Minimal surfaces from a complex analytic viewpoint

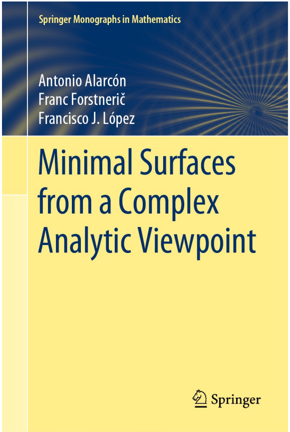
Franc Forstnerič

Univerza v Ljubljani

European Research Council
Executive Agency
Established by the European Commission

Kyushu University, Fukuoka
5 September 2024

In this lecture. . .



I will describe some recent developments in the theory of minimal surfaces in Euclidean spaces which have been obtained by complex analytic methods.
My collaborators on these projects:

- Antonio Alarcón and Francisco J. López, Granada
- Barbara Drinovec Drnovšek, Ljubljana
- David Kalaj, Podgorica
- Finnur Lárusson, Adelaide

The fundamentals go back to **Kiyoshi Oka** and **Shoshichi Kobayashi**.

1744 **Euler** A surface in $\mathbb{R}^3$ is called **minimal** if it locally minimizes the area among all nearby surfaces with the same boundary.

The only minimal surfaces of rotation are planes and catenoids.

$x^2 + y^2 = \cosh^2 z$

$(t, z) \mapsto (\cos t \cdot \cosh z, \sin t \cdot \cosh z, z)$

The catenoid is a paradigmatic example in the theory. Besides **Enneper's surface**, it is the only complete nonflat orientable minimal surface in $\mathbb{R}^3$ with the smallest absolute total Gaussian curvature 4π.

1762 **Lagrange:** Minimal surfaces are stationary points of the area functional. Small pieces of such surfaces are area minimizers. Every minimal graph in $\mathbb{R}^3$ is an area minimizer.

1776 **Meusnier An immersed surface in $\mathbb{R}^3$ is a minimal surface if and only if its mean curvature vanishes at every point.**

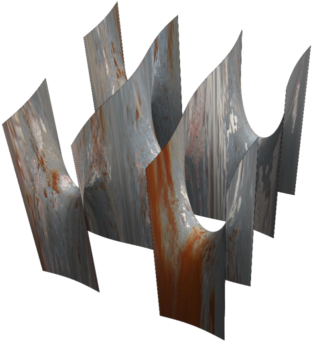
The helicoid is a minimal surface.

It is obtained by rotating a line and displacing it along the axis of rotation.

$(u, v) \mapsto$
$(\cos u \cdot \sinh v, \sin u \cdot \sinh v, u)$

1842 **Catalan** The helicoid and the plane are the only ruled minimal surfaces in $\mathbb{R}^3$.

1835 **Scherk** The first two new minimal surfaces since Meusnier (1776).



The first Scherk's surface is doubly periodic.

Its main branch is a graph over the square $P = (-\pi/2, \pi/2)^2$ given by

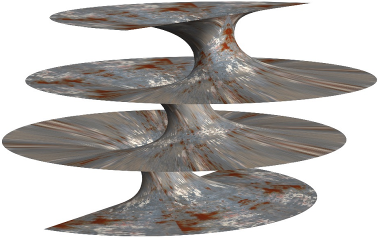
$$x_3 = \log \frac{\cos x_2}{\cos x_1}$$

Finn and Osserman (1964)
Sherk's surface S has the biggest absolute Gaussian curvature at $0 \in \mathbb{R}^3$ over all minimal graphs over P tangent to S at 0.

1917 **Bernstein** A minimal graph in $\mathbb{R}^3$ over the whole plane is a plane.

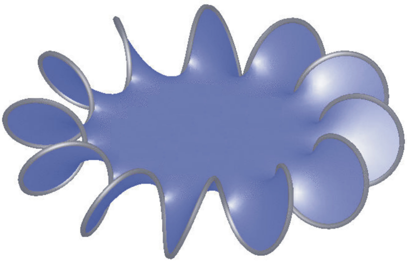
Riemann's minimal examples

Bernhard Riemann discovered a family R_λ, $\lambda > 0$, of periodic planar domains, properly embedded as minimal surfaces in $\mathbb{R}^3$ such that every horizontal plane intersects each R_λ in a circle or a line. As $\lambda \to 0$ his surfaces converge to a vertical catenoid, and as $\lambda \to \infty$ they converge to a vertical helicoid.



2015 **Meeks, Pérez, Ros** Planes, catenoids, helicoids, and Riemann's examples are the only planar domains which can be properly embedded as minimal surfaces in $\mathbb{R}^3$.

1873 **Plateau** Soap films are minimal surfaces. A conformally parameterized minimal disc with a given Jordan boundary curve minimizes the internal tension within the surface.

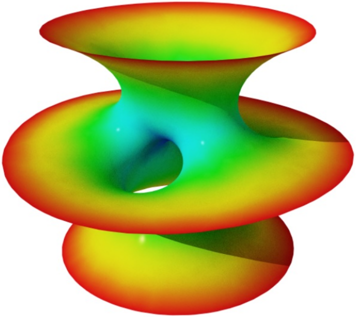


1932 **Douglas, Radó** Every Jordan curve Γ in $\mathbb{R}^3$ spans a minimal surface.

1976 **Meeks, S.-T. Yau** If Γ lies in the boundary of a convex domain, then the disc surface of smallest area with boundary Γ is embedded.

Costa's minimal surface

1982 **Celso J. Costa** discovered a complete embedded minimal surface in $\mathbb{R}^3$ of genus one, a middle planar end and two catenoidal ends, and total Gaussian curvature -12π. Its conformal type is that of a thrice-punctured torus. Hoffman and Meeks proved in 1985 that it is embedded.



1991 **Costa** The only complete embedded minimal surfaces in $\mathbb{R}^3$ having genus one and three ends are the 1-parameter family of Costa–Hoffman–Meeks surfaces.

