Oka tubes in holomorphic line bundles

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Oka manifolds

A complex manifold Y is called an **Oka manifold** if maps $X \to Y$ from any Stein manifold (or reduced Stein space) X satisfy the following conditions:

- every continuous map $f_0 : X \to Y$ can be homotopically deformed to a holomorphic map $f : X \to Y$.
- If in addition $f_0 : X \to Y$ is holomorphic on a compact $\mathscr{O}(X)$-convex set K and on a closed complex subvariety X' of X, then there is a homotopy $\{f_t\}_{t\in[0,1]}$ from f_0 to a holomorphic map $f_1 : X \to Y$ consisting of maps which are holomorphic near K, close to f_0 on K, and agree with f_0 on X'.
- The analogous properties hold for continuous families of maps $X \to Y$.

These properties say that, in the absence of topological obstructions, holomorphic maps from Stein manifolds to Oka manifolds satisfy the same conditions as holomorphic functions on Stein manifolds.

F. Forstnerič, Oka Manifolds. C. R. Acad. Sci. Paris **347:17-18**, 2009
MSC 2020: 32Q56 Oka principle and Oka manifolds

A brief history

The concept of an Oka manifold evolved during the 70-year period 1939–2009.

- **Oka 1939** The holomorphic classification of line bundles on a domain of holomorphy coincide with the topological one. ($\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is Oka.)
- **Grauert 1958** **Every complex Lie group G and every complex homogeneous manifold G/H is an Oka manifold.**

  The holomorphic classification of principal G-bundles on a Stein space coincides with the topological classification.
- **Gromov 1989** **Every elliptic complex manifold is an Oka manifold.**
  If Y admits complete holomorphic vector fields which span the tangent space at every point (such a manifold is called **flexible** after **Arzhantsev et al.**), then Y is elliptic and hence Oka.
- **F. 2006, 2009** **A complex manifold Y is Oka iff it satisfies the**

  Convex approximation property (CAP): Every holomorphic map $K \to Y$ from a compact convex set K in $\mathbb{C}^n$ is a limit of entire maps $\mathbb{C}^n \to Y$.

  I proved that most Oka-type conditions are equivalent.

Another characterization of Oka manifolds

Kusakabe 2021 A complex manifold Y is Oka iff every holomorphic map $f : K \to Y$ from a compact convex set $K \subset \mathbb{C}^n$ is the core map of a dominating holomorphic spray $F : K \times \mathbb{C}^N \to Y$ for some $N \geq \dim Y$:

$$F(\cdot\,, 0) = f \text{ and } \quad \frac{\partial}{\partial z}\Big|_{z=0} F(\zeta, z) : \mathbb{C}^N \to T_{f(\zeta)} Y \text{ is surjective for every } \zeta \in K.$$

This is a restricted version of **condition** Ell_1 introduced and studied by Gromov in 1986 and 1989. Kusakabe proved that

$$\mathrm{Ell}_1 \Longrightarrow \mathrm{CAP}.$$

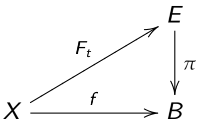
The implications $\mathrm{CAP} \Longleftrightarrow \mathrm{Oka} \Longrightarrow \mathrm{Ell}_1$ were known before (F. 2006).

As an application, Kusakabe proved in the same paper

The localization theorem for Oka manifolds: If a complex manifold Y is a union of Zariski-open Oka domains $Y \setminus A_i$, with every A_i a closed complex subvariety of Y, then Y is Oka.

Oka maps

A holomorphic map $\pi : E \to B$ is said to have the **Oka property** if for any holomorphic map $f : X \to B$ from a Stein manifold, liftings $X \to E$ of f satisfy the Oka property with approximation and interpolation.



The Oka property of a holo. submersion $E \to B$ is local on the base B.

A holomorphic map $\pi : E \to B$ of complex manifolds is an **Oka map** if it satisfies the Oka property and is a topological (Serre) fibration.

F, 2006–2010 A holomorphic fibre bundle with an Oka fibre is an Oka map.
If $E \to B$ is an Oka map then E is Oka iff B is Oka.

Kusakabe, 2021 If a complex manifold Y admits holomorphic maps $\pi_i : Y \to B_i$ with the Oka property such that the kernels of the differentials $d\pi_i$ span TY at every point, then Y is an Oka manifold.

