New developments in Oka theory

Franc Forstnerič

Univerza v Ljubljani

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Kiyoshi Oka, 1901–1978

Kiyoshi Oka was a Japanese mathematician who, during 1937–53, solved some of the most important contemporary problems in complex analysis.

One of his works from 1939 marks the beginning of a major theory in complex analysis in geometry, now called the **Oka theory**.

In his homeland, Oka is better known as a poet and a philosopher.

Flexibility versus rigidity in complex geometry

A central question of complex geometry is to understand the space $\mathscr{O}(X, Y)$ of holomorphic maps $X \to Y$ between a pair of complex manifolds. Are there many maps, or few maps? Which properties can they have?

There are many holomorphic maps $\mathbb{C} \to \mathbb{C}$ and $\mathbb{C} \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, but there are no nonconstant algebraic maps $\mathbb{C} \to \mathbb{C}^*$ or holomorphic maps $\mathbb{C} \to \mathbb{C} \setminus \{0, 1\}$. Manifolds with the latter property are called **hyperbolic**.

Hyperbolicity has been studied since 1967 when **Shoshichi Kobayashi** introduced his intrinsic pseudometric on complex manifold. A vast majority of complex manifolds are close to hyperbolic. Hyperbolicity is a major obstruction to solving global complex analytic problems. For example, there are continuous maps $\mathbb{C}^* \to \mathbb{C} \setminus \{0, 1\}$ which are not homotopic to a holomorphic map.

On the opposite side, **Oka theory** studies special complex manifolds which admit many holomorphic maps from all **Stein manifolds**, i.e., closed complex submanifolds of affine spaces $\mathbb{C}^N$. Oka theory provides solutions to a variety of complex analytic problems in the absence of topological obstructions.

OKA THEORY = THE h-PRINCIPLE IN COMPLEX GEOMETRY

First instances of the h-principle in complex geometry

Kiyoshi Oka 1939 For complex line bundles on Stein manifolds, the holomorphic classification agrees with the topological classification.

Hans Grauert 1958 The same is true for principal and associated fibre bundles (e.g. for vector bundles) on Stein manifolds and Stein spaces.

An equivalence between two such bundles is a section of an associated fibre bundle with a complex Lie group fibre.

The proof amounts to showing that every Stein manifold X admits many holomorphic maps $X \to Y$ to any complex homogeneous manifold Y, and many holomorphic sections $X \to Z$ of any holomorphic fibre bundle $Z \to X$ with Lie group fibre.

What is a good way to interpret 'many holomorphic maps'?

Look at properties of holomorphic functions $X \to \mathbb{C}$ on Stein manifolds.

Oka–Weil and Oka–Cartan

A compact set K in a complex manifold X is called $\mathscr{O}(X)$-convex if for every point $p \in X \setminus K$ there is a holomorphic function $f \in \mathscr{O}(X)$ such that

$$|f(p)| > \sup_{x \in K} |f(x)|.$$

Oka–Weil 1936 If K is a compact $\mathscr{O}(X)$-convex subset of a Stein manifold X, then every holomorphic function on (a neighbourhood of) K can be approximated uniformly on K by holomorphic functions $X \to \mathbb{C}$.

Oka–Cartan 1951 If T is a closed complex subvariety of a Stein manifold X, then every holomorphic function on T extends to a holomorphic function on X.

These results can be combined to approximation and (jet) interpolation.

These are fundamental properties of Stein manifolds and Stein spaces.

A twist of philosophy: We can view them as properties of the complex number field $\mathbb{C}$. We now formulate them as properties of an arbitrary target manifold Y in the absence of topological obstructions.

Oka properties of a complex manifold Y

BOPA — the basic Oka property with approximation: For every compact $\mathscr{O}(X)$-convex subset K of a Stein space X and continuous map $f_0 : X \to Y$ which is holomorphic on K there is a homotopy $f_t : X \to Y$ $(t \in [0,1])$ of maps of the same type such that $f_t|_K \approx f_0|_K$ for all t and f_1 is holomorphic on X.