Analytic description of minimal surfaces in $\mathbb{R}^n$

Assume that $D \subset \mathbb{R}^2_{(u,v)}$ is a bounded domain with smooth boundary and $X : \overline{D} \to \mathbb{R}^n$ is a smooth immersion. Precomposing X with a diffeomorphism from another such domain in $\mathbb{R}^2$, we may assume that X is **conformal**:

$$|X_u| = |X_v|, \qquad X_u \cdot X_v = 0.$$

Digression: If M is a smooth surface and $X : M \to N$ is an immersion into a Riemannian manifold (N, ds^2), then X induces on M a unique structure of a conformal surface (and of a Riemann surface if M is oriented) such that X is a conformal immersion.

Indeed, let $g = X^* ds^2$ be the induced Riemannian metric on M. At every point of M there exists a local **isothermal coordinate** $z = x + \imath y$ in which

$$g = \lambda |dz|^2 = \lambda (dx^2 + dy^2), \quad \lambda > 0.$$

(Solve a **Beltrami equation**.) The transition map between any pair of isothermal charts is an conformal diffeomorphism between plane domains.

The area and the energy functionals

Consider the **area functional**

$$\mathrm{Area}(X) = \int_D |X_u \times X_v| \, dudv = \int_D \sqrt{|X_u|^2 |X_v|^2 - |X_u \cdot X_v|^2} \, dudv$$

and the **Dirichlet energy functional**

$$\mathscr{D}(X) = \frac{1}{2} \int_D |\nabla X|^2 \, dudv = \frac{1}{2} \int_D \left(|X_u|^2 + |X_v|^2 \right) \, dudv.$$

From the elementary inequalities

$$|x|^2 |y|^2 - |x \cdot y|^2 \le |x|^2 |y|^2 \le \frac{1}{4} \left(|x|^2 + |y|^2 \right)^2, \quad x, y \in \mathbb{R}^n,$$

which are equalities iff (x, y) is a conformal frame, we infer that

$$\mathrm{Area}(X) \le \mathscr{D}(X), \quad \text{with equality iff } X \text{ is conformal.}$$

Hence, these two functionals have the same critical points.

A conformal immersion is minimal iff it is harmonic

If $G : \overline{D} \to \mathbb{R}^n$ is a smooth map vanishing on bD, then

$$\frac{d}{dt}\Big|_{t=0} \mathscr{D}(X + tG) = \int_D (X_u \cdot G_u + X_v \cdot G_v)\, dudv = -\int_D \Delta X \cdot G\, dudv.$$

We integrated by parts and used $G|_{bD} = 0$.

This vanishes for all G iff $\Delta X = 0$, thereby proving (a)⇔(b) in

Theorem

The following are equivalent for a smooth conformal immersion $X : \overline{D} \to \mathbb{R}^n$.

(a) X *parameterizes a minimal surface.*

(b) X *is a harmonic map:* $\Delta X = 0$.

(c) *the mean curvature vector field of* X *vanishes.*

The equivalence (b)⇔(c) is seen by an elementary calculation, showing that the metric Laplacian $\Delta_g X$ with respect to $g = X^* ds^2$ equals the mean curvature vector field of X. Note also that $\Delta_g X = 0$ iff $\Delta X = 0$.

Let $z = u + \mathrm{i}v$ be a complex coordinate on $\mathbb{C}$. Then,

$$dX = \partial X + \bar{\partial} X = X_z dz + X_{\bar{z}} d\bar{z} = \frac{1}{2}\left(X_u - \mathrm{i}X_v\right) dz + \frac{1}{2}\left(X_u + \mathrm{i}X_v\right) d\bar{z}.$$

From $\Delta X = 4(X_z)_{\bar{z}}$ we see that

X is harmonic iff X_z is holomorphic.

Furthermore, it is elementary to see that

$$X \text{ is conformal} \iff X_u \cdot X_v = 0,\ |X_u|^2 = |X_v|^2 \iff \sum_{k=1}^{n} (\partial X_k)^2 = 0.$$

The analogous conclusions hold if D is replaced by any **open Riemann surface**.

Hence, a smooth immersion $X=(X_1,X_2,\ldots,X_n):M\to\mathbb{R}^n$ from an open Riemann surface M is conformal and parameterizes a minimal surface if and only if

$$\partial X=(\partial X_1,\ldots,\partial X_n) \text{ is a holomorphic 1-form and } \sum_{k=1}^n(\partial X_k)^2=0.$$

Conversely: a nowhere vanishing holomorphic 1-form $\Phi=(\phi_1,\ldots,\phi_n)$ on M satisfying the nullity condition $\sum_{k=1}^n\phi_k^2=0$ and the period vanishing conditions

$$\Re\int_C\Phi=0\in\mathbb{R}^n \quad \text{for every closed curve } C \text{ in } M$$

determines a conformal minimal immersion

$$X=\Re\int\Phi:M\to\mathbb{R}^n, \qquad 2\partial X=\Phi.$$

Since Φ is holomorphic, it suffices to test the period conditions on the basis of the first homology group $H_1(M,\mathbb{Z})\cong\mathbb{Z}^\ell$, $\ell\in\mathbb{Z}_+\cup\{\infty\}$.

Let us introduce the **null quadric**:

$$\mathbf{A} = \left\{ z = (z_1, \dots, z_n) \in \mathbb{C}^n : \sum_{j=1}^{n} z_j^2 = 0 \right\}.$$

Fix a nowhere vanishing holomorphic 1-form θ on M. With Φ as above,

$$\Phi = f\theta$$

where

$$f : M \to \mathbf{A}_* = \mathbf{A} \setminus \{0\}$$

is a holomorphic map having vanishing real periods:

$$\Re \int_C f\theta = 0 \in \mathbb{R}^n \quad \text{for every } [C] \in H_1(M, \mathbb{Z}).$$

