ISOTOPIES OF COMPLETE MINIMAL SURFACES OF FINITE TOTAL CURVATURE

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Abstract

Complete minimal surfaces of finite total Gaussian curvature in Euclidean spaces $\mathbb{R}^n$ are among the most classical and widely studied objects in the theory. They are analogues of algebraic curves in complex Euclidean spaces.

In this talk, I shall presents the results on the approximation and structure theory of complete minimal surfaces of finite total curvature:

A. Alarcón, F. Forstnerič, and F. Lárusson: Isotopies of complete minimal surfaces of finite total curvature.
Preprint, June 2024. `https://arxiv.org/abs/2308.12637`

An elementary introduction to minimal surfaces:

F. Forstnerič: Minimal surfaces in Euclidean spaces by way of complex analysis. European Congress of Mathematics, 9–43. EMS Press, Berlin, ©2023.

A more comprehensive treatment:

A. Alarcón, F. Forstnerič, F. J. López: Minimal surfaces from a complex analytic viewpoint. Springer, Cham, 2021.

The area and the Dirichlet functionals

Assume that M is a domain in $\mathbb{C}$ with coordinate $z = x + \imath y$. Given a $\mathscr{C}^1$ map $u : M \to \mathbb{R}^n$, consider the **area functional**

$$\mathrm{Area}(u) = \int_M |u_x \times u_y|\, dxdy = \int_M \sqrt{|u_x|^2 |u_y|^2 - |u_x \cdot u_y|^2}\, dxdy$$

and the **Dirichlet energy functional**

$$\mathscr{D}(u) = \frac{1}{2} \int_M |\nabla u|^2\, dxdy = \frac{1}{2} \int_M \left(|u_x|^2 + |u_y|^2 \right) dxdy.$$

From the elementary inequalities

$$|a|^2 |b|^2 - |a \cdot b|^2 \le |a|^2 |b|^2 \le \frac{1}{4} \left(|a|^2 + |b|^2 \right)^2, \quad a, b \in \mathbb{R}^n,$$

which are equalities iff a, b is a **conformal frame**

$$a \cdot b = 0 \quad \text{and} \quad |a|^2 = |b|^2,$$

we infer that

$$\mathrm{Area}(u) \le \mathscr{D}(u), \quad \text{with equality iff } u \text{ is conformal.}$$

The map u is conformal iff (u_x, u_y) is a conformal frame at every point.

Hence, these two functionals have the same critical points on the space of conformal immersions.

Joseph-Louis de Lagrange (Giuseppe Lodovico Lagrangia), 1762 Minimal surfaces are critical points of these functionals.

Characterizations of minimal surfaces

Assume now that M is a bounded domain in $\mathbb{R}^2$ with piecewise smooth boundary and $u, v : \overline{M} \to \mathbb{R}^n$ are smooth maps such that u is conformal and $v\big|_{bM} = 0$. Integration by parts gives

$$\frac{d}{dt}\Big|_{t=0} \mathscr{D}(u+tv) = \int_M (u_x \cdot v_x + u_y \cdot v_y)\, dxdy = -\int_M \Delta u \cdot v\, dxdy.$$

This vanishes for all such v iff $\Delta u = 0$, thereby proving the first part of the following theorem. The characterization of minimal surfaces by vanishing mean curvature is due to Meusnier.

Theorem (Lagrange 1762, Meusnier 1776)

Let M be an open Riemann surface (or a conformal surface). The following are equivalent for a smooth conformal immersion $u : M \to \mathbb{R}^n$ $(n \geq 3)$:

- *u parametrizes a minimal surface in $\mathbb{R}^n$ (it is a critical point of the area and the Dirichlet functionals on every smoothly bounded domain in M).*
- *u is a harmonic map, i.e., $\Delta u = 0$.*
- *The mean curvature of the surface $u(M)$ vanishes at every point.*

Let M be an open Riemann surface. An immersion $u = (u_1, \ldots, u_n) : M \to \mathbb{R}^n$ for $n \geq 2$ is **conformal** if and only if the $\mathbb{C}^n$-valued $(1,0)$-form $\phi = (\phi_1, \ldots, \phi_n) = \partial u$, with $\phi_i = \partial u_i$, satisfies the nullity condition

$$\sum_{i=1}^{n} \phi_i^2 = \sum_{i=1}^{n} (\partial u_i)^2 = 0.$$

A conformal immersion $u : M \to \mathbb{R}^n$ $(n \geq 3)$ parametrizes a minimal surface in $\mathbb{R}^n$ if and only if $\phi = \partial u$ is a holomorphic 1-form.