BOPI — the basic Oka property with interpolation: For every closed complex subvariety T of a Stein space X and continuous map $f_0 : X \to Y$ such that $f_0|_T$ is holomorphic there is a homotopy $f_t : X \to Y$ $(t \in [0,1])$ of maps of the same type such that $f_t|_T = f_0|_T$ for all t and f_1 is holomorphic on X.

BOPAI = BOPA + BOPI.

POPA, POPI, POPAI — the parametric Oka properties: The analogous properties for families of maps $f_p : X \to Y$ depending continuously on a parameter $p \in P$ in a compact Hausdorff space, with f_p holomorphic for p in a compact subset $Q \subset P$. We ask for the existence of a homotopy $f^t : X \to Y$, $t \in [0,1]$, fixed for $p \in Q$, to a family of holomorphic maps $f^1_p : X \to Y$, $p \in P$.

70 years after Oka's fundamental paper

A complex manifold satisfying all these properties is called an

***** OKA MANIFOLD *****

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COMPTES RENDUS MATHEMATIQUE

Complex Analysis

Oka manifolds

Franc Forstnerič

Faculty of Mathematics and Physics, University of Ljubljana, and Institute of Mathematics, Physics and Mechanics, Jadranska 19, 1000 Ljubljana, Slovenia

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Presented by Mikhaël Gromov

To Mikhaël Gromov on the occasion of his receiving the Abel Prize

MSC 2020: New subfield **32Q56 Oka principle and Oka manifolds**

Basic properties of Oka manifolds

- **The weak homotopy equivalence principle:** For every Stein manifold X and Oka manifold Y, the inclusion

$$(*) \qquad \mathscr{O}(X,Y) \hookrightarrow \mathscr{C}(X,Y)$$

is a weak homotopy equivalence.

- **Lárusson 2015** If X is a Stein manifold of finite type (this holds in particular for every affine manifold), then $(*)$ is a homotopy equivalence.

- **A Riemann surface is Oka iff it is not hyperbolic.**

- **Every Oka manifold is Liouville**, i.e., every bounded plurisubharmonic function on the manifold is constant.

- **Kobayashi and Ochiai 1977** A compact complex manifold Y of maximal Kodaira dimension $\kappa(Y) = \dim Y$ (i.e., of general type) is not dominable by the affine space. Hence, no such manifold is Oka.

Note that $\kappa(Y) \in \{-\infty, 0, 1, \ldots, \dim Y\}$ is the smallest integer k such that $\dim H^0(Y, K_Y^d) \leq cd^k$ for some $c > 0$. Here, $K_Y = \wedge^{\dim Y} T^* Y$.

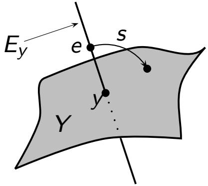
Gromov elliptic manifolds

Gromov 1989 Every elliptic complex manifold is an Oka manifold. Furthermore, the Oka principle holds in all forms for sections of elliptic holomorphic submersions $Z \to X$ over a Stein base X.

A complex manifold Y is **elliptic** if it admits a **dominating holomorphic spray**, i.e., a holomorphic map $s: E \to Y$ from the total space of a holomorphic vector bundle $E \to Y$ such that for all $y \in Y$,

$$s(0_y) = y \quad \text{and} \quad s: E_y \to Y \text{ is a submersion at } 0_y \in E_y.$$

Gromov's seminal paper from 1989 marks the beginning of modern Oka theory twenty years after the last major works on the Oka–Grauert theory by the German school.



Examples of elliptic manifolds

- **Every complex homogeneous manifold Y is elliptic.**
  Let G be a complex Lie group with the Lie algebra $\mathfrak{g} = T_1 G \cong \mathbb{C}^{\dim G}$.
  A dominating spray on a G-homogeneous manifold Y is given by
  $Y \times \mathfrak{g} \to Y$, $(y, g) \mapsto \mathrm{e}^g y$.
- **A flexible complex manifold Y is elliptic.** Such Y admits $\mathbb{C}$-complete holomorphic vector fields $V_1, \ldots, V_k$ spanning the tangent space $T_y Y$ at every point. Let ϕ_t^j denote the flow of V_j for time $t \in \mathbb{C}$. The following map $s \colon Y \times \mathbb{C}^k \to Y$ is then a dominating spray on Y:

$$s(y, t_1, \ldots, t_k) = \phi_{t_1}^1 \circ \cdots \circ \phi_{t_k}^k(y)$$

- A spray of this type exists on $\mathbb{C}^n \setminus A$, where A is algebraic subvariety with $\dim A \le n-2$. We can use complete vector fields $f(\pi(z))v$, where $v \in \mathbb{C}^n$, $\pi : \mathbb{C}^n \to \mathbb{C}^{n-1}$ is a linear projection with $\pi(v) = 0$, and f is a polynomial on $\mathbb{C}^{n-1}$ that vanishes on the subvariety $\pi(A) \subset \mathbb{C}^{n-1}$.
- **An algebraically flexible manifold is algebraically elliptic.** Every uniformly rational projective manifold $Y \subset \mathbb{P}^n$ is algebraically elliptic (**Arzhantsev, Kaliman and Zaidenberg 2024**).

The convex approximation property (CAP), 2005

A complex manifold Y enjoys the **convex approximation property (CAP)** if every holomorphic map $K \to Y$ from a compact convex set $K \subset \mathbb{C}^n$ is a uniform limit of entire maps $\mathbb{C}^n \to Y$.

F. 2005–2009, 2017

- A complex manifold is an Oka manifold iff it enjoys CAP.
- It is easily seen that Gromov's ellipticity implies CAP.
- If $Y \to Z$ is a holomorphic fibre bundle with Oka fibre, then Y is Oka iff Z is Oka. In particular, the class of Oka maps is invariant under unramified holomorphic coverings and quotients.
- The Oka properties described above are pairwise equivalent.
- Every Oka manifold Y is the image of a strongly dominating holomorphic map $\mathbb{C}^{\dim Y} \to Y$.

The proof of the main result CAP $\Rightarrow$ OKA amounts to inductively extending holomorphic maps $X \supset D \to Y$ to bigger and bigger subsets of X, using approximation and gluing techniques for manifold-valued maps.

Kusakabe's characterization of Oka manifolds (2021)

After 80 years, the Oka theory returned to Japan.

A complex manifold Y enjoys the **convex extension property (CEP)** if for every compact convex set $L \subset \mathbb{C}^n$ and holomorphic map $f : L \to Y$ there is a holomorphic map $F : L \times \mathbb{C}^N \to Y$ such that $F(\cdot, 0) = f$ and

$$\frac{\partial}{\partial \zeta}\Big|_{\zeta=0} F(z, \zeta) : \mathbb{C}^N \to T_{f(z)} Y \ \text{ is surjective for every } z \in L.$$

Condition CEP is a special case of Gromov's Condition Ell_1 (1986).

Yuta Kusakabe 2021 CEP $\Rightarrow$ CAP. Thus:

$$\mathrm{CEP} \iff \mathrm{CAP} \iff \mathrm{OKA}.$$

Corollary (Kusakabe 2021) A complex manifold Y which is a union of Zariski open Oka domains is an Oka manifold.

A Zariski open domain in a complex manifold Y is the complement $Y \setminus A$ of a closed complex subvariety $A \subset Y$.

Complements of polynomially convex sets are Oka

Kusakabe 2020; Ann. Math. 2024
If K is a compact polynomially convex set in $\mathbb{C}^n$ $(n > 1)$ then $\mathbb{C}^n \setminus K$ satisfies CEP, and hence is an Oka manifold. The same holds if K is an closed unbounded polynomially convex set in $\mathbb{C}^n$ with a proper projection to $\mathbb{C}^{n-2}$.

The analogous result holds in any Stein manifold having the following property.

Varolin's density property (DP), 2000 Every holomorphic vector field on X can be approximated on compacts by sums and commutators of complete holomorphic vector fields. The Euclidean spaces $\mathbb{C}^n$ $(n > 1)$ and many complex Lie groups and homogeneous manifolds have DP.

Andrist, Shcherbina & Wold, 2016 If K is a compact set in $\mathbb{C}^n$ $(n \geq 3)$ with infinitely many limit points, then $\mathbb{C}^n \setminus K$ is not elliptic.

Kusakabe's theorem provides a huge class of nonelliptic Oka manifolds, thereby solving a longstanding open problem.

Problem: is there a non-elliptic compact (or even projective) Oka manifold?