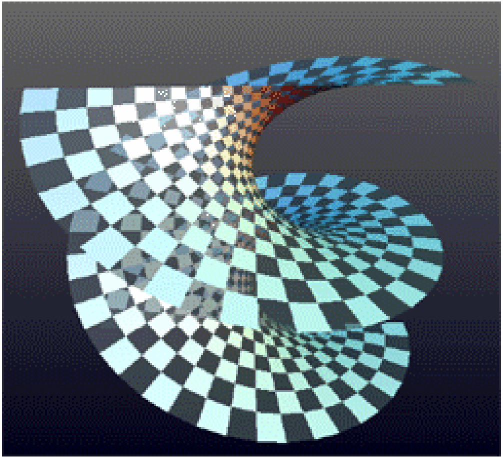
If

$$\int_C f\theta = 0 \in \mathbb{C}^n \quad \text{for every } [C] \in H_1(M, \mathbb{Z}),$$

then $Z = \int f\theta : M \to \mathbb{C}^n$ is a holomorphic curve with $dZ = f\theta$, called a **holomorphic null curve**. The real and the imaginary part of a null curve in $\mathbb{C}^n$ are conjugate minimal surfaces in $\mathbb{R}^n$.

Helicatenoid (Source: Wikipedia)

Helicatenoid is a holomorphic null curve in $\mathbb{C}^3$ whose real part is a catenoid and whose imaginary part is a helicoid.



Dimension $n = 3$

Let $X = (X_1, X_2, X_3) : M \to \mathbb{R}^3$ be a conformal minimal immersion, and let $(N_1, N_2, N_3) : M \to S^2 \subset \mathbb{R}^3$ denote its **classical Gauss map**. Then,

$$\mathfrak{g} = \frac{N_1 + \mathrm{i}N_2}{1 - N_3} = \frac{\partial X_3}{\partial X_1 - \mathrm{i}\partial X_2} : M \to \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$$

is a holomorphic map, called the **complex Gauss map** of X, and we have the **classical Enneper-Weierstrass formula** from 1860s:

$$X = 2\Re \int \left(\frac{1}{2}\left(\frac{1}{\mathfrak{g}} - \mathfrak{g}\right), \frac{\mathrm{i}}{2}\left(\frac{1}{\mathfrak{g}} + \mathfrak{g}\right), 1 \right) \partial X_3.$$

Properties of minimal surfaces in terms of their Gauss map were studied by Nevanlinna theory (**Fujimoto, Ossermann, Ru**, ...).

Alarcón, López, F. 2019 Every holomorphic map $M \to \mathbb{CP}^1$ is the Gauss map of a conformal minimal immersion $M \to \mathbb{R}^3$. The analogous result holds in higher dimensions.

Runge-Mergelyan approximation theorem for minimal surfaces

Theorem (Alarcón, López, F., 2012–2017)

Let K be a compact smoothly bounded domain without holes in an open Riemann surface M (such K is called a **Runge compact** *in M). Then:*

- **Runge approximation theorem for minimal surfaces:** *Every conformal minimal immersion $X : K \to \mathbb{R}^n$ $(n \geq 3)$ can be approximated uniformly on K by (proper) conformal minimal immersions $\widetilde{X} : M \to \mathbb{R}^n$.*
- **Mergelyan approximation on admissible Runge sets**.
- **General position theorem:** *$\widetilde{X}$ can be chosen to have only simple double points if $n = 4$ and to be an embedding if $n \geq 5$.*
- *Analogous results hold for* **nonorientable minimal surfaces** *in $\mathbb{R}^n$ and for* **holomorphic null curves** *in $\mathbb{C}^n$, $n \geq 3$.*

2019 **Alarcón, Castro-Infantes**: In addition, one can prescribe the values of $\widetilde{X}$ on any closed discrete subset of M (Weierstrass-type interpolation).

2019– **Alarcón, López, Lárusson, F.**: The analogous result holds for complete minimal surfaces of finite total Gaussian curvature. In this case, M is a finitely punctured compact Riemann surface and the 1-form $\partial\widetilde{X}$ on M is algebraic, with an effective pole at every puncture.

Techniques used in the proof

Fix a nonvanishing holomorphic 1-form θ on M.

By Enneper–Weierstrass, it suffices to prove the Runge–Mergelyan approximation theorem for holomorphic maps $f : M \to \mathbf{A}_*$ satisfying the period vanishing conditions

$$\Re \int_C f\theta = 0 \quad \text{for all } [C] \in H_1(M, \mathbb{Z}).$$

We use two main properties of punctured null quadric. The first one is:

The punctured null quadric $\mathbf{A}_*$ is an Oka manifold.

In fact, $\mathbf{A}_*$ is a homogeneous space of the complex orthogonal group $O_n(\mathbb{C})$. Every complex homogenous manifold is Oka by a theorem of Grauert (1958).

$\mathbf{A}_*$ is also **flexible**, in the sense that its tangent bundle is spanned by locally nilpotent derivations (i.e., algebraic vector fields with algebraic flows).

What is an Oka manifold?

A complex manifold Y is called an **Oka manifold** (F. 2009) if maps $S \to Y$ from any Stein manifold (or reduced Stein space) S satisfy the following:

- every continuous map $S \to Y$ is homotopic to a holomorphic map.
- If a continuous map $f_0 : S \to Y$ is holomorphic on a neighbourhood of a compact $\mathscr{O}(S)$-convex set K and on a closed complex subvariety S' of S, then there is a homotopy $f_t : S \to Y$ $(t \in [0, 1]$ of maps which are holomorphic near K, close to f_0 on K, they agree with f_0 on S', and such that the map f_1 is holomorphic on S.
- The analogous properties hold for continuous families of maps $S \to Y$.

Model Oka manifolds are the complex Euclidean spaces.

These properties imply that, for X Stein and Y Oka, the inclusion

$$\mathscr{O}(X, Y) \hookrightarrow \mathscr{C}(X, Y)$$

is a weak homotopy equivalence. It is a genuine homotopy equivalence if the Stein manifold X is of finite type.