Conversely, a nowhere vanishing holomorphic 1-form $\phi = (\phi_1, \ldots, \phi_n)$ on M satisfying the above nullity condition and the period vanishing condition

$$\Re \int_C \phi = 0 \;\text{ for all closed curves } C \text{ in } M$$

integrates to a conformal minimal immersion $u = \Re \int \phi : M \to \mathbb{R}^n$.

If in addition $\int_C \phi = 0$ holds for all closed curves C in M, then ϕ integrates to an immersed **holomorphic null curve** $h = \int \phi : M \to \mathbb{C}^n$ whose real and imaginary parts are conjugate minimal surfaces.

Every holomorphic curve in $\mathbb{C}^n$, $n \geq 2$, is a conformal minimal surface in $\mathbb{R}^{2n}$.

The null quadric $\mathbf{A}$ and the subbundle $\mathcal{A} \subset (T^*M)^{\oplus n}$

Define the (punctured) null quadric

$$\mathbf{A} = \big\{(z_1, \ldots, z_n) \in \mathbb{C}^n_* = \mathbb{C}^n \setminus \{0\} : z_1^2 + z_2^2 + \cdots + z_n^2 = 0\big\}.$$

This manifold is **flexible** (LND's on $\mathbf{A}$ generate the tangent space at every point), hence **algebraically elliptic** and an **Oka manifold**.

Given a Riemann surface M, we have a holomorphic subbundle $\mathcal{A}$ of the vector bundle $(T^*M)^{\oplus n}$, with fibre isomorphic to $\mathbf{A}$, whose sections are n-tuples $(\phi_1, \ldots, \phi_n)$ of $(1,0)$-forms on M without common zeros such that the map

$$G(\phi) = [\phi_1 : \phi_2 : \cdots : \phi_n] : M \to \mathbb{P}^{n-1}$$

takes values in the projective quadric $\mathbb{P}(\mathbf{A}) \subset \mathbb{P}^{n-1}$ defined by the same equation $z_1^2 + z_2^2 + \cdots + z_n^2 = 0$. Transition maps on $\mathcal{A}$ are given by fibre multiplication by nonvanishing holomorphic functions.

Hence, a smooth map $u : M \to \mathbb{R}^n$ is a conformal minimal immersion if and only if $\phi = \partial u$ is a holomorphic section of the bundle $\mathcal{A} \to M$.

The holomorphic map $G(\partial u) : M \to \mathbb{P}(\mathbf{A})$ is called the **Gauss map** of the conformal minimal immersion $u : M \to \mathbb{R}^n$.

Let $u = (u_1, u_2, u_3) : M \to \mathbb{R}^3$ be a conformal minimal immersion, and let $(N_1, N_2, N_3) : M \to S^2 \subset \mathbb{R}^3$ denote its **classical Gauss map**. Then, the stereographic projection

$$\mathfrak{g} = \frac{N_1 + \mathrm{i}N_2}{1 - N_3} = \frac{\partial u_3}{\partial u_1 - \mathrm{i}\partial u_2} : M \to \mathbb{P}^1 \cong \mathbb{P}(\mathbf{A}) \subset \mathbb{P}^2$$

is a holomorphic map, called the **complex Gauss map** of u, and we have the classical **Enneper–Weierstrass formula** for minimal surfaces:

$$u = 2\Re \int \left(\frac{1}{2}\left(\frac{1}{\mathfrak{g}} - \mathfrak{g}\right), \frac{\mathrm{i}}{2}\left(\frac{1}{\mathfrak{g}} + \mathfrak{g}\right), 1 \right) \partial u_3.$$