Oka complements of holomorphically convex sets

Kusakabe 2020 [Annals of Math, to appear]

- If K is compact polynomially convex set in $\mathbb{C}^n$ $(n > 1)$ then $\mathbb{C}^n \setminus K$ is an Oka domain. The same holds for complements of holomorphically convex sets in any Stein manifold having Varolin's density property.

- If S is a closed polynomially convex subset of $\mathbb{C}^n$ such that

$$S \subset \left\{(z, w) \in \mathbb{C}^{n-2} \times \mathbb{C}^2 : |w| \leq c(1 + |z|)\right\}$$

for some $c > 0$, then $\mathbb{C}^n \setminus S$ is an Oka manifold.

- Let $\pi : Y \to Z$ be a holomorphic fibre bundle whose fibre is a Stein manifold with the density property, and let $S \subset Y$ be a family of compact holomorphically convex sets. Then, $\pi : Y \setminus S \to Z$ has the Oka property.

Wold & F., 2023 For most closed convex sets $K \subset \mathbb{C}^n$ $(n > 1)$, $\mathbb{C}^n \setminus K$ is Oka. If K is a compact polynomially convex set in $\mathbb{C}^n$ and H is a complex hyperplane in $\mathbb{C}^n$, then $\mathbb{C}^n \setminus (H \cup K)$ is Oka.

F. 2023 If $K \subset \mathbb{C}^n$ $(n > 1)$ is polynomially convex then $\mathbb{CP}^n \setminus K$ is Oka.

Our new work with Kusakabe

Let $\pi : E \to X$ be a holomorphic vector bundle on a connected compact complex manifold X, and let h be a hermitian metric on E. Denote by $|e|_h$ the norm of $e \in E$. If E is a **line bundle** then $h : E \to [0, \infty)$ is a function.

Problem

Given a hermitian line bundle $(E, h) \to X$, when is the disc bundle

$$\Delta_h(E) = \{e \in E : |e|_h < 1\}$$

an Oka manifold? In particular, when does the zero section $E(0) = \{e \in E : |e|_h = 0\}$ admit a basis of open Oka neighbourhoods?

What about vector bundles of rank > 1?

Some observations

Let $\pi : E \to X$ be a holomorphic vector bundle.

- The total space E is Oka if and only if the base X is Oka.
- For any $c > 0$ the disc bundle $\Delta_{h,c}(E) = \{|e|_h < c\}$ is biholomorphic to $\Delta_h(E)$ by a dilation in the fibres, and hence an affirmative answer to the first question implies the same for the second one.
- Since $\Delta_h(E)$ admits a holomorphic deformation retraction onto the zero section $E(0) \cong X$, we infer that if $\Delta_h(E)$ is an Oka manifold then so is X.
- If X is compact then the answer to both questions is negative for any hermitian metric h on the trivial bundle $E = X \times \mathbb{C}^r$.

  Indeed, $\Delta_h(E) \subset X \times c\mathbb{B}^r$ for some $c > 0$, where $\mathbb{B}^r$ is the unit ball.

  Clearly, $X \times c\mathbb{B}^r$ admits a bounded plurisubharmonic function which is nonconstant on every open subset, so it does not contain any Oka domains.

Some basic facts about line bundles

Holomorphic isomorphism classes of line bundles on a complex manifold X form the **Picard group**

$$\mathrm{Pic}(X) = H^1(X, \mathscr{O}^*).$$

There is a natural homomorphism $c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ coming from the exponential sheaf sequence. We have

$$\mathrm{Pic}(\mathbb{CP}^n) \cong H^2(\mathbb{CP}^n, \mathbb{Z}) \cong \mathbb{Z}.$$

We write $\mathscr{O}_{\mathbb{CP}^n}(k)$ for the line bundle corresponding to $k \in \mathbb{Z}$.

$\mathscr{O}_{\mathbb{CP}^n}(1)$ is the **hyperplane section bundle**, and
$\mathscr{O}_{\mathbb{CP}^n}(-1)$ is the **universal bundle**.

The bundle $\mathscr{O}_{\mathbb{CP}^n}(k)$ is the k-th tensor power of $\mathscr{O}_{\mathbb{CP}^n}(1)$.

The dual bundle of $\mathscr{O}_{\mathbb{CP}^n}(k)$ is $\mathscr{O}_{\mathbb{CP}^n}(-k)$.

The bundle $\mathscr{O}_{\mathbb{CP}^n}(k)$ is **positive** if $k > 0$ and **negative** if $k < 0$.