MSC 2020: 32Q56 Oka principle and Oka manifolds

A brief history of Oka theory

- **Oka–Weil, Oka–Cartan** Euclidean spaces $\mathbb{C}^n$ are Oka manifolds.
- **Oka 1939** $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is an Oka manifold.
- **Grauert 1958** Every complex homogeneous manifold is an Oka manifold.
- **Forster & Ramspott 1966** Oka pairs of sheaves.
- **Gromov 1989** Every elliptic complex manifold is an Oka manifold. (A complex manifold is elliptic if it admits a dominating holomorphic spray.)
- **F. 2005–2009** A complex manifold Y is Oka iff it satisfies the

  Convex approximation property (CAP): Every holomorphic map $K \to Y$ from a compact convex set K in $\mathbb{C}^n$ is a limit of entire maps $\mathbb{C}^n \to Y$.

  By using this axiom, I showed that most of the Oka-type conditions are pairwise equivalent.
- **Kusakabe 2017–** A new characterization of Oka manifolds, a localization theorem, many new examples, ...

The role of convex integration theory

To control the periods of maps $M \to \mathbf{A}_* \subset \mathbb{C}^n$ on closed curves in M, we also use the following elementary observation:

the convex hull of $\mathbf{A}_*$ equals $\mathbb{C}^n$.

Lemma

The convex hull $\mathrm{Co}(A)$ of an connected algebraic subvariety $A \subset \mathbb{C}^n$ is the smallest affine subspace containing A.

If $\mathrm{Co}(A) = \mathbb{C}^n$ then for any pair of points $p, q \in A$ and vector $v \in \mathbb{C}^n$ there is a path $\gamma : [0,1] \to A$ with $\gamma(0) = p$, $\gamma(1) = q$, and $\int_0^1 \gamma(t)dt = v$.

The first part of the lemma follows easily from the theory of plurisubharmonic functions and the **Hironaka desingularization theorem**.

The second part is elementary, and is a special case of **Gromov's lemma on convex integration** (1973).

Outline of proof of the approximation theorem

Fix a nowhere vanishing holomorphic 1-form θ on the open Riemann surface M. Assume that

- $K \subset L$ are connected, smoothly bounded Runge compacts in M,
- $X : K \to \mathbb{R}^n$ is a conformal minimal surface, and
- $f = 2\partial X/\theta : K \to \mathbf{A}_*$.

We may assume that $f(K)$ is not contained in a ray of $\mathbf{A}_*$, for in such case the result follows from Runge's theorem for maps $M \to \mathbb{C}^*$.

The noncritical case: there is no change of topology from K to L.

We have $H_1(K, \mathbb{Z}) = \mathbb{Z}^\ell$ for some $\ell \in \mathbb{Z}_+$.

Choose smooth closed curves $C_1, \ldots, C_\ell \subset K$ forming a basis of $H_1(K, \mathbb{Z})$ such that $C = \bigcup_{j=1}^{\ell} C_j$ is a Runge set.

The noncritical case

Let $\mathbb{B}^n$ denote the unit ball of $\mathbb{C}^n$. By using flows of vector fields tangent to $\mathbf{A}_*$ and the fact that $\mathrm{Co}(\mathbf{A}_*) = \mathbb{C}^n$, we construct a holomorphic map

$$F : K \times \mathbb{B}^{n\ell} \to \mathbf{A}_*, \quad F(\cdot\,, 0) = f = 2\partial X/\theta$$

such that the **period map**

$$\mathbb{B}^{n\ell} \ni t \longmapsto \left(\int_{C_j} F(\cdot\,, t)\theta \right)_{j=1}^{\ell} \in \mathbb{C}^{n\ell}$$

is **biholomorphic onto its image**. Such a map can be found of the form

$$F(p,t) = \phi^1_{g_1(p)t_1} \circ \phi^2_{g_2(p)t_2} \circ \cdots \circ \phi^{n\ell}_{g_{n\ell}(p)t_{n\ell}}(f(p)) \in \mathbf{A}_*, \quad p \in K,$$

where each ϕ^j is the flow of a holomorphic vector field tangent to $\mathbf{A}$ and $g_j \in \mathcal{O}(K)$.

The noncritical case

Since $\mathbf{A}_*$ is an Oka manifold, we can approximate F by a holomorphic map

$$\widetilde{F} : M \times \mathbb{B}^{n\ell} \to \mathbf{A}_*.$$

Assuming that the approximation is close enough, the implicit function theorem shows that there exists $\tilde{t} \in \mathbb{B}^{n\ell}$ close to 0 such that the map

$$\tilde{f} = \widetilde{F}(\cdot\,, \tilde{t}) : M \to \mathbf{A}_*$$

has vanishing real periods on the curves $C_1, \ldots, C_\ell$.

Hence, fixing a point $p_0 \in K$, the map $\widetilde{X} : L \to \mathbb{R}^n$ given by

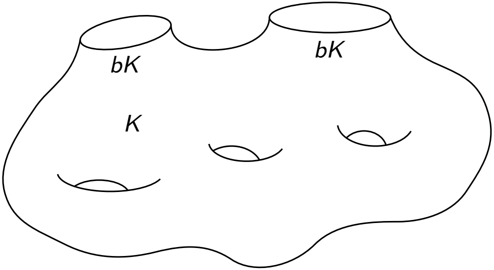
$$\widetilde{X}(p) = X(p_0) + \Re \int_{p_0}^{p} \tilde{f}\theta, \qquad p \in L$$

is a conformal minimal immersion which approximates $X : K \to \mathbb{R}^n$ on K.

The critical case

Assume now that $E \subset L \setminus \mathring{K}$ is an arc attached with its endpoints to K such that $K \cup E$ is a deformation retract of L. This situation arises when passing a critical point of index 1 of an exhaustion function on M. We illustrate **attaching a pair of pants**, which increases the genus by one.

