MINIMAL SURFACES WITH SYMMETRIES

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Abstract

Objects with symmetries are of special interest in any mathematical theory.

In this work, we study the existence of orientable minimal surfaces in Euclidean spaces $\mathbb{R}^n$, $n \geq 3$, with a given finite group of symmetries induced by orthogonal transformations of the ambient space.

We show in particular that any finite group is a group of symmetries of a minimal surface.

F. Forstnerič: Minimal surfaces with symmetries. Preprint, August 2023.
`https://arxiv.org/abs/2308.12637`

What is a minimal surface?

Euler 1744; Lagrange 1762 Let $(\mathbb{R}^n, ds^2)$ be the flat Euclidean space. A smooth immersed surface $F : X \to \mathbb{R}^n$ $(n \geq 3)$ is a **minimal surface** if it is a **stationary point of the area functional**. Any small enough piece of such a surface has the smallest area among all surfaces with the same boundary.

Meusnier, 1776 A surface in $\mathbb{R}^n$ is a minimal surface if and only if its **mean curvature vector vanishes at every point**.

Let X be a smooth surface. An immersion $F : X \to \mathbb{R}^n$ determines on X a Riemannian metric $g = F^* ds^2$, which makes F an isometry, and hence a **conformal map**. By **Gauss**, there are local **isothermal coordinates** (x, y) at any point of X in which

$$g = \lambda(dx^2 + dy^2) \text{ for some function } \lambda > 0.$$

Transition maps between isothermal charts are conformal diffeomorphisms of plane domains, hence holomorphic or antiholomorphic. This endows X with the structure of a **conformal surface**, and of a **Riemann surface** if X is oriented.

Minimal surfaces are given by conformal harmonic immersions

If $F : X \to \mathbb{R}^n$ is a **conformal immersion**, then

$$F \text{ parameterizes a minimal surface} \iff F \text{ is a harmonic map}$$
$$\iff F \text{ is a stationary point of the energy functional.}$$

In any isothermal coordinate $z = x + \imath y$ on X, this is the **Laplace equation**

$$\Delta F = F_{xx} + F_{yy} = 4\frac{\partial^2 F}{\partial \bar{z}\, \partial z} = 0. \tag{1}$$

Write $\partial F = \frac{\partial F}{\partial z} dz = \frac{1}{2}\left(\frac{\partial F}{\partial x} - \imath \frac{\partial F}{\partial y}\right)(dx + \imath dy)$. Then, $\Re(2\partial F) = dF$ and

$$\Delta F = 0 \iff \partial F = (\partial F_1, \ldots, \partial F_n) \text{ is a holomorphic 1-form on } X.$$

It is elementary see that an immersion $F = (F_1, \ldots, F_n)$ is conformal iff

$$\sum_{i=1}^{n} (\partial F_i)^2 = 0. \tag{2}$$

Minimal surfaces are solutions of the nonlinear elliptic PDE (1), (2).

Let $A \subset \mathbb{C}^n$ denote the **null quadric**

$$A = \{z = (z_1, \ldots, z_n) : z_1^2 + z_2^2 + \cdots + z_n^2 = 0\},$$

and let $\overline{A} \subset \mathbb{CP}^n$ denote its projective closure. Pick a nontrivial holomorphic 1-form θ on X (possibly with zeros). If $F : X \to \mathbb{R}^n$ is a minimal surface then

$$2\partial F = f\theta,$$

where $f = (f_1, \ldots, f_n) : X \to \overline{A} \setminus \{0\}$ is a holomorphic map such that

$$\Re \oint_C f\theta = \oint_C dF = 0 \ \text{ for every closed curve } C \subset X. \tag{3}$$

Conversely, given f as above such that $f\theta$ is a nowhere vanishing holomorphic 1-form on X satisfying (3), the map $F : X \to \mathbb{R}^n$ given by

$$F(x) = \Re \int_*^x f\theta$$

is a conformal harmonic immersion.