Curvature and pseudoconvexity

Let $\phi_{i,j} \in \mathscr{O}^*(U_{i,j})$ be a transition 1-cocycle of a line bundle $E \to X$. A hermitian metric h on E is given on any line bundle chart $(x, t) \in U_i \times \mathbb{C}$ by

$$h(x, t) = h_i(x)|t|^2,$$

where the functions $h_i : U_i \to (0, +\infty)$ satisfy the compatibility conditions

$$h_i(x)|\phi_{i,j}(x)|^2 = h_j(x) \quad \text{for } x \in U_{i,j} = U_i \cap U_j.$$

In the tensor power $(E^{\otimes k}, h^{\otimes k})$, the functions $\phi_{i,j}$ and h_i are raised to power k.

The bundle (E, h) is curved positively (resp. negatively) if the real $(1, 1)$-form

$$\mathrm{i}\Theta_h = -\mathrm{i}\partial\bar{\partial} \log h_i = -\mathrm{i}\,\partial\bar{\partial} \log h = -\frac{1}{2} dd^c \log h$$

is positive (resp. negative). The following conditions are equivalent.

1. The metric h is semipositive: $\mathrm{i}\Theta_h \geq 0$.
2. The function $-\log h$ is plurisubharmonic on E.
3. The disc bundle $\Delta_h(E) = \{h < 1\}$ is pseudoconcave along $\{h = 1\}$.

The first result: Oka tubes on projective spaces

Kusakabe & F., 2023 Given a positive holomorphic line bundle $E = \mathscr{O}_{\mathbb{CP}^n}(k)$ on $\mathbb{CP}^n$ $(n \geq 1,\ k \geq 1)$ and a semipositive hermitian metric h on E $(\mathrm{i}\Theta_h \geq 0)$, the following assertions hold.

- (a) The punctured disc bundle $\Delta_h^*(E) = \{e \in E : 0 < |e|_h < 1\}$ is an Oka manifold, and the disc bundle $\Delta_h(E) = \{e \in E : |e|_h < 1\}$ is an Oka-1 manifold (i.e., it satisfies the basic Oka property for maps from open Riemann surfaces).
- (b) **If $n \geq 2$ or $E = \mathscr{O}_{\mathbb{CP}^n}(1)$ then $\Delta_h(E)$ is an Oka manifold.**
- (c) The domain $D_h(E) = E \setminus \overline{\Delta_h(E)} = \{|e|_h > 1\}$ is Kobayashi hyperbolic.

Given a negative holomorphic line bundle $E = \mathscr{O}_{\mathbb{CP}^n}(k)$ $(k \leq -1)$ and a seminegative hermitian metric h on E $(\mathrm{i}\Theta_h \leq 0)$, the following hold.

- (a′) The punctured disc bundle $\Delta_h^*(E)$ is Kobayashi hyperbolic.
- (b′) The domain $D_h(E) = E \setminus \overline{\Delta_h(E)}$ is Oka.

These results hold if the metric h is continuous and semipositive (resp. seminegative) in the weak sense. They also hold for the restrictions of these bundles to any affine Euclidean chart in $\mathbb{CP}^n$.

Proof of the theorem for the hyperplane section bundle

Let $\pi : E = \mathscr{O}_{\mathbb{CP}^n}(1) \to \mathbb{CP}^n$ be the hyperplane section bundle.

Consider $H = \mathbb{CP}^n$ as the hyperplane at infinity in $\mathbb{CP}^{n+1} = \mathbb{C}^{n+1} \cup H$. Let $0 \in \mathbb{C}^{n+1}$ denote the origin. Then:

- the total space of E is $\mathbb{CP}^{n+1} \setminus \{0\}$,
- the zero section is $E(0) = H$, and
- the fibres of π are lines through 0, punctured at 0.

If h is a semipositive hermitian metric on E then $\Delta_h(E) = \{h < 1\}$ is a pseudoconcave domain in $\mathbb{CP}^{n+1}$ containing $H = \{h = 0\}$, and

$$K = \{h \geq 1\} = \{1/h \leq 1\} \subset \mathbb{C}^{n+1}$$

is a compact set with disc fibres, containing 0 in the interior.
Since $1/h = \mathrm{e}^{-\log h}$ is plurisubharmonic on $\mathbb{C}^{n+1}$, K is polynomially convex, so

$$\Delta_h(E) = \mathbb{CP}^{n+1} \setminus K \quad \text{and} \quad \Delta_h^*(E) = \mathbb{C}^{n+1} \setminus K$$

are Oka domains.