1° The domain K

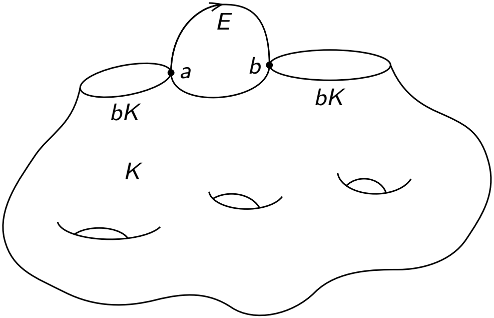


The critical case

Assume now that $E \subset L \setminus \mathring{K}$ is an arc attached with its endpoints to K such that $K \cup E$ is a deformation retract of L. This situation arises when passing a critical point of index 1 of an exhaustion function on M. We illustrate **attaching a pair of pants**, which increases the genus by one.

1° The domain K

2° Add the arc E – the seam of the pants

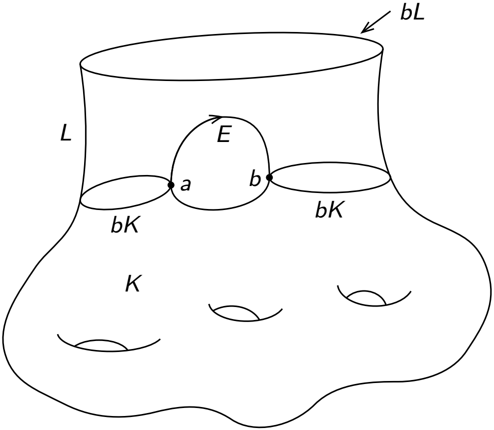


Assume now that $E \subset L \setminus \mathring{K}$ is an arc attached with its endpoints to K such that $K \cup E$ is a deformation retract of L. This situation arises when passing a critical point of index 1 of an exhaustion function on M. We illustrate **attaching a pair of pants**, which increases the genus by one.

1° The domain K

2° Add the arc E – the seam of the pants

3° Add the pair of pants



The critical case

Let $a, b \in bK$ denote the endpoints of the arc E. Fix a point $p_0 \in K$ and write

$$X(p) = X(p_0) + \Re \int_{p_0}^{p} f\theta, \qquad p \in K.$$

Extend f smoothly across E to a map $f : K \cup E \to \mathbf{A}_*$ such that

$$\Re \int_E f\theta = X(b) - X(a) \in \mathbb{R}^n.$$

This is possible by the convex integration lemma.

Now, proceed as in the noncritical case, constructing a period dominating spray and applying Mergelyan approximation on the admissible Runge set $K \cup E$.

The proof of the approximation theorem follows by induction on an exhaustion of M such that every step is of one of the two types described above. Interpolation on a discrete set of points is obtained by the same scheme.

However, general position theorems and the existence of proper conformal minimal immersions or embeddings require substantial additional work.

Topological structure of the space of conformal minimal immersions

By using the parametric extensions of our techniques, we showed that all maps in the following diagram are weak homotopy equivalences, and genuine homotopy equivalences if $H_1(M,\mathbb{Z})$ is finitely generated.

$$\begin{array}{ccccc} \Re\mathrm{NC}_*(M,\mathbb{C}^n) & \hookrightarrow & \mathrm{CMI}_*(M,\mathbb{R}^n) & & \\ & \overset{\partial}{\searrow} & \downarrow \partial & & \\ & & \mathscr{O}(M,\mathbf{A}_*) & \hookrightarrow & \mathscr{C}(M,\mathbf{A}_*) \end{array}$$

$\mathrm{NC}_*(M,\mathbb{C}^n)$ is the space of nonflat holomorphic null curves $M \to \mathbb{C}^n$
$\mathrm{CMI}_*(M,\mathbb{R}^n)$ is the space of nonflat conformal minimal immersions $M \to \mathbb{R}^n$

Furthermore, the map $\mathrm{CMI}(M,\mathbb{R}^n) \to \mathscr{C}(M,\mathbf{A}_*)$ induces a bijection of path components of the two spaces. Let $H_1(M,\mathbb{Z}) = \mathbb{Z}^\ell$. Then,

$$\pi_0(\mathrm{CMI}(M,\mathbb{R}^n)) = \begin{cases} \mathbb{Z}_2^\ell, & n = 3; \\ 0, & n > 3. \end{cases}$$

Complete minimal surfaces of finite total curvature

Let $M = \overline{M} \setminus \{p_1, \ldots, p_m\}$ be a compact Riemann surface with finitely many punctures. Consider the diagram

$$\begin{array}{ccc} \Re\mathrm{NC}_*(M, \mathbb{C}^n) & \hookrightarrow & \mathrm{CMI}_*(M, \mathbb{R}^n) \\ & \searrow^{\partial} & \downarrow \partial \\ & & \mathscr{A}^1(M, \mathbf{A}_*) \end{array}$$

- ∂ is the $(1, 0)$-differential,
- $\mathrm{CMI}_*(M, \mathbb{R}^n)$ is the space of nonflat complete conformal minimal immersions of finite total Gaussian curvature,
- $\mathrm{NC}_*(M, \mathbb{C}^n)$ is the space of nonflat complete algebraic null immerson,
- $\mathscr{A}^1(M, \mathbf{A})$ is the space of algebraic 1-forms on M with values in $\mathbf{A}_*$.

Alarcón, Lárusson, F. 2024 The maps in the above diagram are weak homotopy equivalences.

Problem: What can be said about the inclusion $\mathscr{A}^1(M, \mathbf{A}_*) \longrightarrow \mathscr{O}^1(M, \mathbf{A}_*)$?

Another major theme of our work is the **Calabi–Yau problem for minimal surfaces**. Our central results in this field are:

Theorem (Alarcón, Drinovec, F., López, 2015)

Let M be any bordered Riemann surface. Every conformal minimal immersion $X_0 : \overline{M} \to \mathbb{R}^n$ $(n \geq 3)$ can be uniformly approximated by continuous maps $X : \overline{M} \to \mathbb{R}^n$ (embeddings if $n \geq 5$) such that

$X : M \to \mathbb{R}^n$ *is a* **complete conformal minimal immersion, and the boundary $X(bM) \subset \mathbb{R}^n$ is a union of Jordan curves.**

Theorem (Alarcón and F., 2021)

The same holds if M is a bordered Riemann surface of the form

$$M = R \setminus \bigcup_i D_i$$

where R is a compact Riemann surface and D_i is a finite or countable family of pairwise disjoint compact geometric discs in R.