Symmetries and G-equivariant maps

A smooth map $T : \mathbb{R}^n \to \mathbb{R}^n$ maps minimal surfaces to minimal surfaces iff T is a **rigid map** — a composition of orthogonal maps, dilations, and translations.

Let G be a group acting on $\mathbb{R}^n$ by rigid transformations. A surface $S \subset \mathbb{R}^n$ is ***G*-invariant** if

$$g(S) = S \quad \text{for every } g \in G.$$

If $F : X \to S = F(X) \subset \mathbb{R}^n$ is an injective conformal immersion, then G also acts on X by conformal diffeomorphisms such that F is ***G*-equivariant**:

$$F \circ g = g \circ F \ \ \text{for every } g \in G.$$

If X is a Riemann surface and every $g \in G$ preserves the orientation on $S = F(X)$, then G acts on X by holomorphic automorphisms.

Conversely, the image of a G-equivariant immersion is a G-invariant surface.

$* * *$ **Which groups arise in this way for minimal surfaces?** $* * *$

Euler 1744 The only minimal surfaces of rotation in $\mathbb{R}^3$ are planes and catenoids.

$x^2 + y^2 = \cosh^2 z$

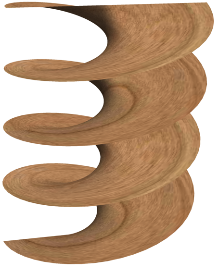
$(t, z) \mapsto (\cos t \cdot \cosh z, \sin t \cdot \cosh z, z)$

The symmetries consist of rotations in the (x, y)-plane and the reflection $z \mapsto -z$.

The helicoid (Archimedes' screw)

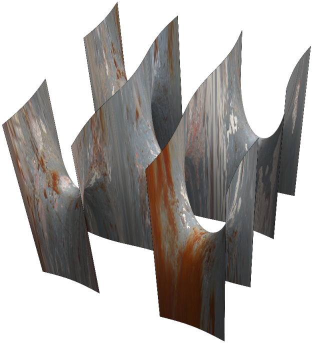
Meusnier 1776 The helicoid is a ruled minimal surface.

$$x = \rho \cos(\alpha z), \quad y = \rho \sin(\alpha z); \quad (z, \rho) \in \mathbb{R}^2.$$



The group $\mathbb{Z}$ acts on the helicoid by translations $z \mapsto z + k2\pi/\alpha$.
Also, $\mathbb{R}$ acts by translations and simultaneous rotations.

Scherk, 1835 The first Scherk's surface is doubly periodic, with the symmetry group $\mathbb{Z}^2$ of translations.



Its main branch is a graph over the square $P = (-\pi/2, \pi/2)^2$ given by

$$x_3 = \log \frac{\cos x_2}{\cos x_1}$$

Finn and Osserman, 1964
Sherk's surface S has the biggest absolute Gaussian curvature at $0 \in \mathbb{R}^3$ over all minimal graphs over P tangent to S at 0.

The main theorem

Let X be a connected open Riemann surface and $G \subset \mathrm{Aut}(X)$ be a **finite group of holomorphic automorphisms**. The **stabiliser** of $x \in X$ is

$$G_x = \{g \in G : gx = x\}.$$

Assume that G also acts on $\mathbb{R}^n$ by orthogonal transformations in $O(n,\mathbb{R})$.

Theorem

The following are equivalent:

- (a) *For every nontrivial stabiliser G_x $(x \in X)$ there is a G_x-invariant 2-plane $\Lambda_x \subset \mathbb{R}^n$ on which G_x acts effectively by rotations.*
- (b) *There exists a G-equivariant conformal minimal immersion $F : X \to \mathbb{R}^n$:*

$$F(gx) = gF(x), \quad x \in X, \ g \in G.$$

In particular, such F exists if the group G acts freely on X.

(b) $\implies$ (a)

Let $x \in X$ be a point with a nontrivial stabiliser G_x of order $k = |G_x| > 1$.