We do not know a comparably simple proof for bundles $\mathscr{O}_{\mathbb{CP}^n}(k)$ with $k > 1$.

Special hermitian line bundles on $\mathbb{CP}^n$

Denote by $z = (z_0, z_1, \ldots, z_n)$ the Euclidean coordinates on $\mathbb{C}^{n+1}$ and by $[z] = [z_0 : z_1 : \cdots : z_n]$ the homogeneous coordinates on $\mathbb{CP}^n$.

On the affine chart $U_i = \{[z] \in \mathbb{CP}^n : z_i \neq 0\} \cong \mathbb{C}^n$ $(i = 0, 1, \ldots, n)$ we have the affine coordinates $z^i = (z_0/z_i, \ldots, z_n/z_i)$, where $z_i/z_i = 1$ omitted.

On $E = \mathscr{O}_{\mathbb{CP}^n}(k)$, we have $\phi_{i,j}(z) = (z_j/z_i)^k$. Hence,

$$h([z], t) = \frac{|t|^2}{(1 + |z^i|^2)^k} = \frac{|z_i|^{2k}}{|z|^{2k}} |t|^2 \quad \text{for } [z] \in U_i \text{ and } t \in \mathbb{C}. \qquad (1)$$

is a hermitian metric on E. Note that $h = \tilde{h}^{\otimes k}$ where $\tilde{h}$ is the metric (1) on $\mathscr{O}_{\mathbb{CP}^n}(1)$. We have

$$\mathrm{i}\Theta_h = k\, \mathrm{i}\, \partial\bar{\partial} \log\bigl(|z|^2\bigr),$$

which is k-times the Fubini–Study form. The disc tube

$$\Delta_h(E)|_{U_i} = \{(z, t) \in \mathbb{C}^n \times \mathbb{C} : |t| < (1 + |z|^2)^{k/2}\}$$

is a Hartogs domain with radius of order $|z|^k$ as $|z| \to \infty$.

Hence, every hermitian metric on $\mathscr{O}_{\mathbb{CP}^n}(k)$ grows/decays at this rate at infinity.

Oka Hartogs domains

This shows that for every semipositive hermitian metric h on $E = \mathscr{O}_{\mathbb{CP}^n}(k)$ $(k \geq 1)$ and affine chart $\mathbb{C}^n \cong U \subset \mathbb{CP}^n$, the restricted disc bundle $\Delta_h(E)|_U$ is a pseudoconcave Hartogs domain

$$\Omega = \{(z, t) \in \mathbb{C}^n \times \mathbb{C} : |t| < \phi(z)\},$$

where $\phi > 0$ is a positive function on $\mathbb{C}^n$ such that $\log \phi$ is plurisubharmonic and there is a constant $c > 0$ such that

$$\phi(z) \geq c\,(1 + |z|) \text{ holds for all } z \in \mathbb{C}^n.$$

Lemma

If $n \geq 2$ then every domain Ω as above is an Oka domain.

Since $\Delta_h(E)$ is covered by Zariski open Oka domains $\Delta_h(E)|_U$ for affine charts $U \subset \mathbb{CP}^n$, it follows that $\Delta_h(E)$ is Oka by Kusakabe's localization theorem.

Proof of the lemma, 1

Let $T : \mathbb{C}^{n+1} = \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$ denote the projection $T(z,t) = t$. Set

$$\begin{aligned} S &= \mathbb{C}^{n+1} \setminus \Omega = \{(z,t) \in \mathbb{C}^n \times \mathbb{C} : |t| \geq \phi(z)\} \\ &= \{(z,t) \in \mathbb{C}^n \times \mathbb{C}^* : \log \phi(z) - \log|t| \leq 0\}. \end{aligned}$$

Since $\log|t|$ is harmonic on $t \in \mathbb{C}^*$,

$$\psi(z,t) = \log \phi(z) - \log|t| \quad \text{is plurisubharmonic on } \mathbb{C}^n \times \mathbb{C}^*.$$