Many quantities of a minimal surface can be expressed by its Gauss map:

$$g = u^* ds^2 = \frac{(1 + |\mathfrak{g}|^2)^2}{4|\mathfrak{g}|^2} |\partial u_3|^2 \qquad \text{the induced metric on } M$$

$$Kg = -\frac{4|d\mathfrak{g}|^2}{(1 + |\mathfrak{g}|^2)^2} = -\mathfrak{g}^*(\sigma^2_{\mathbb{P}^1}) \qquad \text{the Gauss curvature function}$$

$$\mathrm{TC}(u) = \int_M K\, dA = -\mathrm{Area}_{\mathbb{P}^1}(\mathfrak{g}(M)) \qquad \text{the total Gaussian curvature}$$

Complete FTC minimal surfaces

A minimal surface $u : M \to \mathbb{R}^n$ is said to be **complete** if the Riemannian metric $g = u^* ds^2 = |du|^2$ on M is a complete metric, and **of finite total curvature** if

$$\mathrm{TC}(u) = \int_M K \, dA > -\infty.$$

If $u : M \to \mathbb{R}^n$ is a conformally immersed minimal surface of finite total curvature, then $M = \overline{M} \setminus E$ where $\overline{M}$ is a compact Riemann surface and $E = \{x_1, \ldots, x_m\}$ is a nonempty finite set (**Huber 1957**). Such M admits a biholomorphism onto a closed embedded algebraic curve in $\mathbb{C}^3$, so it will be called an **affine Riemann surface**.

The bundle $(T^*M)^{\oplus n} \supset \mathcal{A} \to M$ with fibre $\mathbf{A}$ is algebraic, and ∂u is a rational 1-form on $\overline{M}$ without zeros or poles on M, that is, a regular algebraic section of $\mathcal{A} \to M$.

The Gauss map of such a surface is also algebraic. The total Gaussian curvature of any complete minimal surface in $\mathbb{R}^3$ in a nonnegative integer multiple of $-4\pi = -\mathrm{Area}(\mathbb{P}^1)$, where the integer is the degree of the Gauss map $\mathfrak{g} : M \to \mathbb{P}^1$. For surfaces in $\mathbb{R}^n$ with $n > 3$, it is an integer multiple of -2π.

The surface $u : M \to \mathbb{R}^n$ is complete iff ∂u has an effective pole at every point of $E = \overline{M} \setminus M$. In such case, $u(M)$ is properly immersed in $\mathbb{R}^n$ and has a fairly simple and well-understood asymptotic behaviour at every end of M, described by the **Jorge–Meeks theorem, 1983**.

Example: the catenoid

The catenoid is obtained by rotating the catenal curve in $\mathbb{R}^2$ (the graph of the hyperbolic cosine function) around a suitable axis in $\mathbb{R}^3$. It was first described by Euler in 1744. For example, by rotating the catenal curve $\mathbb{R} \ni v \mapsto (\cosh v, 0, v) \in \mathbb{R}^3$ around the x_3-axis we obtain the vertical catenoid in $\mathbb{R}^3$ with the axis $x_1 = x_2 = 0$ and the implicit equation

$$x_1^2 + x_2^2 = \cosh^2 x_3.$$

A conformal parametrization is given by $u = (u_1, u_2, u_3) : \mathbb{R}^2 \to \mathbb{R}^3$,

$$u(x, y) = (\cos x \cdot \cosh y, \sin x \cdot \cosh y, y) .$$

It is 2π-periodic in the x variable. Introducing the variable $\zeta = x + \mathrm{i}y$ and $z = \mathrm{e}^{\mathrm{i}\zeta} = \mathrm{e}^{-y+\mathrm{i}x} \in \mathbb{C}^*$, we obtain a single sheeted parametrization $u : \mathbb{C}^* \to \mathbb{R}^3$ having the Weierstrass representation

$$u(z) = (1, 0, 0) - \Re \int_1^z \left(\frac{1}{2}\left(\frac{1}{\zeta} - \zeta\right), \frac{\mathrm{i}}{2}\left(\frac{1}{\zeta} + \zeta\right), 1 \right) \frac{d\zeta}{\zeta}.$$