There is a local holomorphic coordinate z on X around x, with $z(x) = 0$, in which a generator of $G_x = \langle g \rangle$ is the rotation

$$gz = \mathrm{e}^{\mathrm{i}\phi} z, \quad \phi = 2\pi/k.$$

Assume that $F : X \to \mathbb{R}^n$ is an immersion. Differentiating $g \circ F = F \circ g$ gives

$$g \circ dF_x = dF_x \circ dg_x : T_x X \to \Lambda_x := dF_x(T_x X) \subset \mathbb{R}^n.$$

Since $dF_x : T_x X \to \Lambda_x$ is a linear isomorphism, we see that $\Lambda_x \subset \mathbb{R}^n$ is a G_x-invariant plane on which g acts as the rotation R_ϕ, so condition (a) holds.

Remark: Condition (a) implies the existence of a flat (linear) G_x-equivariant conformal minimal immersion from a neighbourhood of $x \in X$ to $\mathbb{R}^n$.

The main work is to globalize this construction, thereby proving (a) $\implies$ (b).

Corollary

Assume that G is a finite subgroup of the orthogonal group $O(n, \mathbb{R})$, $n \geq 3$.

Let $X \subset \mathbb{R}^n$ be a smoothly embedded, connected, oriented, noncompact, G-invariant surface such that every $g \in G$ preserves the orientation on X, and g induces the identity map on X only if $g = 1 \in G$.

Then, X endowed with the complex structure induced by the embedding $X \hookrightarrow \mathbb{R}^n$ admits a G-equivariant conformal minimal immersion $F : X \to \mathbb{R}^n$.

Proof.

The given embedding $F_0 : X \hookrightarrow \mathbb{R}^n$ induces on X a unique structure of a Riemann surface and an action of G by holomorphic automorphisms so that $F_0 : X \hookrightarrow \mathbb{R}^n$ is conformal and G-equivariant.

Hence, condition (a) in the Theorem holds by the argument on the previous page, so we can change F_0 to a conformal minimal immersion. □

Every finite group in $\mathrm{Aut}(X)$ is a symmetry group of a minimal surface

Corollary

For every connected open Riemann surface X and finite subgroup $G \subset \mathrm{Aut}(X)$ of order $n \geq 2$ there are an effective action of G by orthogonal transformations on $\mathbb{R}^{2n}$ and a G-equivariant conformal minimal immersion $F : X \to \mathbb{R}^{2n}$.

Proof.

Consider the representation of G on $\mathbb{C}^n$ with the basis vectors e_g, $g \in G$, where $h \in G$ acts by $he_g = e_{hg}$. For a fixed $g \in G$ of order $k > 1$ let Σ denote the k-dimensional $\mathbb{C}$-linear subspace of $\mathbb{C}^n$ spanned by the vectors e_{g^j} for $j = 0, 1, \ldots, k-1$, corresponding to the elements of the cyclic group $\langle g \rangle$.

Then, Σ is g-invariant, and the eigenvalues of the $\mathbb{C}$-linear isomorphism $g : \Sigma \to \Sigma$ are precisely all the k-th roots of 1. In particular, there is a vector $0 \neq w \in \Sigma$ with $gw = \mathrm{e}^{\mathrm{i}2\pi/k} w$. Identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, the 2-plane $\Lambda \subset \mathbb{R}^{2n}$ determined by the complex line $\mathbb{C}w$ is g-invariant, and g acts on it as a rotation by the angle $2\pi/k$. Hence, condition (a) in the Theorem holds. □

Equivariant minimal surfaces of genus zero

Let $S \cong \mathbb{CP}^1$ be the unit sphere in $\mathbb{R}^3$. The group $SO(3,\mathbb{R})$ acts on S by orientation preserving isometries, hence by holomorphic automorphisms, and it forms a real 3-dimensional subgroup of the holomorphic automorphism group

$$\mathrm{Aut}(S) = \left\{ z \mapsto \frac{az+b}{cz+d} : \quad a,b,c,d \in \mathbb{C},\ ad-bc=1 \right\}.$$

Finite subgroups of $SO(3,\mathbb{R})$ are called **spherical von Dyck groups**. Besides **the cyclic and the dihedral groups**, we have the symmetry groups of Platonic solids, the so-called **crystallographic groups**:

- the alternating group A_4 of order 12 is the group of symmetries of the tetrahedron,
- the symmetric group S_4 of order 24 is the group of symmetries of the cube and the octahedron, and
- the alternating group A_5 of order 60 is the group of symmetries of the icosahedron and the dodecahedron.