Sketch of proof of the complements theorem

Assume that K is a compact polynomially convex set in $\mathbb{C}^n$, $n > 1$. We shall verify that $\mathbb{C}^n \setminus K$ satisfies CEP, so it is Oka.

Let $L \subset \mathbb{C}^N$ be a compact convex set and $f\colon U \to \mathbb{C}^n \setminus K$ be a holomorphic map from an open neighbourhood $U \subset \mathbb{C}^N$ of L. Let

$$\Gamma = \{(z, f(z)) : z \in L\} \subset \mathbb{C}^N \times \mathbb{C}^n.$$

The compact set

$$(L \times K) \cup \Gamma \subset \mathbb{C}^N \times \mathbb{C}^n$$

is polynomialy convex. Let

$$G(z, \zeta) = (z, \psi(z, \zeta))$$

be the identity map on a neighborhood of $L \times K$ and the contraction

$$\psi(z, \zeta) = f(z) + \frac{1}{2}(\zeta - f(z))$$

to the point $f(z) \in \mathbb{C}^n$ for (z, ζ) in a neighbourhood of Γ.

Sketch of proof, 2

Rosay & F. 1993 We can approximate the map G, uniformly on a neighbourhood of $(L \times K) \cup \Gamma$, by holomorphic automorphisms $\Phi \in \mathrm{Aut}(U \times \mathbb{C}^n)$ of the form

$$\Phi(z, \zeta) = \big(z, \phi(z, \zeta)\big), \quad z \in U,\ \zeta \in \mathbb{C}^n.$$

Hence, $\phi(z, \cdot\,) \in \mathrm{Aut}(\mathbb{C}^n)$ is close to the identity on a neighbourhood of K and has an attractive fixed point at $f(z)$ for every $z \in U$.

Iteration of this procedure yields a holomorphic family of Fatou–Bieberbach domains $f(z) \in \Omega_z \subset \mathbb{C}^n \setminus K$ for $z \in L$, and hence a holomorphic map $F : L \times \mathbb{C}^n \to \mathbb{C}^n$ such that for all $z \in L$,

$$F(z, 0) = f(z) \ \text{ and } \ F(z, \cdot\,) : \mathbb{C}^n \stackrel{\cong}{\to} \Omega_z \subset \mathbb{C}^n \setminus K \text{ is a Fatou–Bieberbach map.}$$

Thus, $\mathbb{C}^n \setminus K$ satisfies condition CEP, so it is Oka by Kusakabe's theorem.

Strictly concave domains in $\mathbb{C}^n$ are Oka

F. & Wold, 2023 Complements of most closed convex sets $E \subset \mathbb{C}^n$ for $n > 1$ are Oka. This holds in particular if E does not contain any affine real line.

We first show that for any polynomially convex set $K \subset \mathbb{C}^n$ and complex hyperplane $H \subset \mathbb{C}^n$, the domain $\mathbb{C}^n \setminus (H \cup K)$ is Oka. This is similar to the proof that $\mathbb{C}^n \setminus K$ is Oka since $\mathbb{C}^n \setminus H$ has the density property and the approximation theorem of Rosay & F. applies.

Let $E \subset \mathbb{C}^n$ be as above. Consider the projective closure

$$K := \bar{E} \subset \mathbb{CP}^n = \mathbb{C}^n \cup H, \qquad H \cong \mathbb{CP}^{n-1}.$$

Pick a projective hyperplane $\Lambda \subset \mathbb{CP}^n$ with $K \cap \Lambda = \varnothing$. Then, $K = \bar{E}$ is a compact polynomially convex set in $\mathbb{CP}^n \setminus \Lambda \cong \mathbb{C}^n$, and hence

$$(\mathbb{CP}^n \setminus \Lambda) \setminus (H \cup K) \ \text{ is Oka.}$$

Finitely many such domains (varying Λ) cover $\mathbb{CP}^n \setminus (H \cup K) = \mathbb{C}^n \setminus K$, so this domain is Oka by Kusakabe's localization theorem.