The complex Gauss map of this catenoid is

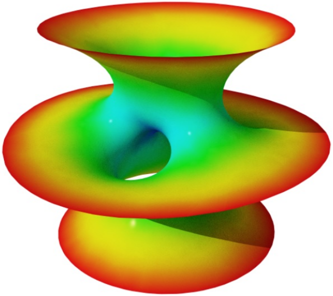
$$\mathfrak{g}(z) = z, \qquad z \in \mathbb{C}^*,$$

so it extends to the identity map on $\overline{\mathbb{C}^*} = \mathbb{P}^1$. Hence, this catenoid is complete and has total Gaussian curvature $-4\pi = -\mathrm{Area}(\mathbb{P}^1)$.

Osserman 1986 Together with Enneper's surface, the catenoid is the only complete minimal surface in $\mathbb{R}^3$ with total Gaussian curvature -4π.

Example: Costa's minimal surface

Celso J. Costa (1982) discovered a complete embedded minimal surface in $\mathbb{R}^3$ of genus one, a middle planar end and two catenoidal ends, of total Gaussian curvature -12π. Its conformal type is that of a thrice-punctured torus. It has the D_4 dihedral group of symmetries.



Costa 1991 The only complete embedded minimal surfaces in $\mathbb{R}^3$ having genus one and three ends are the 1-parameter family of Costa–Hoffman–Meeks surfaces.

Runge approximation theorem for minimal surfaces

Alarcón, Forstnerič, López, 2012–2016 Let M be an open Riemann surface, K be a compact Runge set in M, and $u : U \to \mathbb{R}^n$ $(n \geq 3)$ be a conformal minimal immersion from an open neighbourhood of K. Then, u can be approximated uniformly on K by proper conformal minimal immersions $\tilde{u} : M \to \mathbb{R}^n$. The analogous result holds for holomorphic null curves and for nonorientable minimal surfaces.

The proof uses methods of **Oka theory** (maps from Stein manifolds into the Oka manifold $\mathbf{A}$ satisfy the Oka principle) and of **convex integration theory** (to arrange vanishing periods of holomorphic maps $M \to \mathbf{A}$).

Alarcón & Lopez 2021; Alarcón & Lárusson 2023
If $M = \overline{M} \setminus E$ is an affine Riemann surface and K is a compact Runge set in M, then a conformal minimal immersion $u : U \to \mathbb{R}^n$ from a neighbourhood of K can be approximated, uniformly on K, by complete conformal minimal immersions $\tilde{u} : M \to \mathbb{R}^n$ of finite total curvature.

That is, $\partial\tilde{u}$ is algebraic and has an effective pole in every end of M.

The first proof of the theorem for FTC minimal surfaces uses special divisors on Riemann surfaces and is technically very involved.

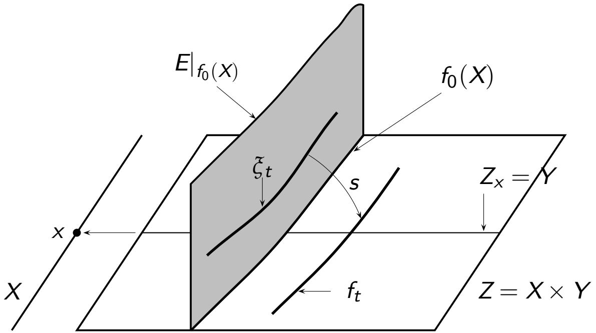
The second proof is more conceptual, and it relies on the following algebraic approximation theorem.