Xu 1995 Any closed subgroup G of $SO(3,\mathbb{R})$, which is not isomorphic to $SO(2,\mathbb{R})$ or $SO(3,\mathbb{R})$, is the symmetry group of a complete immersed minimal surface in $\mathbb{R}^3$ of genus zero with finite total curvature and embedded ends. Examples by **Jorge and Meeks 1983**, **Rossman 1995**, **Small 1999**, and others.

Every finite group is a symmetry group of a minimal surface

Every Riemann surface of genus ≥ 2 is uniformized by $\mathbb{H} = \{x + \imath y : y > 0\}$. The projective special linear group $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm I\}$ is the group of orientation preserving isometries (holomorphic automorphisms) of the hyperbolic plane $\mathbb{H}$ with the metric $\frac{dx^2+dy^2}{y^2}$ of constant negative curvature:

$$\mathrm{Aut}(\mathbb{H}) = \left\{ z \mapsto \frac{az+b}{cz+d} \quad \text{for } a,b,c,d \in \mathbb{R},\ ad-bc=1 \right\}.$$

Hurwitz 1893, Maskit 1968 If X is a Riemann surface of genus ≥ 2 then $|\mathrm{Aut}(X)| \leq 84(\mathfrak{g}-1)$. Most such surfaces have no nontrivial automorphisms.

Greenberg 1960, 1974 Every countable group G is the automorphism group of a Riemann surface X. If G is finite then X can be taken compact.

Corollary

For every finite group G of order $n>1$ there exist an open connected Riemann surface X, effective actions of G by holomorphic automorphisms on X and by orthogonal transformations on $\mathbb{R}^{2n}$, and a G-equivariant conformal minimal immersion $X \to \mathbb{R}^{2n}$. The surface X can be chosen to be the complement of n points in a compact Riemann surface.

Notation and the setup used in the proof

Let G be a finite group acting on an open Riemann surface X by holomorphic automorphisms.

$$\mathrm{Fix}(g) = \{x \in X : gx = x\}, \quad g \in G$$

$$X_0 = \bigcup_{g \in G \setminus \{1\}} \mathrm{Fix}(g) = \{x \in X : G_x \neq \{1\}\}$$

X_0 is a closed, discrete, G-invariant subset of X. Set

$$X_1 = X \setminus X_0 = \{x \in X : gx \neq x \text{ for all } g \in G \setminus \{1\}\}.$$

The orbit space X/G is an open Riemann surface, the quotient projection

$\pi : X \to X/G$ is a holomorphic map which branches precisely on X_0

$\pi : X_1 \to X_1/G$ is a holomorphic covering projection of degree $|G|$.

Choose a holomorphic immersion $\tilde{h} : X/G \to \mathbb{C}$. The holomorphic map

$$h = \tilde{h} \circ \pi : X \to \mathbb{C}$$

is G-invariant ($h \circ g = h$), and it branches precisely at the points of X_0.

Notation, 2

The holomorphic 1-form

$$\theta = dh = d(\tilde{h} \circ \pi) = \pi^* d\tilde{h}$$

on X satisfies the following invariance condition for every $g \in G$:

$$\theta_{gx} \circ dg_x = \theta_x \ \text{ for all } x \in X, \ \text{ and } \{\theta_x = 0\} \text{ if and only if } x \in X_0.$$

