighbourhood of $(L \times K) \cup \Gamma$ by a holomorphic automorphism $\Phi \in \mathrm{Aut}(U \times \mathbb{C}^n)$ of the form

$$\Phi(\zeta, z) = (\zeta, \phi(\zeta, z)), \quad \zeta \in U, \ z \in \mathbb{C}^n.$$

Iteration of this procedure leads to a holomorphic maps $F \colon U \times \mathbb{C}^n \to \mathbb{C}^n$ such that for all $\zeta \in U$ we have $F(\zeta, 0) = f(\zeta)$ and

$$F(\zeta, \cdot\,) : \mathbb{C}^n \to \mathbb{C}^n \setminus K \text{ is a Fatou–Bieberbach map.}$$

Hence, $F$ is a dominating holomorphic spray with the core $f$ and taking values in $\mathbb{C}^n \setminus K$.

Thus, $\mathbb{C}^n \setminus K$ satisfies condition $\mathrm{C}\text{-}\mathrm{Ell}_1$, so it is Oka by Kusakabe's theorem.

# Oka domains in $\mathbb{CP}^n$

## Corollary

*If $K$ is a compact polynomially convex set in $\mathbb{C}^n$ for $n > 1$, then $\mathbb{CP}^n \setminus K$ is Oka.*

## Proof.

Let $H = \mathbb{CP}^n \setminus \mathbb{C}^n$. Let $B$ be a connected open set of complex hyperplanes $\Lambda \subset \mathbb{CP}^n$ not intersecting $K$, with $H \in B$. Then, the compact set $L = \mathbb{CP}^n \setminus \bigcup_{\Lambda \in B} \Lambda$ is polynomially convex in $\mathbb{CP}^n \setminus \Lambda_0$ for any $\Lambda_0 \in B$.

Note that $K \subset L$. Since $K$ is polynomially convex in $\mathbb{C}^n = \mathbb{CP}^n \setminus H$, it is also $\mathcal{O}(L)$-convex, and hence polynomially convex in $\mathbb{CP}^n \setminus \Lambda \cong \mathbb{C}^n$ for any $\Lambda \in B$. Hence, $\mathbb{CP}^n \setminus (K \cup \Lambda)$ is an Oka domain for every $\Lambda \in B$.

Taking $\Lambda_0, \Lambda_1, \ldots, \Lambda_n \in B$ with $\bigcap_{j=0}^n \Lambda_j = \varnothing$, we have that

$$\mathbb{CP}^n \setminus K = \bigcup_{j=0}^{n} \mathbb{CP}^n \setminus (K \cup \Lambda_j).$$

Since each domain in the union on the right hand side is Oka and Zariski open in $\mathbb{CP}^n \setminus K$, it follows from the localization theorem that $\mathbb{CP}^n \setminus K$ is Oka. □

# A generalization

## Theorem

*If $\Lambda$ is a closed complex hypersurface in $\mathbb{CP}^n$ $(n > 1)$ such that the Stein domain $\Omega = \mathbb{CP}^n \setminus \Lambda$ has the density property, then for any compact $\mathcal{O}(\Omega)$-convex set $K \subset \Omega$ the complement $\mathbb{CP}^n \setminus K$ is an Oka domain.*

*In particular, $\Lambda$ has a basis of open Oka neighbourhoods in $\mathbb{CP}^n$.*

*This holds if $\Lambda$ is a quadric, or a union of $\leq n$ hyperplanes in general position.*

The complement of $n+1$ hyperplanes in general position is $(\mathbb{C}^*)^n$, which is Oka but it is not known to have the density property. The complement of more than $n+1$ hyperplanes is not Oka.

The proof is similar to that of the corollary. $\Lambda$ is the zero set of a homogeneous polynomial $P$ of degree $k = \deg \Lambda$. With respect to the Veronese embedding $\mathbb{CP}^n \hookrightarrow \mathbb{CP}^N$ (where $N = \binom{n+k}{k} - 1$), whose components are all homogeneous monomials of degree $k$ in $n+1$ variables, $\Lambda$ is the intersection of the image of $\mathbb{CP}^n$ with a hyperplane $H \subset \mathbb{CP}^N$, so $\mathbb{CP}^n \setminus \Lambda$ is a closed affine submanifold of $\mathbb{CP}^N \setminus H = \mathbb{C}^N$. Furthermore, $\mathbb{CP}^n \setminus (K \cup \Lambda)$ is Oka by Kusakabe's theorem. Moving $\Lambda$ among nearby hypersurfaces we get $\mathbb{CP}^n \setminus K = \bigcup_{i=0}^m \mathbb{CP}^n \setminus (K \cup \Lambda_i)$, so $\mathbb{CP}^n \setminus K$ is Oka by the localization theorem.