F. 2006 Let X be an affine algebraic manifold and Y be an algebraically elliptic manifold (i.e., Y admits a dominating algebraic spray $s : E \to Y$ from the total space of an algebraic vector bundle $\pi : E \to Y$). Assume that

- $f_0 : X \to Y$ is an algebraic map,
- $K \subset X$ is a compact $\mathscr{O}(X)$-convex set, and
- $f_t : U \to Y$ $(t \in I = [0, 1])$ is a homotopy of holomorphic maps from a neighbourhood $U \supset K$.

Then, $\{f_t\}_{t \in I}$ can be approximated uniformly on $K \times I$ by algebraic maps $F : X \times \mathbb{C} \to Y$.

A simplified idea of proof



We replace maps $X \to Y$ by sections of $Z = X \times Y \to X$ and consider E as an algebraic vector bundle over Z. The homotopy of sections $f_t : U \to Z$ is lifted to a homotopy of sections $\xi_t : f_0(U) \to E$ of the restricted algebraic bundle $E|_{f_0(X)} \to f_0(X)$ such that $s \circ \xi_t \circ f_0 = f_t$ and ξ_0 is the zero section.

Then, ξ_t is approximated by a homotopy of algebraic sections $\widetilde{\xi}_t : f_0(X) \to E|_{f_0(X)}$. The homotopy $F_t = s \circ \widetilde{\xi}_t : X \to Z$ does the job.

Let $M = \overline{M} \setminus E$ be an affine Riemann surface.

- $\mathrm{CMI}_*(M, \mathbb{R}^n)$ denotes the space of complete nonflat conformal minimal immersions $M \to \mathbb{R}^n$ of finite total curvature.
- $\mathrm{NC}_*(M, \mathbb{C}^n)$ is the space of complete nonflat holomorphic null immersions $M \to \mathbb{C}^n$ of finite total curvature (that is, proper and algebraic).
- $\Re\mathrm{NC}_*(M, \mathbb{C}^n) = \{\Re f \colon f \in \mathrm{NC}_*(M, \mathbb{C}^n)\} \subset \mathrm{CMI}_*(M, \mathbb{R}^n)$.
- $\mathscr{A}^1(M, \mathbf{A})$ denotes the space of $\mathbb{C}^n$-valued rational 1-forms $\phi = (\phi_1, \ldots, \phi_n)$ on $\overline{M}$ having no zeros or poles in M and satisfying the nullity condition, that is, regular algebraic sections of the bundle $\mathcal{A} \to M$.
- $\mathscr{A}^1_*(M, \mathbf{A})$ is the subspace of $\mathscr{A}^1(M, \mathbf{A})$ consisting of nonflat 1-forms.

A minimal surface $u = (u_1, \ldots, u_n) : M \to \mathbb{R}^n$ is said to be nonflat iff $u(M)$ is not contained in an affine plane.

This holds iff the Gauss map $G = [\partial u_1 : \cdots : \partial u_n] : M \to \mathbb{P}^{n-1}$ is nonconstant.

The main theorem

Consider the diagram

$$\begin{array}{ccc} \Re\mathrm{NC}_*(M,\mathbb{C}^n) & \hookrightarrow & \mathrm{CMI}_*(M,\mathbb{R}^n) \\ & \searrow^{\partial} & \downarrow{\scriptstyle \partial} \\ & & \mathscr{A}^1(M,\mathbf{A}) \end{array}$$

where ∂ is the $(1,0)$-differential.

Theorem (Alarcón, F., Lárusson, 2024)

If M is an affine Riemann surface, then the maps in the above diagram are weak homotopy equivalences.

Recall that a continuous map $\alpha : X \to Y$ between topological spaces is said to be a **weak homotopy equivalence** (WHE) if it induces a bijection of path components of the two spaces and an isomorphism $\pi_k(\alpha) : \pi_k(X) \to \pi_k(Y)$ of their homotopy groups for $k = 1, 2, \ldots$ and arbitrary base points.

Thus, our theorem says that the three mapping spaces in the above diagram have the same rough topological shape.

A comment

Note that the images of the maps ∂ in the above diagram are contained in the subspace $\mathscr{A}^1_\infty(M, \mathbf{A}) \subset \mathscr{A}^1_*(M, \mathbf{A})$ consisting of nonflat 1-forms that have effective poles at all ends of M.