Oka tubes in ample line bundles

Kusakabe & F. 2024 Let $L \to X$ be a holomorphic line bundle on a compact complex manifold X. Assume that for each $x \in X$ there exists a divisor $D \in |L|$ whose complement $X \setminus D$ is a Stein neighbourhood of x with the density property. Then, for any semipositive hermitian metric h on L the disc bundle $\Delta_h(L) = \{h < 1\}$ is an Oka manifold while $L \setminus \overline{\Delta_h(L)} = \{h > 1\}$ is Kobayashi hyperbolic.

It is easily seen that every line bundle $L \to X$ as above is ample, and there is a finite holomorphic map $\Phi : X \to \mathbb{CP}^N$ such that $L = \Phi^* \mathscr{O}_{\mathbb{CP}^N}(1)$.

Examples: ample line bundles on projective spaces, Grassmannians, flag manifolds,.... This can be contrasted with the following classical result:

Grauert 1961 If (L, h) is a negatively curved holomorphic hermitian line bundle on a compact complex manifold X, then the tube $\{0 < h < 1\}$ is Kobayashi hyperbolic.

The inverse Levi problem

The classical **Levi problem** in complex analysis asks for a geometric characterization of domains of holomorphy $\Omega \subset \mathbb{C}^n$, i.e., domains with a holomorphic function which does not extend holomorphically across any boundary point of Ω. This holds iff Ω is a Stein domain.

The Levi problem was solved by **Oka (1942–53)**, who showed that Ω is a domain of holomorphy iff it is pseudoconvex iff the function $-\log \mathrm{dist}(\cdot\,, b\Omega)$ is plurisubharmonic on Ω.

Since Oka manifolds are in a precise sense dual to Stein manifolds, the following is a natural problem.

Problem

Is every Oka domain $X \subset \mathbb{C}^n$ pseudoconcave (i.e., is $\mathbb{C}^n \setminus \overline{X}$ pseudoconvex)?

Is the complement $X = \mathbb{C}^n \setminus \overline{\Omega}$ of every smoothly bounded pseudoconvex domain $\Omega \subset \mathbb{C}^n$ $(n > 1)$ an Oka domain?

A new direction in Oka theory

I am currently developing the **Oka principle for families of Stein structures** on the source manifold. The following is the first instance. Assume that

- B is a Euclidean neighbourhood retract (every finite CW complex is such),
- X is a smooth open oriented surface,
- $\{J_b\}_{b\in B}$ is a continuous family of complex structures on X,
- $K \subset X$ is a compact Runge set (i.e., $X \setminus K$ has no holes),
- $U \subset B \times X$ is an open set containing $B \times K$,
- Y is an Oka manifold, and
- $f\colon B \times X \to Y$ is a continuous map such that $f_b = f(b,\cdot\,) : X \to Y$ is J_b-holomorphic on $U_b = \{x \in X\colon (b,x) \in U\}$ for every $b \in B$.

Given a continuous function $\epsilon : B \to (0,+\infty)$, there is a continuous map $F\colon B \times X \to Y$ such that for every $b \in B$ the map $F_b = F(b,\cdot\,) : X \to Y$ is J_b-holomorphic and satisfies $\sup_{x\in K} \mathrm{dist}_Y(F_b(x), f_b(x)) < \epsilon(b)$.

If B is a smooth manifold and the data (J_b, f_b) depend smoothly on $b \in B$, then so do the approximants F_b.

Oka properties and metric positivity

A challenging open problem is to understand the relationship between Oka properties and metric positivity of compact hermitian or Kähler manifolds.

The specialness of manifolds of low Kodaira dimension is analogous to the specialness of Riemannian manifolds of positive curvature, while general type (maximal Kodaira dimension) corresponds to the genericity of non-positive curvature.

The result on tubes of line bundles, mentioned above, is an example of this phenomenon.

We mention a few known results in this direction, beginning with the following.

Grauert & Reckziegel 1965 A hermitian manifold with negative holomorphic sectional curvature is Kobayashi hyperbolic.

This is a generalization of the Ahlfors–Schwarz lemma. There are many further results on this subject (**Wu 1967, Kobayashi 1970, Greene and Wu 1979**).