# Non-polynomially convex sets with Oka complements

**Theorem**

*If $C$ is a closed rectifiable Jordan curve in $\mathbb{C}^n$ for $n > 1$ then $\mathbb{C}^n \setminus C$ and $\mathbb{CP}^n \setminus C$ are Oka.*

*More generally, if $K$ is a compact polynomially convex set in $\mathbb{C}^n$ for $n > 1$ and $C$ is a finite union of rectifiable curves such that the complex curve $A = \widehat{C \cup K} \setminus (C \cup K)$ has at most finitely any irreducible components, then $\mathbb{C}^n \setminus (C \cup K)$ and $\mathbb{CP}^n \setminus (C \cup K)$ are Oka.*

To see this, we prove that there are holomorphic coordinates $z = (z', z_n)$ on $\mathbb{C}^n$ such that the hyperplane $H = \{z_n = 0\}$ intersects every irreducible component of $A$ but $H \cap (C \cup K) = \varnothing$. Hence, $C \cup K$ is $\mathcal{O}(\Omega)$-convex in the Stein manifold $\Omega = \mathbb{C}^n \setminus H = \mathbb{C}^{n-1} \times \mathbb{C}^*$. Since $\Omega$ has the density property, the result follows.

**Problem**

*Is the complement of the standard torus in $\mathbb{C}^2$ an Oka domain? Is the complement of $\mathbb{R}^2$ in $\mathbb{C}^2$ Oka?*

# Oka complements of unbounded convex sets in $\mathbb{C}^n$

In a recent work, E. F. Wold and I (2022) proved that complements of most closed convex sets in $\mathbb{C}^n$ for $n > 1$ are Oka.

**Theorem**

*If $E$ is a closed convex set in $\mathbb{C}^n$ for $n > 1$ with $\mathcal{C}^1$ boundary such that $E \cap T_p^{\mathbb{C}} bE$ does not contain any real halfline, then $\mathbb{C}^n \setminus E$ is an Oka domain.*

This result provides many model concave Oka domains of the form

$$\Omega = \{z = (z', z_n) \in \mathbb{C}^n : \Im z_n < \phi(z', \Re z_n)\}, \qquad (1)$$

where $\phi \geq 0$ is a strictly convex function, which are only slightly bigger than a halfspace, the latter being neither Oka nor hyperbolic.

Let $K = \overline{E} \subset \mathbb{CP}^n$ and $H = \mathbb{CP}^n \setminus \mathbb{C}^n$. The condition in the theorem implies that there is a complex hyperplane $\Lambda \subset \mathbb{CP}^n$ such that $K \cap \Lambda = \varnothing$ and $K$ is polynomially convex in the affine chart $\mathbb{CP}^n \setminus \Lambda \cong \mathbb{C}^n$. (Indeed, $\mathbb{CP}^n \setminus K$ is the union of a connected family of complex hyperplanes.) Choose affine coordinates $z = (z_1, \ldots, z_n)$ on $\mathbb{CP}^n \setminus \Lambda$ such that $H \setminus \Lambda = \{z_n = 0\}$. The result now follows from the following:

# Oka complements of unbounded convex sets, 2

**Theorem (F. and Wold, 2022)**

*If $K$ is a compact polynomially convex set in $\mathbb{C}^n$ $(n > 1)$ then*

$$\Omega = (\mathbb{C}^{n-1} \times \mathbb{C}^*) \setminus K \quad \textit{is an Oka domain.}$$

The proof is similar to the proof that $\mathbb{C}^n \setminus K$ is Oka. By Varolin (2001), the Lie algebra of algebraic vector fields on $\mathbb{C}^n$ vanishing on the hyperplane $\mathbb{C}^{n-1} \times \{0\} = \{z_n = 0\}$ is generated by complex algebraic vector fields. This allows us to verify condition $\mathrm{C\text{-}Ell}_1$ for $\Omega$ in a similar way as we did for $\mathbb{C}^n \setminus K$. Specifically, we get the following result of independent interest.