We show that $\mathscr{A}^1_\infty(M, \mathbf{A})$ and $\mathscr{A}^1_*(M, \mathbf{A})$ are open everywhere dense subsets of $\mathscr{A}^1(M, \mathbf{A})$, and the inclusions

$$\mathscr{A}^1_\infty(M, \mathbf{A}) \hookrightarrow \mathscr{A}^1_*(M, \mathbf{A}) \hookrightarrow \mathscr{A}^1(M, \mathbf{A})$$

are weak homotopy equivalences.

Hence, our theorem also holds if ∂ is considered as a map to any of these two smaller spaces of 1-forms on M.

Topological structure of these mapping spaces

In order to understand the topological structure of the relevant mapping spaces, one may consider the following extended diagram

$$
\begin{array}{ccccc}
\mathrm{CMI}_*(M,\mathbb{R}^n) & \overset{\iota}{\hookrightarrow} & \mathrm{CMI}_{\mathrm{nf}}(M,\mathbb{R}^n) & & \\
\downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \\
\mathscr{A}^1(M,\mathbf{A}) & \overset{\alpha}{\hookrightarrow} & \mathscr{O}^1(M,\mathbf{A}) & \overset{\beta}{\longrightarrow} & \mathscr{C}(M,\mathbf{A}),
\end{array}
$$

where

- ι is the inclusion of $\mathrm{CMI}_*(M,\mathbb{R}^n)$ in the space $\mathrm{CMI}_{\mathrm{nf}}(M,\mathbb{R}^n)$ of nonflat conformal minimal immersions $M \to \mathbb{R}^n$,
- α is the inclusion of the space of algebraic 1-forms in the space $\mathscr{O}^1(M,\mathbf{A})$ of holomorphic 1-forms with values in $\mathbf{A}$, and
- β is the map $\mathscr{O}^1(M,\mathbf{A}) \ni \phi \mapsto \phi/\theta \in \mathscr{O}(M,\mathbf{A})$, where θ is a fixed nowhere vanishing holomorphic 1-form on M, followed by the inclusion $\mathscr{O}(M,\mathbf{A}) \hookrightarrow \mathscr{C}(M,\mathbf{A})$. In general, θ cannot be chosen algebraic.

Topological structure, 2

$$
\begin{array}{ccccc}
\mathrm{CMI}_*(M,\mathbb{R}^n) & \overset{\iota}{\hookrightarrow} & \mathrm{CMI}_{\mathrm{nf}}(M,\mathbb{R}^n) & & \\
\downarrow \partial & & \downarrow \partial & & \\
\mathscr{A}^1(M,\mathbf{A}) & \overset{\alpha}{\hookrightarrow} & \mathscr{O}^1(M,\mathbf{A}) & \overset{\beta}{\longrightarrow} & \mathscr{C}(M,\mathbf{A}),
\end{array}
$$

The map β is a WHE by the Oka–Grauert principle since **A** is an Oka manifold.

The left-hand vertical map ∂ is a WHE by our theorem, while the right-hand one is a WHE by a theorem of Lárusson and myself (2019).

In order to understand the inclusion ι, it thus remains to understand the inclusion α.

The **limitations of the algebraic Oka principle, discovered by Lárusson and Truong (2019)**, suggest that α may fail to be a weak homotopy equivalence. Nevertheless, it has recently been shown by Alarcón and Lárusson that α induces a surjection of the path components, so ι does as well.

We expect that α and hence ι induce bijections of path components.

An h-principle for $\partial : \mathrm{CMI}_*(M, \mathbb{R}^n) \to \mathscr{A}^1_*(M, \mathbf{A})$

The WHE of this map is a consequence of the following h-principle.