$$\begin{array}{rcl} A & = & \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : z_1^2 + z_2^2 + \cdots + z_n^2 = 0\} \\ A_* & = & A \setminus \{0\} \ \text{ the punctured null quadric} \\ \overline{A} & = & \text{the closure of } A \text{ in } \mathbb{CP}^n = \mathbb{C}^n \cup \mathbb{CP}^{n-1} \\ Y & = & \overline{A} \setminus \{0\} = A_* \cup Y_0 \\ Y_0 & = & Y \setminus A_* = \left\{[z_1 : \cdots : z_n] \in \mathbb{CP}^{n-1} : z_1^2 + z_2^2 + \cdots + z_n^2 = 0\right\} \\ p & : & \mathbb{C}^n \setminus \{0\} \to \mathbb{CP}^{n-1}, \ \ p(z_1, \dots, z_n) = [z_1 : \cdots : z_n] \end{array}$$

Then, $p : A_* \to Y_0$ is a holomorphic $\mathbb{C}^*$-bundle, and $p : Y \to Y_0$ is a holomorphic line bundle with the zero section Y_0.

The action of $O(n, \mathbb{R}) \subset O(n, \mathbb{C})$ on $\mathbb{C}^n$ extends to an action on $\mathbb{CP}^n$, with Y and the hyperplane at infinity $\mathbb{CP}^n \setminus \mathbb{C}^n \cong \mathbb{CP}^{n-1}$ being invariant submanifolds.

Conformal frames

To any oriented 2-plane $0 \in \Lambda \subset \mathbb{R}^n$ we associate a complex line $L \subset \mathbb{C}^n$ in the null quadric A, by choosing an oriented basis (u, v) of Λ such that $\|u\| = \|v\| \neq 0$ and $u \cdot v = 0$ (a **conformal frame**) and setting

$$L = L(\Lambda) = \mathbb{C}(u - \mathrm{i}v) \subset A \subset \mathbb{C}^n.$$

A rotation R_ϕ on Λ corresponds to the multiplication by $\mathrm{e}^{\mathrm{i}\phi}$ on $L(\Lambda)$.

If $F : X \to \mathbb{R}^n$ is a conformal immersion then, in any local holomorphic coordinate $z = x + \mathrm{i}y$ on X,

$$\text{the vectors } \frac{\partial F}{\partial x}(z) \text{ and } \frac{\partial F}{\partial y}(z) \text{ form a conformal frame in } \mathbb{R}^n.$$

The corresponding null complex line $L(z) \subset A$ is spanned by the vector

$$\frac{\partial F}{\partial x}(z) - \mathrm{i}\frac{\partial F}{\partial y}(z) = 2\frac{\partial F}{\partial z}(z).$$

The chain rule applied to $F \circ g = g \circ F$ gives

$$\partial F_{gx} \circ dg_x = g\,\partial F_x \quad \text{for every } x \in X \text{ and } g \in G.$$

Basic properties of G-equivariant conformal minimal immersions

If $F : X \to \mathbb{R}^n$ is a G-equivariant conformal minimal immersion then

$$f = 2\partial F/\theta : X \to Y = A_* \cup Y_0$$

is a holomorphic G-equivariant map satisfying $f^{-1}(Y_0) = X_0$:

$$f(gx) = \frac{2\partial F_{gx}}{\theta_{gx}} = \frac{2\partial F_{gx} \circ dg_x}{\theta_{gx} \circ dg_x} = \frac{g\, 2\partial F_x}{\theta_x} = gf(x) \text{ for every } x \in X_1 \text{ and } g \in G.$$

The following conditions hold for every point $x_0 \in X_0$.

(a) The stabiliser $G_{x_0} = \langle g_0 \rangle$ is a cyclic group. There is a local holomorphic coordinate z on X, with $z(x_0) = 0$, such that $g_0(z) = \mathrm{e}^{\mathrm{i}\phi} z$, where $\phi = 2\pi/k$ with $k = |G_{x_0}|$.

(b) The tangent plane $\Lambda = dF_{x_0}(T_{x_0}X) \subset \mathbb{R}^n$ is G_{x_0}-invariant, g_0 acts on Λ by the rotation R_ϕ, and g_0 acts on the null line $L = L(\Lambda) \subset A$ as multiplication by $\mathrm{e}^{\mathrm{i}\phi}$.