**Theorem**

*Let $L$ be a Stein compact in $\mathbb{C}^N$, and let $\Omega \subset \mathbb{C}^n$ be as in the corollary. Given a holomorphic map $f \colon L \to \Omega$, there is a holomorphic map $F \colon L \times \mathbb{C}^n \to \Omega$ such that for every $\zeta \in L$, the map $F(\zeta, \cdot\,) \colon \mathbb{C}^n \to \Omega$ is injective (a Fatou–Bieberbach map) with $F(\zeta, 0) = f(\zeta)$.*

# Complements of closed convex sets without affine lines are Oka

**Corollary**

*If $E$ is a closed convex set in $\mathbb{C}^n$ for $n > 1$ which does not contain any affine real line, then $\mathbb{C}^n \setminus E$ is an Oka domain.*

To see this, we show by tools of convex geometry that for every closed convex set $E$ which does not contain any affine real line there is a decreasing sequence of closed strongly convex domains

$$E_1 \supset E_2 \supset \cdots \supset \bigcap_{j=1}^{\infty} E_j = E.$$

Note that $\Omega_j = \mathbb{C}^n \setminus E_j$ is Oka by the theorem for every $j \in \mathbb{N}$.

Thus, $\mathbb{C}^n \setminus E = \bigcup_{j=1}^{\infty} \Omega_j$ is an increasing union of Oka domains $\Omega_1 \subset \Omega \subset \cdots$, hence it is Oka.

# Lifting boundaries of images of SPSC domains

These results can be combined with another important tool, developed by several authors (Løw, F., Hakim, Dor) and culminating in the following result.

## Lemma (Drinovec Drnovšek & F., Amer. Math. J. 2010)

*Let $\rho$ be a strongly plurisubharmonic exhaustion function on a Stein manifold $Y$, let $X$ be a Stein manifold with $\dim Y \geq 2\dim X$, $D$ be a smoothly bounded strongly pseudoconvex domain in $X$, and $f\colon \overline{D} \to Y$ be a holomorphic map. Assume that $a < \rho(f(x)) < b$ for some $a < b$ and for all $x \in bD$.*

*Given $\epsilon > 0$ and a compact set $K \subset D$, there is a holomorphic immersion $F\colon \overline{D} \to Y$ satisfying*

- a) $\rho(F(x)) > b$ *for all* $x \in bD$,
- b) $\rho(F(x)) > \rho(f(x)) - \epsilon$ *for all* $x \in \overline{D}$, *and*
- c) $\mathrm{dist}_Y(F(x), f(x)) < \epsilon$ *for all* $x \in K$.

*If $\dim Y > 2\dim X$ then $F$ can be chosen an embedding.*

*The conclusion also holds (without demanding that $F$ be an immersion) if $\dim Y > \dim X$ and $\rho$ has no critical values in $[a, b]$.*

# An application to proper holomorphic maps

By using this lemma inductively, we obtain the following result. In the second part we also use the result of Kusakabe on Oka complements.

## Theorem

*Let $Y$ be a Stein manifold, and let $D$ be a smoothly bounded strongly pseudoconvex domain in a Stein manifold $X$ such that $\dim Y \geq 2 \dim X$.*

*Then, every holomorphic map $f : \overline{D} \to Y$ can be approximated uniformly on compacts in $D$ by proper holomorphic immersions $F : D \to Y$ (embeddings if $\dim Y > 2 \dim X$).*

*If in addition $Y$ has the density property then every continuous map $f : X \to Y$ is homotopic to a proper holomorphic immersion $F : X \to Y$ (embedding if $\dim Y > 2 \dim X$), with approximation on a compact $\mathcal{O}(X)$-convex sets.*

To prove the second statement (Andrist, F., Ritter, and Wold, 2014-2019), we alternate the use of the lifting lemma (to push the boundary of $f(\overline{D})$ into $\{\rho > b\} \subset Y$ for a given number $b$) with the fact that $\{\rho > b\}$ is Oka, so $f$ can be approximated on $\overline{D}$ by a holomorphic map $X \to Y$ sending $X \setminus D$ to $\{\rho > b\}$. An inductive use of these two steps leads to proper holomorphic maps $X \to Y$.