Since ϕ grows at least linearly, the restricted projection $T|_S : S \to \mathbb{C}$ is proper. It follows that for every $r > 0$ the set

$$S_r = \{(z,t) \in S : |t| \leq r\} = \{(z,t) \in \mathbb{C}^n \times \mathbb{C}^* : \psi(z,t) \leq 0, \ \ \log|t| \leq \log r\}$$

is compact and $\mathscr{O}(\mathbb{C}^n \times \mathbb{C}^*)$-convex. By a theorem of Kusakabe,

$$T : (\mathbb{C}^n \times \mathbb{C}^*) \setminus S \to \mathbb{C}^* \text{ has the Oka property.}$$

Since $S \cap \{t = 0\} = \varnothing$, the projection $T : \mathbb{C}^{n+1} \setminus S \to \mathbb{C}$ has the Oka property as well. (For a holomorphic submersion, the Oka property is local on the base.) Unfortunately, this is not a topological fibration, hence not an Oka map.

Proof of the lemma, 2

To complete the proof, we consider tilted projections $\mathbb{C}^{n+1} \to \mathbb{C}$.

Since the function ϕ grows at least linearly at infinity, we have $\Lambda \cap S = \varnothing$ for every complex hyperplane $\Lambda \subset \mathbb{C}^{n+1}$ sufficiently close to $\Lambda_0 = \{t = 0\}$, and there is a path Λ_s $(s \in [0,1])$ of such hyperplanes connecting Λ_0 to Λ.

For any such Λ the set S_r is also $\mathscr{O}(\mathbb{C}^{n+1} \setminus \Lambda)$-convex. (Apply Oka's criterion for holomorphic convexity.) Hence, S is $\mathscr{O}(\mathbb{C}^{n+1} \setminus \Lambda)$-convex.

Let $T_\Lambda : \mathbb{C}^{n+1} \to \mathbb{C}$ be a linear projection with $(T_\Lambda)^{-1}(0) = \Lambda$. If Λ is close to Λ_0 then the restricted projection $T_\Lambda|_S : S \to \mathbb{C}$ is still proper. As before, we infer that $T_\Lambda : \mathbb{C}^{n+1} \setminus S \to \mathbb{C}$ has the Oka property.

Applying this conclusion with two linearly independent projections, we see that $\mathbb{C}^{n+1} \setminus S = \Omega$ is an Oka domain.

This proves that $\Delta_h(E)$ is Oka in all bundles $\mathscr{O}_{\mathbb{CP}^n}(k)$ for $n \geq 2,\ k \geq 1$.

The Oka property of $\Delta_h^*(E)$

We have seen that the puncture disc bundle $\Delta_h^*(E) = \{0 < h < 1\}$ is Oka when $E = \mathscr{O}_{\mathbb{CP}^n}(1)$, $n \geq 1$, is the hyperplane section bundle.

For every holomorphic line bundle $E \to X$ the map $t \mapsto t^k$ in the fibres defines a surjective holomorphic map $\Psi_k : E \to E^{\otimes k}$ which is branched along the zeros section $E(0)$, and

$$\Psi_k : E \setminus E(0) \to E^{\otimes k} \setminus (E^{\otimes k})(0)$$

is a k-sheeted unbranched holomorphic covering.

Furthermore,

$$\Psi_k : \Delta_h^*(E) \to \Delta_{h^{\otimes k}}^*(E^{\otimes k})$$

is a holomorphic covering projection.

Since the class of Oka manifolds is closed under nonbranched holomorphic coverings, the theorem for $\mathscr{O}_{\mathbb{CP}^n}(k)$ $(k > 1)$ follows from the one for $\mathscr{O}_{\mathbb{CP}^n}(1)$.

The dual bundle

Let $\phi_{i,j}$ be a transition 1-cocycle defining a line bundle $E \to X$. We can represent the dual bundle $E^* = E^{-1}$ of as follows.

Compactifying each fibre $E_x \cong \mathbb{C}$ $(x \in X)$ with the point at infinity yields a holomorphic fibre bundle $\widehat{E} \to X$ with fibre $\mathbb{CP}^1$ having a well-defined ∞-section $E(\infty) \cong X$, disjoint from $E(0)$.

Set $\widetilde{E} = \widehat{E} \setminus E(0) \to X$. If $t \in \mathbb{C}$ is a coordinate on a fibre E_x then $u = t^{-1}$ is a coordinate on $\widetilde{E}_x$, and the transition functions are $\phi_{i,j}^{-1} = 1/\phi_{i,j}$.

Hence, $(\widetilde{E}, h^{-1})$ is a hermitian holomorphic line bundle on X with zero section $\widetilde{E}(0) = E(\infty)$, isomorphic to the dual line bundle (E^*, h^*).