(c) $g_0 F(x_0) = F(g_0 x_0) = F(x_0)$, and the vector $F(x_0)$ is orthogonal to Λ.

(d) $f(x_0) = p(L) \in Y_0 \subset \mathbb{CP}^{n-1}$, and f has a pole of order $|G_{x_0}| - 1$ at x_0.

The converse

Conversely: Let X be a connected open Riemann surface and $f : X \to Y = A_* \cup Y_0$ be a holomorphic map such that the 1-form $f\theta$ has no zeros or poles (i.e., the poles of f on X_0 exactly cancel the zeros of θ) and

$$\Re \int_C f\theta = 0 \quad \text{for every } [C] \in H_1(X, \mathbb{Z}), \tag{4}$$

We obtain a conformal minimal immersion $F : X \to \mathbb{R}^n$ by fixing any pair $x_0 \in X$ and $v \in \mathbb{R}^n$ and setting

$$F(x) = v + \int_{x_0}^{x} \Re(f\theta) \quad \text{for all } x \in X. \tag{5}$$

Claim: The immersion F is G-equivariant if and only if f is G-equivariant and

$$gv = v + \int_{x_0}^{gx_0} \Re(f\theta) \quad \text{holds for all } g \in G. \tag{6}$$

Proof of the Claim

Suppose that $F : X \to \mathbb{R}^n$ (5) is G-equivariant. We have seen that the map $f = 2\partial F/\theta : X \to Y$ is then also G-equivariant, and

$$gv = gF(x_0) = F(gx_0) = v + \int_{x_0}^{gx_0} \Re(f\theta) \quad \text{for all } g \in G.$$

Conversely, assume that $f : X \to Y$ is a G-equivariant holomorphic map such that the 1-form $f\theta$ on X is holomorphic and nowhere vanishing. Given a path $\gamma : [0,1] \to X$, we have for any $g \in G$ that

$$\int_{g\gamma} f\theta = \int_0^1 f(g\gamma(t))\, \theta_{g\gamma(t)}(dg_{\gamma(t)}\dot\gamma(t))\, dt = \int_0^1 gf(\gamma(t))\, \theta_{\gamma(t)}(\dot\gamma(t))\, dt = g\int_\gamma f\theta.$$

(We used that $\theta_{gx} \circ dg_x = \theta_x$ for all $x \in X$.) If f also satisfies conditions (4) and (6), then the integral of $\Re(f\theta)$ is well-defined and we get

$$\begin{aligned} F(gx) &= v + \int_{x_0}^{gx} \Re(f\theta) = \left(v + \int_{x_0}^{gx_0} \Re(f\theta)\right) + \int_{gx_0}^{gx} \Re(f\theta) \\ &\overset{(6)}{=} gv + g\int_{x_0}^{x} \Re(f\theta) = gF(x). \end{aligned}$$

Step 1: We find a G-equivariant conformal minimal immersion $F_0 : V \to \mathbb{R}^n$ from a neighbourhood of the closed discrete subset $X_0 \subset X$.

Fix $x_0 \in X_0$ and set $k = |G_{x_0}| > 1$. Let $G_{x_0} = \langle g_0 \rangle$. There is a holomorphic coordinate z on a disc $x_0 \in \Delta \subset X$, with $z(x_0) = 0$, such that

$$g_0 z = \mathrm{e}^{\mathrm{i}\phi} z, \quad \phi = 2\pi/k.$$

Let $\Lambda \subset \mathbb{R}^n$ be a G_{x_0}-invariant plane on which g_0 acts as the rotation R_ϕ. Then, g_0 acts on the null line $L = L(\Lambda)$ as multiplication by $\mathrm{e}^{\mathrm{i}\phi}$.