Oka properties and metric positivity

Mori 1979, Siu and Yau 1980, Mok 1988 The universal cover of a compact Kähler manifold with nonnegative holomorphic bisectional curvature is biholomorphic to

$$\mathbb{C}^k \times \mathbb{CP}^{n_1} \times \cdots \times \mathbb{CP}^{n_l} \times M_1 \times \cdots \times M_p$$

where each M_j is a compact hermitian symmetric space.
Every such manifold is Oka.

Campana & Peternell 1991 A compact projective manifold with dim ≤ 3 with numerically effective tangent bundle is an Oka manifold.

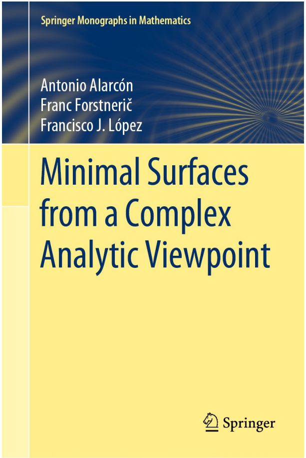
Lárusson and F. 2024 A projective manifold that is birationally equivalent to an algebraically elliptic projective manifold is an **aOka-1 manifold**, i.e., it has the Oka properties for algebraic (regular) maps $X \to Y$ from any affine algebraic curve X. This holds in particular for all rational manifolds.

Conjecture Every compact rationally connected manifold is an Oka-1 and aOka-1 manifold. (If the holomorphic sectional curvature of a compact Kähler manifold is positive then, by **Yau's conjecture solved X. Yang 2018**, the manifold is rationally connected.)

Selected applications of Oka theory

- **Grauert 1958** Classification of principal G-bundles on Stein spaces.
- **Eliashberg & Gromov 1992; Schürmann 1996** Every Stein manifold X of dimension $n > 1$ admits a proper holomorphic embedding in $\mathbb{C}^{[3n/2]+1}$ and immersion in $\mathbb{C}^{[(3n+1)/2]}$. These dimensions are optimal.
- **Eliashberg & Gromov 1985** Holomorphic immersions $X^n \to \mathbb{C}^N$ for $N > n \geq 1$ satisfy the basic h-principle with respect to their tangent maps.
- **F. 2003** Holomorphic subersions $X^n \to \mathbb{C}^m$ for $n > m \geq 1$ satisfy the basic h-principle with respect to their tangent maps. In particular, every Stein manifold admits many holomorphic functions without critical points.
- **Wold & F. 2009, 2013** Proper holomorphic embeddings of bordered Riemann surfaces in $\mathbb{C}^2$.
- **Ivarsson & Kutzschebauch 2012** Null homotopic holomorphic maps $X \to SL_m(\mathbb{C})$ satisfy the Vaserstein factorisation theorem into upper and lower triangular matrix-valued maps.
- **Ivarsson, Kutzschebauch, Løw, Schott, 2019–2022** Factorization of holomorphic symplectic matrices into elementary factors.

Applications to minimal surfaces



A minimal surface in $\mathbb{R}^n$, $n \geq 3$, is given by a conformal harmonic immersion $F : X \to \mathbb{R}^n$ from an open Riemann surface.

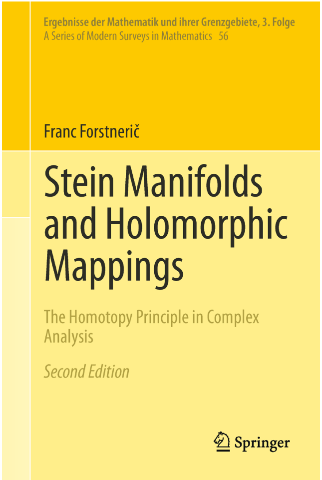
The $(1,0)$-differential $\Phi = \partial F$ is a holomorphic 1-form with exact real part $2\Re\Phi = dF$ and values in the cone

$$A = \{z_1^2 + z_2^2 + \cdots + z_n^2 = 0\} \setminus \{0\},$$

and vice versa.

The cone A is an Oka manifold.
(It is also algebraically flexible.)

Applications of Oka theory yield a variety of new results on minimal surfaces in Euclidean spaces.



The main results on this subject up to 2017, discussed in this talk, are presented in my Ergebnisse monograph.

Developments after 2017 are summarised in my survey

Recent developments on Oka manifolds.
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Thank you for your attention.