# Proper embeddings in $\mathbb{C}^n$ avoiding large convex sets

Here is a recent application of these techniques.

## Definition

A closed convex set $E$ in $\mathbb{R}^n$ has **bounded convex exhaustion hulls (BCEH)** if for every compact convex set $K \subset \mathbb{R}^n$,

$$\text{the set} \quad h(E,K) = \mathrm{Conv}(E \cup K) \setminus E \quad \text{is bounded.}$$

## Theorem (B. Drinovec Drnovšek & F., 2023)

*Let $E$ be an unbounded closed convex set in $\mathbb{C}^n$ $(n > 1)$ having bounded convex exhaustion hulls.*

*Given a Stein manifold $X$ with $\dim X < n$, a compact $\mathcal{O}(X)$-convex set $K$ in $X$, and a holomorphic map $f_0 : K \to \Omega = \mathbb{C}^n \setminus E$, we can approximate $f_0$ uniformly on $K$ by proper holomorphic maps $f : X \to \mathbb{C}^n$ satisfying $f(X) \subset \Omega$.*

*The map $f$ can be chosen an embedding if $2 \dim X < n$ and an immersion if $2 \dim X \leq n$.*

# Proper embeddings in $\mathbb{C}^n$ avoiding large convex sets

If $E$ has BCEH then the convex hull $\mathrm{Conv}(E \cup K)$ is closed for any compact convex set $K$. If $E$ is unbounded, which is the main case of interest, there are affine coordinates $z = (z', z_n)$ on $\mathbb{C}^n$ such that

$$E = \{z = (z', z_n) \in \mathbb{C}^n : \Im z_n \geq \phi(z', \Re z_n)\},$$

where $\phi$ is a convex function in $(z', \Re z_n)$ which grows to $+\infty$ at least linearly as $|(z', \Re z_n)| \to +\infty$. Such a domain does not contain any affine real line, and hence for $n > 1$ its complement $\Omega = \mathbb{C}^n \setminus E$ is an Oka domain.

For any closed ball $B \subset \mathbb{C}^n$, the convex hull $\mathrm{Conv}(E \cup B)$ is of the same form.

The theorem is proved by exhausting $\mathbb{C}^n$ by an increasing family of such sets $E_0 \subset E_1 \subset E_2 \subset \cdots$ with $E_{k+1} = \mathrm{Conv}(E_k \cup r_k\overline{\mathbb{B}})$, and exhausting $X$ by an increasing family of strongly pseudoconvex domains $D_0 \subset D_1 \subset D_2 \subset \cdots$.

In the inductive step, we assume that $f_k(\overline{D_k \setminus D_{k-1}}) \subset \Omega_k := \mathbb{C}^n \setminus E_k$. Keeping this property, we use the lemma to push the image of $bD_k$ into $\Omega_{k+1}$, with approximation on $\overline{D}_{k-1}$. Next, we use the Oka property of $\Omega_{k+1}$ to approximate $f_k$ on $\overline{D}_k$ by a map $f_{k+1} : X \to \mathbb{C}^n$ with $f_{k+1}(X \setminus D_k) \subset \Omega_{k+1}$.

# Characterization of BCEH in the plane

The following lemma characterizes sets with BCEH in the plane $\mathbb{R}^2$. For radially symmetric functions $\phi$, the characterization holds in any dimension.

## Lemma

*Let $\phi : \mathbb{R} \to \mathbb{R}_+$ be a convex function of class $\mathscr{C}^1$ with at least linear growth. Then the epigraph $E = \{(x, y) : y \geq \phi(x)\} \subset \mathbb{R}^2$ has BCEH if and only if*

$$\lim_{x \to +\infty} \left( x - \frac{\phi(x)}{\phi'(x)} \right) = +\infty \quad \textit{and} \quad \lim_{x \to -\infty} \left( x - \frac{\phi(x)}{\phi'(x)} \right) = -\infty.$$

## Example

Let $g : \mathbb{R} \to (-1, 1)$ be an odd continuous strictly increasing function with $\lim_{x \to +\infty} g(x) = 1$ and $\int_0^\infty (1 - g(x))dx = +\infty$. Then, $\phi(x) = \int_0^x g(t)dt$ $(x \in \mathbb{R})$ grows linearly and satisfies the condition in the lemma.