The conformal linear map $F_0 : \Delta \to \Lambda$ is equivariant, and $2\partial F_0 = f_0\theta$ where

$$f_0(z) = \frac{y_0}{z^{k-1}} \quad \text{for some } y_0 \in L \text{ and all } z \in \Delta.$$

We extend F_0 and f_0 by G-equivariance to the orbit $G \cdot \Delta$ and perform the same construction on all G-orbits of X_0. This defines a G-equivariant map $f_0 : V \to Y$ on a G-invariant neighbourhood $V \subset X$ of X_0, with $f_0^{-1}(Y_0) = X_0$.

Step 2: We find a G-equivariant holomorphic map $f : X \to Y$ which agrees with f_0 on X_0, it satisfies $f(X_1) \subset A_*$, and the period conditions (4) and (6) hold. The map $F : X \to \mathbb{R}^n$ given by (5) then solves the problem.

Consider the action of G on $X \times Y$ by

$$g(x, y) = (gx, gy), \quad x \in X, \ g \in G.$$

The projection $X \times Y \to X$ is then G-equivariant, so it induces a projection

$$\rho : Z = (X \times Y)/G \to X/G. \tag{7}$$

Note that Z is a reduced complex space, the map ρ is holomorphic, it is branched over the closed discrete subset X_0/G of X/G, and the restriction

$$\rho : Z_1 = \rho^{-1}(X_1/G) \to X_1/G$$

is a holomorphic G-bundle with fibre $Y = A_* \cup Y_0$. The subset

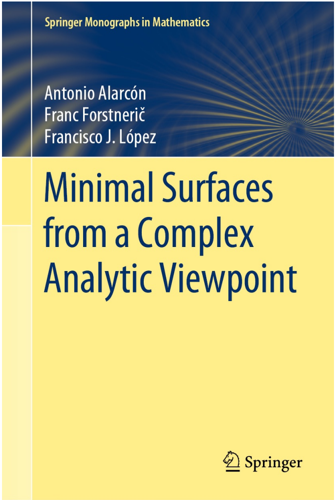
$$\Omega := (X_1 \times A_*)/G \subset Z_1 \subset Z$$

is a G-invariant Zariski open domain without singularities.

Observations and facts:

1. The restricted projection
$$\rho : \Omega = (X_1 \times A_*)/G \to X_1/G \tag{8}$$
is a holomorphic G-bundle with fibre A_*.
2. A G-equivariant map $f : X_1 \to A_*$ is the same thing as a section $\tilde{f} : X_1/G \to \Omega$ of the G-bundle $\rho : \Omega \to X_1/G$ (8).
3. The map f_0 from step 1 gives a local holomorphic section $\tilde{f}_0$ of (7) on a neighbourhood $V/G \subset X/G$ of X_0/G, and
$$\tilde{f}_0((V \setminus X_0)/G)) \subset \Omega.$$
4. The fibre A_* of (8) is $O_n(\mathbb{C})$-homogeneous, hence an **Oka manifold**. Therefore, sections of $\rho : Z = (X \times Y)/G \to X/G$ mapping X_1/G to Ω satisfy the Oka principle (**F. 2003**). This gives global holomorphic sections $\tilde{f} : X/G \to Z$ with $\tilde{f}(X_1/G) \subset \Omega$ which agrees with $\tilde{f}_0$ on X_0/G.
5. $\tilde{f}$ can be chosen such that the corresponding G-equivariant map $f : X \to Y$ integrates to a G-equivariant conformal minimal immersion.

The main reference



Our book (2021) includes proofs of the Runge/Mergelyan approximation theorem, the Weierstrass interpolation theorem, and related results in the classical theory of minimal surfaces in Euclidean spaces. They are obtained by combining **Oka-theoretic methods** with **convex integration theory.**

Since the convex hull of the null quadric $A \subset \mathbb{C}^n$ equals $\mathbb{C}^n$, the holomorphic map $f : X \to A_* \cup Y_0$ can be chosen such that the value of the integral $\int_\gamma f\theta$ on any given curve $\gamma \subset X$ assumes an arbitrary value in $\mathbb{C}^n$. Hence, we can arrange the desired period conditions.

In a galaxy of minimal surfaces

Thank your for your attention