Under this identification, the identity map on $\widehat{E}$ induces a fibre preserving biholomorphism

$$\mathcal{I} : E \setminus E(0) \to E^* \setminus E^*(0)$$

mapping $\Delta_h^*(E) = \{0 < h < 1\}$ onto $D_{h^*}(E^*) = \{h^* > 1\}$, and

mapping $D_h(E) = \{h > 1\}$ onto $\Delta_{h^*}^*(E^*) = \{0 < h^* < 1\}$.

Conclusion of proof of the theorem

Note that (E, h) is semipositive iff (E^*, h^*) is seminegative; in fact,

$$\Theta_{h^{-1}} = -\Theta_h.$$

If $E = \mathscr{O}_{\mathbb{CP}^n}(k)$ with $k > 0$ then $E^* = \mathscr{O}_{\mathbb{CP}^n}(-k)$ is negative.

Hence, in a seminegative line bundle (E^*, h^*) on $\mathbb{CP}^n$, the exterior tube $D_{h^*}(E^*) = \{h^* > 1\}$ is Oka (since it is biholomorphic to $\Delta_h^*(E)$).

Furthermore, the disc tube $\Delta_{h^*}(E^*)$ is pseudoconvex, and the zero section $E^*(0)$ is the maximal compact complex subvariety of E^*, which can be blown down to a point according to Grauert.

This shows that $\Delta^*_{h^*}(E^*)$ (and hence $D_h(E)$) is Kobayashi hyperbolic.

Oka disc bundles on certain projective manifolds

Varolin 2001 A Stein manifold X has the **density property** if every holomorphic vector field on X can be approximated uniformly on compacts by sums (and commutators) of $\mathbb{C}$-complete holomorphic vector fields.

Theorem

Let $\pi : E \to \mathbb{CP}^n$ be a positive holomorphic line bundle with a hermitian metric h, and let $U_i \cong \mathbb{C}^n$ $(i = 0, \ldots, n)$ be affine charts covering $\mathbb{CP}^n$. If $X \subset \mathbb{CP}^n$ is a compact complex submanifold such that $\mathrm{i}\Theta_h(x) \geq 0$ for all $x \in X$ and $X \cap U_i$ has the density property for every $i =$, then

$$\Delta_h(E)|_X = \{e \in E : \pi(e) \in X, \quad |e|_h < 1\} \quad \textit{is an Oka manifold.}$$

is an Oka manifold, while $D_h(E)|_X = \{e \in E|_X : |e|_h > 1\}$ is hyperbolic.

An example is the quadric

$$X = \{[z_0 : z_1 : \cdots : z_n] \in \mathbb{CP}^n : z_0^2 + z_1^2 + \cdots + z_n^2 = 0\}, \quad n \geq 3.$$

The intersection of X with any affine chart $z_i = 1$ is the complexified sphere, which has the density property (Kaliman and Kutzschebauch 2008).

Line bundles on Grassmannians

Given integers $1 \leq m < n$ we denote by $G_{m,n}$ the Grassmann manifold of complex m-dimensional subspaces of $\mathbb{C}^n$.

The Plücker embedding $P : G_{m,n} \hookrightarrow \mathbb{CP}^N$, with $N = \binom{n}{m} - 1$, sends an m-plane $\operatorname{span}(v_1, \ldots, v_m) \in G_{m,n}$ to the complex line in $\mathbb{C}^{N+1}$ given by the vector $v_1 \wedge \cdots \wedge v_m \in \Lambda^m(\mathbb{C}^n) \cong \mathbb{C}^{N+1}$. The intersection of the image $P(G_{m,n}) \subset \mathbb{CP}^N$ with an affine chart $\mathbb{C}^N \cong U \subset \mathbb{CP}^N$ is biholomorphic to $\mathbb{C}^{m(n-m)}$, which has the density property if $m(n-m) > 1$.

Furthermore, the map

$$P^* : \operatorname{Pic}(\mathbb{CP}^N) \to \operatorname{Pic}(G_{m,n}) \cong \mathbb{Z}$$

is an isomorphism, and positive line bundles on $\mathbb{CP}^N$ pull back to positive line bundles on $G_{m,n}$.