Schwarz–Pick lemma for harmonic maps which are conformal at a point

Theorem (Kalaj and F. 2021; Anal. & PDE 2024)

Let $\mathbb{D}$ denote the unit disc in $\mathbb{C}$ and $\mathbb{B}^n$ denote the unit ball of $\mathbb{R}^n$. Assume that $f : \mathbb{D} \to \mathbb{B}^n$ $(n \geq 2)$ is a harmonic map which is conformal at a point $z \in \mathbb{D}$. Denote by $R \in (0, 1]$ the radius of the affine disc $\Sigma = (f(z) + df_z(\mathbb{R}^2)) \cap \mathbb{B}^n$. Then

$$\|df_z\| \leq \frac{1}{R} \cdot \frac{1 - |f(z)|^2}{1 - |z|^2}.$$

Equality holds if and only if f is a conformal diffeomorphism of $\mathbb{D}$ onto Σ, and in this case equality holds at every point of $\mathbb{D}$.

The case $n = 2$, $R = 1$ generalizes the classical **Schwarz–Pick lemma** due to **H. A. Schwarz** 1869, **H. Poincaré** 1884, and **G. Pick** 1915.

The classical lemma pertains only to maps which are **conformal at every point** of $\mathbb{D}$, i.e., they are **holomorphic or antiholomorphic**.

Corollary: Conformal harmonic maps $\mathbb{D} \to \mathbb{B}^n$ are distance decreasing from the Poincaré metric $\frac{|dz|}{1-|z|^2}$ on $\mathbb{D}$ to the **Cayley–Klein metric** on $\mathbb{B}^n$:

$$\mathcal{CK}^2 = \frac{(1-|x|^2)|dx|^2 + |x \cdot dx|^2}{(1-|x|^2)^2} = \frac{|dx|^2}{1-|x|^2} + \frac{|x \cdot dx|^2}{(1-|x|^2)^2}.$$

Extremal maps are the conformal embeddings of $\mathbb{D}$ onto affine discs in $\mathbb{B}^n$.

The same is true if we replace the disc by any hyperbolic Riemann surface (universally covered by the disc) endowed with the Poincaré metric.

The **Beltrami–Cayley–Klein model of hyperbolic geometry** was introduced by **Arthur Cayley** (1859) and **Eugenio Beltrami** (1868), and it was developed by **Felix Klein** (1871–1873). The underlying space is the ball $\mathbb{B}^n$, geodesics are line segments with endpoints on the sphere, and the distance is given by the cross ratio. This is a special case of a metric on convex domains in $\mathbb{R}^n$ introduced by **David Hilbert** in 1895.

Up to constants, the Cayley–Klein metric is the restriction to $\mathbb{B}^n$ of the Kobayashi metric and the Bergman metric on the complex ball $\mathbb{B}^n_{\mathbb{C}} \subset \mathbb{C}^n$.

The extremal holomorphic discs for the Kobayashi metric on the complex ball $\mathbb{B}^n_{\mathbb{C}} \subset \mathbb{C}^n$ are the complex affine discs in $\mathbb{B}^n_{\mathbb{C}}$. The standard proof uses the fact that the group of holomorphic automorphisms of $\mathbb{B}^n_{\mathbb{C}}$ acts transitively.

Comparison with the new result shows that, up to orientation, the Kobayashi-extremal holomorphic discs in $\mathbb{B}^n_{\mathbb{C}}$are precisely those extremal conformal minimal discs for the minimal metric on $\mathbb{B}^n_{\mathbb{C}}$ whose images are complex.

The biggest group preserving the set of all conformal minimal discs in $\mathbb{B}^n$ (under postcomposition) is the orthogonal group, which does not act transitively. Our proof also gives a new proof of the complex Schwarz lemma without using the group $\mathrm{Aut}(\mathbb{B}^n_{\mathbb{C}})$.

It suffices to prove it for $z = 0$. Indeed, with f and z as in the theorem, let $\phi_z \in \mathrm{Aut}(\mathbb{D})$ be such that $\phi_z(0) = z$. The harmonic map $\tilde{f} = f \circ \phi_z : \mathbb{D} \to \mathbb{B}^n$ is then conformal at the origin. Since $|\phi_z'(0)| = 1 - |z|^2$, the estimate follows from the same estimate for the map $\tilde{f}$ applied at $z = 0$.

We find an explicit conformal parameterization of affine discs in $\mathbb{B}^n$. Fix a point $\mathbf{q} \in \mathbb{B}^n$ and a 2-plane $0 \in \Lambda \subset \mathbb{R}^n$, and consider the affine disc $\Sigma = (\mathbf{q} + \Lambda) \cap \mathbb{B}^n$. Let $\mathbf{p} \in \Sigma$ be the closest point to the origin.

If $n = 2$ then $\mathbf{p} = 0$ and $\Sigma = \mathbb{D}$. Suppose now that $n = 3$; the case $n > 3$ will be the same. By an orthogonal rotation on $\mathbb{R}^3$ we may assume that

$$\mathbf{p} = (0, 0, p) \quad \text{and} \quad \Sigma = \left\{ (x, y, p) : x^2 + y^2 < 1 - p^2 \right\}.$$

Let $\mathbf{q} = (b_1, b_2, p) \in \Sigma$, and let θ denote the angle between $\mathbf{q}$ and Σ. Set

$$R = \sqrt{1 - p^2} = \sqrt{1 - |\mathbf{q}|^2 \sin^2\theta}, \qquad a = \frac{b_1 + \mathrm{i} b_2}{R} \in \mathbb{D}, \qquad |a| = \frac{|\mathbf{q}| \cos\theta}{R}.$$

We orient Σ by the pair of tangent vectors ∂_x, ∂_y.