# The Oka property for sections of elliptic submersions

**Gromov (1989)** outlined a proof of the following Oka principle.
Prezelj and I provided complete details (Math. Ann. 2000 & 2002).

**Theorem (Gromov 1989; F. and Prezelj, 2002)**

*If $h : Z \to X$ is an elliptic holomorphic submersion onto a Stein base, then sections $X \to Z$ satisfy the parametric Oka principle.*

Ellipticity of $h$ means that every point $p \in X$ has an open neighbourhood $U \subset X$ such that the restricted submersion $h : Z|_U \to U$ admits a **fibre-dominating spray**: a holomorphic vector bundle $\pi : E \to Z_U$ and a holomorphic map $s : E \to Z_U$ such that

$$h \circ s = h \quad \text{and} \quad ds_{0_z} : E_z \to VT_z Z = \ker dh_z \quad \text{is surjective for every } z \in Z_U.$$

**F., 2002** The same result holds for sections of subelliptic submersions (replace dominating sprays by finite dominating families of sprays).

**F., 2010** The same result holds for sections of a stratified subelliptic submersion over a reduced Stein space.

# h-Runge theorem for sections of submersions with dominating sprays

The following result is essential in the proof of Gromov's Oka principle. The proof is very similar to the one for maps to elliptic manifolds, given at the beginning of this lecture.

## Theorem (Gromov 1989: HAP)

*Let $h : Z \to X$ be a holomorphic submersion with a holomorphic vector bundle $\pi : E \to Z$ and a holomorphic spray $s : E \to Z$ such that*

$$h \circ s = h \quad \text{and} \quad ds_{0_z} : E_z \to VT_z Z = \ker dh_z \quad \text{is surjective for every } z \in Z.$$

*Let $K \subset L$ be Stein compacts in $X$, and assume that $K$ is $\mathcal{O}(L)$-convex.*

*Given a holomorphic section $f_0 : L \to Z$ and a homotopy of holomorphic sections $f_t : K \to Z$ $(t \in [0,1])$, we can approximate $\{f_t\}$ uniformly on $K$ by a homotopy $\tilde{f}_t : L \to Z$ $(t \in [0,1])$ of holomorphic sections with $\tilde{f}_0 = f_0$.*

*The fully parametric version of* HAP *holds as well.*

# HAP implies the Oka principle for sections

The homotopy approximation property, HAP, is a natural replacement for, and a generalization of the convex approximation property, CAP, that is used for sections holomorphic fibre bundles. We have the following analogue of the main theorem on Oka manifolds.

**Theorem (F., Theorem 6.6.6 in my book)**

*If $h : Z \to X$ is a holomorphic submersion onto a Stein space $X$ which satisfies the parametric version of $\mathrm{HAP}$ on small open subsets of $X$, then sections $X \to Z$ satisfy the parametric Oka principle.*

*A similar result holds in the stratified case.*

The proof uses a considerably more complex inductive procedure than in the case of locally trivial fibrations. One forms a complex of holomorphic sections and homotopies between them, parameterized by the nerve of a covering of $X$, and inductively assembles the complex of sections into a global section.

# Applications

- **Sections avoiding analytic subvarieties:** Let $E \to X$ be a holomorphic vector bundle with fibre $E_x \cong \mathbb{C}^k$, and let $\Sigma \subset E$ be a tame complex subvariety with fibres $\Sigma_x \subset E_x$ of codimension $\geq 2$. (Algebraic subvarieties are tame.) Then, $E \setminus \Sigma \to X$ is an elliptic submersion.

  Hence, sections $X \to E$ avoiding $\Sigma$ satisfy POP.

- **Removal of intersections** of maps $X \to \mathbb{C}^n$ and $X \to \mathbb{CP}^n$ with closed algebraic subvarieties of codimension $\geq 2$.

  Special case: complete intersections.