If $m(n-m) > 1$, it follows that for every positive holomorphic line bundle E on $G_{m,n}$ and semipositive hermitian metric h on E the disc bundle $\Delta_h(E) = \{e \in E : |e|_h < 1\}$ is an Oka manifold

The most general result

Theorem

Let (E, h) be a semipositive hermitian holomorphic line bundle on a compact complex manifold X.

Assume that for each point $x \in X$ there exists a divisor $D \in |E|$ whose complement $X \setminus D$ is a Stein neighbourhood of x with the density property.

Then, the disc bundle $\Delta_h(E)$ is an Oka manifold while $D_h(E) = E \setminus \overline{\Delta_h(E)}$ is a pseudoconvex Kobayashi hyperbolic domain.

Idea of proof: Fix D. It suffices to show that $\Delta_h(E)|_{X \setminus D}$ is Oka. Note that $E|_{X \setminus D} \cong (X \setminus D) \times \mathbb{C}$. Since the complete linear system $|E|$ is base-point-free, there are a holomorphic map $\Phi : X \to \mathbb{CP}^N$ and a hyperplane $H \subset \mathbb{CP}^N$ such that

$$E = \Phi^*(\mathscr{O}_{\mathbb{CP}^N}(1)) \text{ and } \Phi^{-1}(\mathbb{CP}^N \setminus H) = X \setminus D.$$

This shows that the pseudoconvex Hartogs domain $\Delta_h(E)|_{X \setminus D} \subset (X \setminus D) \times \mathbb{C}$ grows at least linearly. Since the Stein manifold $X \setminus D$ has the density property, we can apply the previous proof with projections $T : (X \setminus D) \times \mathbb{C} \to \mathbb{C}$ close to the standard one.

Griffiths seminegative vector bundles of higher rank

Theorem

If (E, h) is a Griffiths seminegative hermitian holomorphic vector bundle of rank > 1 on a (not necessarily compact) Oka manifold X, then $\Omega_h(E) = \{e \in E : |e|_h > 1\}$ is a pseudoconcave Oka domain.

Proof. Let $\pi : E \to X$ be the bundle projection and set

$$S = \{e \in E : |e|_h \leq 1\}.$$

For each holomorphic chart $\psi : U \to \mathbb{B}^n$ from an open set $U \subset X$ onto the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ $(n = \dim X)$ and each $0 < r < 1$,

$$\{e \in S|_U : |\psi \circ \pi(e)| \leq r\} \text{ is } \mathscr{O}(E|_U)\text{-convex}.$$

By a theorem of Kusakabe, the restricted projection $\pi : \Omega_h(E) = E \setminus S \to X$ has the Oka property.

This projection is also a topological fibre bundle, and hence an Oka map.

Since X is an Oka manifold, it follows that $\Omega_h(E)$ is an Oka manifold as well.

The inverse Levi problem for Oka manifolds

Problem

Let (E, h) be a Griffiths semipositive hermitian holomorphic vector bundle of rank > 1 over an Oka manifold X. Let

$$\phi(e) = |e|_h^2, \quad e \in E.$$

Is the tube $\{\phi < 1\}$ an Oka manifold?

The boundary $\{\phi = 1\}$ of this domain is pseudoconcave in the horizontal directions and strongly pseudoconvex in the fibre directions.

There is no example in the literature of a non-pseudoconcave Oka domain.

Problem (The inverse Levi problem for Oka manifolds)

Is every Oka domain with smooth boundary pseudoconcave?

Oka properties and metric positivity

A heuristic principle: Oka properties are related to metric positivity.

Mori 1979, Siu & Yau 1980, Mok 1988 Every compact Kähler manifold with semipositive holomorphic bisectional curvature is Oka.

Ustinovskiy 2019 If (X, h) is a compact hermitian manifold whose holomorphic bisectional curvature is semipositive everywhere and positive at a point, then X is a projective space, hence Oka.

X. Yang 2018 Every compact Kähler manifold with positive holomorphic sectional curvature is rationally connected and projective.

Alarcón & F. 2023 Every rationally connected projective manifold is an Oka-1 manifold (i.e., it has an abundance of open holomorphic curves).

It is not known whether every such manifold is Oka. If it is, then by a result of **Matsumura 2022** and invariance of Oka manifolds under holomorphic fibre bundles with Oka fibres, the same is true for every projective manifold with semipositive holomorphic sectional curvature. However, this is likely false.

THANK YOU

FOR YOUR ATTENTION

At the Savica Watefall, Bohinj, Slovenia

Yuta Kusakabe & F. F., October 2023