Proof, 2

Every orientation preserving conformal parameterization $f : \mathbb{D} \to \Sigma$ with $f(0) = \mathbf{q}$ is then of the form

$$f(z) = \left(R \cdot \Re \frac{e^{it} z + a}{1 + \bar{a} e^{it} z}, R \cdot \Im \frac{e^{it} z + a}{1 + \bar{a} e^{it} z}, p \right) = \left(R \, \frac{e^{it} z + a}{1 + \bar{a} e^{it} z}, p \right)$$

for $z \in \mathbb{D}$ and some $t \in \mathbb{R}$. (If $n = 2$ then $p = 0$, $R = 1$, and we drop the last coordinate.)

We have that

$$\begin{aligned} \|df_0\| &= R(1 - |a|^2) = \frac{R^2 - R^2|a|^2}{R} = \frac{1 - |\mathbf{q}|^2 \sin^2\theta - |\mathbf{q}|^2 \cos^2\theta}{R} \\ &= \frac{1 - |\mathbf{q}|^2}{\sqrt{1 - |\mathbf{q}|^2 \sin^2\theta}} = \frac{1 - |f(0)|^2}{\sqrt{1 - |f(0)|^2 \sin^2\theta}}. \end{aligned}$$

This shows that the conformal parameterizations of the proper affine discs in the ball satisfy the equality in the theorem at every point.

Proof, 3

Let $f : \mathbb{D} \to \mathbb{B}^3$ be as above, where we may assume that $t = 0$.

Suppose that $g : \mathbb{D} \to \mathbb{B}^3$ is a harmonic map such that $g(0) = f(0)$, g is conformal at 0, and $dg_0(\mathbb{R}^2) = df_0(\mathbb{R}^2)$. Up to replacing g by $g(e^{\mathrm{i}t}z)$ or $g(e^{\mathrm{i}t}\bar{z})$ for some $t \in \mathbb{R}$, we may assume that

$$dg_0 = r\, df_0 \quad \text{for some } r > 0.$$

We must prove that $r \le 1$, and that $r = 1$ if and only if $g = f$.

Consider the holomorphic map $F : \mathbb{D} \to \Omega = \mathbb{B}^3 \times \mathrm{i}\mathbb{R}^3$ with $f = \Re F$, given by

$$F(z) = \left(R \cdot \frac{z+a}{1+\bar{a}z}, -R \cdot \mathrm{i} \frac{z+a}{1+\bar{a}z}, p \right), \quad z \in \mathbb{D}.$$

Let $G : \mathbb{D} \to \Omega$ be the holomorphic map with $\Re G = g$ and $G(0) = F(0)$.

By the Cauchy–Riemann equations, the condition $dg_0 = r\, df_0$ implies

$$G'(0) = r\, F'(0).$$

It follows that the map $(F(z) - G(z))/z$ is holomorphic on $\mathbb{D}$ and

$$\lim_{z \to 0} \frac{F(z) - G(z)}{z} = F'(0) - G'(0) = (1 - r)F'(0).$$

Since $g : \mathbb{D} \to \mathbb{B}^3$ is a bounded harmonic map, it has a nontangential boundary value at almost every point of the circle $\mathbb{T} = b\mathbb{D}$. Since the Hilbert transform is an isometry on the Hilbert space $L^2(\mathbb{T})$, the same is true for G.

Denote by $\langle \cdot, \cdot \rangle$ the complex bilinear form on $\mathbb{C}^n$ given by

$$\langle z, w \rangle = \sum_{i=1}^{n} z_i w_i$$

for $z, w \in \mathbb{C}^n$.

For each $z = e^{\mathrm{i}t} \in b\mathbb{D}$ the vector $f(z) \in b\mathbb{B}^3$ is the unit normal vector to the sphere $b\mathbb{B}^3$ at the point $f(z)$. Since $\mathbb{B}^3$ is strongly convex, we have that

$$\Re \left\langle F(z) - G(z), f(z) \right\rangle = \left\langle f(z) - g(z), f(z) \right\rangle \geq 0 \quad a.e.\ z \in b\mathbb{D},$$

and the value is positive for almost every $z \in b\mathbb{D}$ if and only if $g \neq f$.

Consider the following function on the circle $b\mathbb{D}$:

$$\tilde{f}(z) = z|1 + \bar{a}z|^2 f(z), \qquad |z| = 1.$$

A calculation, taking into account $z\bar{z} = 1$, shows that

$$\tilde{f}(z) = \begin{pmatrix} \frac{c}{2}\left(1 + a^2 + 4(\Re a)z + (1 + \bar{a}^2)z^2\right) \\ \frac{c}{2}\left(\mathrm{i}(1 - a^2) + 4(\Im a)z + \mathrm{i}(\bar{a}^2 - 1)z^2\right) \\ p\,(z + a)(1 + \bar{a}z) \end{pmatrix}, \qquad |z| = 1.$$

Conclusion of the proof

We extend $\tilde{f}$ to all $z \in \mathbb{C}$ by letting it equal the holomorphic polynomial map on the right hand side above. Since $|1+\bar{a}z|^2 > 0$ for $z \in \overline{\mathbb{D}}$, we have

$$\begin{aligned} h(z) &:= \Re \langle F(z) - G(z), |1+\bar{a}z|^2 f(z) \rangle \\ &= \langle f(z) - g(z), |1+\bar{a}z|^2 f(z) \rangle \geq 0 \quad a.e.\ z \in b\mathbb{D}, \end{aligned}$$

and $h > 0$ almost everywhere on $b\mathbb{D}$ if and only if $g \neq f$.

From the definition of $\tilde{f}$ we see that

$$h(z) = \Re \left\langle \frac{F(z) - G(z)}{z}, \tilde{f}(z) \right\rangle \quad a.e.\ z \in b\mathbb{D}$$

Since the maps $(F(z) - G(z))/z$ and $\tilde{f}(z)$ are holomorphic on $\mathbb{D}$, h extends to a nonnegative harmonic function on $\mathbb{D}$ which is positive on $\mathbb{D}$ unless $f = g$.