- **Eliashberg and Gromov, 1992; Schürmann, 1997:** Existence of proper holomorphic embeddings $X^n \hookrightarrow \mathbb{C}^{\left[\frac{3n}{2}\right]+1}$ and of proper holomorphic immersions $X^n \hookrightarrow \mathbb{C}^{\left[\frac{3n+1}{2}\right]}$ when $X^n$ is Stein (and $n > 1$ for embeddings).

- **h-principle for holomorphic immersions** $X^n \to \mathbb{C}^N$, $N > n$.

  **Open problem:** Does every Stein manifold $X$ of dimension $n$ with trivial tangent bundle $TX$ admit a holomorphic immersion $X^n \to \mathbb{C}^n$?

  This is true for $n = 1$ by Gunning and Narasimhan (1967).

# Solution of the Gromov–Vaserstein Problem

**Theorem (Ivarsson and Kutzschebauch, Annals of Math. 2013)**

*Let $X$ be a Stein manifold and $f\colon X \to SL_m(\mathbb{C})$ be a null-homotopic holomorphic map. There exist $k \in \mathbb{N}$ and holomorphic maps $G_1, \ldots, G_k : X \to \mathbb{C}^{m(m-1)/2}$ such that*

$$f(x) = \begin{pmatrix} 1 & 0 \\ G_1(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2(x) \\ 0 & 1 \end{pmatrix} \ldots \begin{pmatrix} 1 & G_k(x) \\ 0 & 1 \end{pmatrix}.$$

The proof uses a theorem of **Vaserstein (1988)** on factorization of continuous maps and the Oka principle for sections of stratified elliptic submersions.

**Ivarsson, Kutzschebauch, Løw, Schott, 2019–2022** Factorization of holomorphic symplectic matrices into elementary factors.

**The algebraic case: Cohn 1966** The matrix

$$\begin{pmatrix} 1 - z_1 z_2 & z_1^2 \\ -z_2^2 & 1 + z_1 z_2 \end{pmatrix} \in SL_2(\mathbb{C}[z_1, z_2])$$

does not decompose as a finite product of unipotent matrices.

**Suslin 1977** For $m \geq 3$ (and any $n$) any matrix in $SL_m(\mathbb{C}^{[n]})$ decomposes as a finite product of unipotent matrices.

# Oka-1 manifolds

The following notion was introduced and studied by Alarón and myself in a paper posted to arXiv in March 2023.

## Definition

A connected complex manifold $X$ with a complete distance function $\mathrm{dist}_X$ is an **Oka-1 manifold** if for any open Riemann surface $R$, Runge compact set $K$ in $R$, discrete sequence $a_i \in R$ without repetitions, continuous map $f: R \to X$ which is holomorphic on a neighbourhood of $K \cup \bigcup_i \{a_i\}$, number $\epsilon > 0$, and integers $k_i \in \mathbb{N} = \{1, 2, \ldots\}$ there is a holomorphic map $F: R \to X$ which is homotopic to $f$ and satisfies

1. $\sup_{p \in K} \mathrm{dist}_X(F(p), f(p)) < \epsilon$ and
2. $F$ agrees with $f$ to order $k_i$ at the point $a_i$ for every $i$.

A not necessarily connected manifold $X$ is Oka-1 if every component of $X$ is such.

# Dominability at most points implies Oka-1

**Theorem**

*Assume that $X$ is a complex manifold of dimension $n > 1$ such that for every point $x \in X \setminus E$ in the complement of a closed subset $E$ with $\mathcal{H}^{2n-1}(E) = 0$ there is a holomorphic map $f \colon \mathbb{C}^n \to X$ with $f(0) = x$ and $df_0(\mathbb{C}^n) = T_x X$. Then, $X$ is an Oka-1 manifold.*

The condition on $X$ in the theorem is called **dense dominability**. We have that

$$\text{Oka} \Longrightarrow \text{densely dominable} \Longrightarrow \text{Oka-1} \Longrightarrow \text{nonhyperbolic.}$$

These conditions are pairwise equivalent for a Riemann surface $X$.

**Theorem**

- *Every Kummer surface and every elliptic K3 surface is Oka-1.*
- *Every compact rationally connected manifold is Oka-1.*