Theorem

Assume that

- *M is an affine Riemann surface,*
- *Q is a closed subspace of a compact Hausdorff space P,*
- *$u : M \times Q \to \mathbb{R}^n$, $n \geq 3$, is a continuous map such that $u_p = u(\cdot\,, p) \in \mathrm{CMI}_*(M, \mathbb{R}^n)$ for all $p \in Q$, and*
- *$\phi : M \times P \to \mathcal{A}$ is a continuous map such that*
  - a. *$\phi_p = \phi(\cdot\,, p) \in \mathscr{A}^1_*(M, \mathbf{A})$ for every $p \in P$, and*
  - b. *$\partial u_p = \phi_p$ for every $p \in Q$.*

Then there is a homotopy $\phi^t : M \times P \to \mathcal{A}$, $t \in [0, 1]$, such that $\phi^0 = \phi$ and the following conditions hold.

- i. *$\phi^t_p = \phi^t(\cdot\,, p) \in \mathscr{A}^1_*(M, \mathbf{A})$ for every $p \in P$ and $t \in [0, 1]$.*
- ii. *$\phi^t_p = \phi_p$ for every $p \in Q$ and $t \in [0, 1]$ (the homotopy is fixed on Q).*
- iii. *$\phi^1_p \in \mathscr{A}^1_\infty(M, \mathbf{A})$ for every $p \in P$.*
- iv. *$\Re \int_C \phi^1_p = 0$ for every $p \in P$ and $[C] \in H_1(M, \mathbb{Z})$.*

It follows from (iii) and (iv) that the maps $u^1_p : M \to \mathbb{R}^n$ defined by

$$u^1_p(x) = c_p + \Re \int_{x_0}^{x} 2\phi^1_p \qquad \text{for } x \in M \text{ and } p \in P,$$

with suitably chosen constants $c_p \in \mathbb{R}^n$, form a continuous family

$$P \to \mathrm{CMI}_*(M, \mathbb{R}^n), \quad p \mapsto u^1_p$$

of complete nonflat conformal minimal immersions of finite total curvature such that $u^1_p = u_p$ for all $p \in Q$.

The analogous h-principle for conformal minimal immersions without FTC and completeness was obtained by Lárusson and myself in 2019.

In the present work, we combine their result with a newly developed **parametric algebraic approximation theorem for maps from affine manifolds X to smooth flexible varieties Y** and, more generally, for sections of algebraic submersions $Z \to X$ which admit a dominating algebraic fibre-spray defined on a trivial vector bundle $Z \times \mathbb{C}^N \to Z$. (In our case, $X = M$ is an affine Riemann surface and $Z = \mathcal{A} \to M$ is the algebraic fibre bundle with fibre $\mathbf{A}$.)

We also apply the new method of ensuring effective poles at all ends of the given affine Riemann surface M.

The WHE principle for directed holomorphic immersion

Holomorphic null curves are a special type of directed holomorphic immersions of open Riemann surfaces to Euclidean spaces. A connected compact complex submanifold Y of $\mathbb{P}^{n-1}$, $n \geq 2$, determines the punctured complex cone

$$A = \{(z_1, \ldots, z_n) \in \mathbb{C}_*^n : [z_1 : \cdots : z_n] \in Y\}.$$

A holomorphic immersion $h : M \to \mathbb{C}^n$ from an open Riemann surface is said to be *directed by A*, or an ***A*-immersion**, if the differential $dh = \partial h$ is a section of the subbundle $\mathcal{A}$ with fibre A of the vector bundle $(T^*M)^{\oplus n}$.

Theorem

Let M be an affine Riemann surface, and let $A \subset \mathbb{C}_*^n$, $n \geq 2$, be a flexible smooth connected cone not contained in any hyperplane. Then the map

$$\mathscr{I}_*(M, A) \to \mathscr{A}^1(M, A), \quad h \mapsto dh, \tag{1}$$

from the space $\mathscr{I}_*(M, A)$ of proper nondegenerate algebraic A-immersions $M \to \mathbb{C}^n$ to the space $\mathscr{A}^1(M, A)$ of algebraic 1-forms on M with values in A is a weak homotopy equivalence.

This holds for example for the "big cone" $A = \mathbb{C}^n \setminus \{0\}$, and in this case directed immersions are plain immersions.