At $z = 0$ we have

$$h(0) = \Re \langle F'(0) - G'(0), \tilde{f}(0) \rangle = (1-r) \Re \langle F'(0), \tilde{f}(0) \rangle \geq 0,$$

with equality if and only if $g = f$. Applying this to the constant map $g(z) = f(0)$ (for which $r = 0$) gives $\Re \langle F'(0), \tilde{f}(0) \rangle \geq 0$. It follows that $r \leq 1$, with equality if and only if $g = f$. This completes the proof.

The minimal pseudometric

Let Ω be a domain in $\mathbb{R}^n$, $n \geq 3$. Denote by $\mathrm{CH}(\mathbb{D}, \Omega)$ the space of (not necessarily immersed) conformal harmonic discs $\mathbb{D} \to \Omega$.

We define a **Kobayashi-type Finsler pseudometric** on $T\Omega = \Omega \times \mathbb{R}^n$ by

$$g_\Omega(\mathbf{x}, \mathbf{v}) = \inf\{1/r > 0 : \exists f \in \mathrm{CH}(\mathbb{D}, \Omega),\ f(0) = \mathbf{x},\ f_x(0) = r\mathbf{v}\}. \qquad (1)$$

The **minimal pseudodistance** $\rho_\Omega : \Omega \times \Omega \to \mathbb{R}_+$ is

$$\rho_\Omega(\mathbf{x}, \mathbf{y}) = \inf_\gamma \int_0^1 g_\Omega(\gamma(t), \dot\gamma(t))\, dt, \qquad \mathbf{x}, \mathbf{y} \in \Omega. \qquad (2)$$

The infimum is over all piecewise smooth paths $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = \mathbf{x}$ and $\gamma(1) = \mathbf{y}$. Obviously, ρ_Ω satisfies the triangle inequality, but it need not be a distance function. In particular, $\rho_{\mathbb{R}^n}$ vanishes identically.

Kalaj and F. 2021 $\rho_{\mathbb{B}^n}$ is the Cayley–Klein metric.

This is the only domain on which an explicit formula for ρ is known.

Gaussier & Sukhov, 2023– Definition and basic properties of the minimal metric on any Riemannian manifold.

The distance decreasing property

For every conformal harmonic disc $f : \mathbb{D} \to \Omega$ we have

$$g_\Omega(f(z), df_z(\xi)) \leq \frac{|\xi|}{1-|z|^2} = \mathscr{P}_{\mathbb{D}}(z,\xi), \quad z \in \mathbb{D},\ \xi \in \mathbb{R}^2,$$

and g_Ω is the biggest pseudometric on Ω with this property. For $z = 0$ this follows from the definition of g_Ω. For an arbitrary point $z \in \mathbb{D}$ it is obtained by replacing f by the conformal harmonic disc $f \circ \phi$, where $\phi \in \mathrm{Aut}(\mathbb{D})$ is a holomorphic automorphism of the disc interchanging 0 and z.

It follows that conformal harmonic maps $M \to \Omega$ from any conformal surface M are distance-decreasing in the Poincaré metric on M and the minimal metric on Ω:

$$\rho_\Omega(f(x), f(x')) \leq \mathrm{dist}_{\mathcal{P}_M}(x, x'), \qquad x, x' \in M;$$

furthermore, ρ_Ω is the biggest pseudodistance on Ω having this property.

If $R : \mathbb{R}^n \to \mathbb{R}^n$ is a *rigid transformation* (a composition of an orthogonal map, a dilation, and a translation) and $R(\Omega) \subset \Omega'$, then

$$g_{\Omega'}(R(\mathbf{x}), dR_{\mathbf{x}}(\mathbf{u})) \leq g_\Omega(\mathbf{x}, \mathbf{u}), \quad \mathbf{x} \in \Omega,\ \mathbf{u} \in \mathbb{R}^n.$$

The minimal metric is defined by chains of conformal harmonic discs

Fix a pair of points $\mathbf{x}, \mathbf{y} \in \Omega \subset \mathbb{R}^n$. Given a finite chain of conformal harmonic discs $f_i : \mathbb{D} \to \Omega$ and points $a_i \in \mathbb{D}$ $(i = 1, \dots, k)$ such that

$$f_1(0) = \mathbf{x}, \quad f_{i+1}(0) = f_i(a_i) \text{ for } i = 1, \dots, k-1, \quad f_k(a_k) = \mathbf{y},$$

we define the length of the chain to be the number

$$\sum_{i=1}^{k} \mathrm{dist}_{\mathcal{P}_{\mathbb{D}}}(0, a_i) = \sum_{i=1}^{k} \frac{1}{2} \log \frac{1 + |a_i|}{1 - |a_i|} \geq 0.$$

It turns out that $\rho_\Omega(\mathbf{x}, \mathbf{y})$ is the infimum of the lengths of all such chains.

Note that the Kobayashi pseudodistance on a complex manifold X is defined in the same way by using chains of holomorphic discs $\mathbb{D} \to X$.

Definition

A domain $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ is *hyperbolic* if ρ_Ω is a distance function on Ω, and is *complete hyperbolic* if (Ω, ρ_Ω) is a complete metric space.

A few results on hyperbolicity

Theorem (B. Drinovec Drnovšek & F. 2021, publ. 2024)

The following are equivalent for a convex domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$.

1. Ω *is complete hyperbolic.*
2. Ω *is hyperbolic.*
3. Ω *does not contain any 2-dimensional affine subspaces.*

For comparison: A convex domain in $\mathbb{C}^n$ is Kobayashi hyperbolic if and only if it does not contain any affine complex line (**Barth** 1980, **Harris** 1979).

Theorem (B. Drinovec Drnovšek & F.)

Let Ω *be a bounded domain in* $\mathbb{R}^n$ $(n \geq 3)$ *with smooth boundary* $b\Omega$ *whose principal curvatures* $\nu_1 \leq \nu_2 \leq \ldots \leq \nu_{n-1}$ *at any point* $p \in b\Omega$ *satisfy* $\nu_1 + \nu_2 > 0$. *Then,* Ω *is complete hyperbolic.*

Theorem (Fiacchi 2023)

Every domain as in the previous theorem is Gromov hyperbolic.

∼ Thank you for your attention ∼