MINIMAL SURFACES WITH SYMMETRIES

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Abstract

Objects with symmetries are of special interest in any mathematical theory.

In this talk, I shall discuss the existence of orientable minimal surfaces in Euclidean spaces $\mathbb{R}^n$, $n \geq 3$, with a given group of symmetries.

F. Forstnerič: Minimal surfaces with symmetries. Proc. London Math. Soc., **128:3** (2024) e12590. http://dx.doi.org/10.1112/plms.12590

An elementary introduction to minimal surfaces:

F. Forstnerič: Minimal surfaces in Euclidean spaces by way of complex analysis. European Congress of Mathematics, 9–43. EMS Press, Berlin, ©2023.

What is a minimal surface?

Euler 1744; Lagrange 1762 A smooth immersed surface $F : X \to \mathbb{R}^n$ $(n \geq 3)$ is a **minimal surface** if it is a **stationary point of the area functional**. Small pieces of such a surface are area minimisers.

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Digression: conformal structures on surfaces. Let X be a smooth surface. An immersion $F : X \to \mathbb{R}^n$ determines on X a Riemannian metric $g = F^* ds^2$, which makes F an isometry. By **Gauss**, there are local **isothermal coordinates** (x, y) at any point of X in which

$$g = \lambda(dx^2 + dy^2) \text{ for some function } \lambda > 0.$$

Transition maps between isothermal charts are conformal diffeomorphisms of plane domains, hence holomorphic or antiholomorphic. This endows X with the structure of a **conformal surface**, and of a **Riemann surface** if X is oriented, such that F **is a conformal immersion**.

Minimal surfaces are given by conformal harmonic immersions

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Write $\partial F = \frac{\partial F}{\partial z} dz = \frac{1}{2}\left(\frac{\partial F}{\partial x} - \mathrm{i}\frac{\partial F}{\partial y}\right)(dx + \mathrm{i}dy)$. Then, $\Re(2\partial F) = dF$ and

$$\Delta F = 0 \iff \partial F = (\partial F_1, \ldots, \partial F_n) \text{ is a holomorphic 1-form on } X.$$

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It is elementary to see that an immersion $F = (F_1, \ldots, F_n)$ is conformal iff

$$\sum_{i=1}^{n} (\partial F_i)^2 = 0. \tag{2}$$

Minimal surfaces are solutions of the nonlinear elliptic PDE (1), (2).

The Enneper–Weierstrass representation of minimal surfaces

Let $A \subset \mathbb{C}^n$ denote the **punctured null quadric**

$$A = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \setminus \{0\} : z_1^2 + z_2^2 + \cdots + z_n^2 = 0\},$$

and let $\overline{A} \subset \mathbb{CP}^n$ denote its projective closure.

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and let $\overline{A} \subset \mathbb{CP}^n$ denote its projective closure. Pick a nontrivial holomorphic 1-form θ on X (possibly with zeros). If $F : X \to \mathbb{R}^n$ is a minimal surface then

$$2\partial F = f\theta, \quad dF = \Re(2\partial F) = \Re(f\theta),$$

where $f = (f_1, \ldots, f_n) : X \to \overline{A} \setminus \{0\}$ is a holomorphic map such that

$$\Re \oint_C f\theta = \oint_C dF = 0 \text{ for every closed curve } C \subset X. \tag{3}$$

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Conversely, given f as above such that $f\theta$ is a nowhere vanishing holomorphic 1-form on X satisfying (3), the map $F : X \to \mathbb{R}^n$ given by

$$F(x) = \Re \int_*^x f\theta$$

is a conformal harmonic immersion.

Symmetries and G-equivariant maps

A smooth map $T : \mathbb{R}^n \to \mathbb{R}^n$ maps minimal surfaces to minimal surfaces iff T is a **rigid map** — a composition of orthogonal maps, dilations, and translations.

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If $F : X \to S = F(X) \subset \mathbb{R}^n$ is an injective conformal immersion, then G also acts on X by conformal diffeomorphisms such that F is **G-equivariant**:

$$F \circ g = g \circ F \quad \text{for every } g \in G.$$

If X is a Riemann surface and every $g \in G$ preserves the orientation on $S = F(X)$, then G acts on X by holomorphic automorphisms.

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Conversely, the image of a G-equivariant immersion is a G-invariant surface.

$***$ **Which groups arise in this way for minimal surfaces?** $***$

Most classical minimal surfaces have symmetries

Euler 1744 The only minimal surfaces of rotation in $\mathbb{R}^3$ are planes and catenoids.

$$x^2 + y^2 = \cosh^2 z$$

$$(t, z) \mapsto (\cos t \cdot \cosh z, \sin t \cdot \cosh z, z)$$

The symmetry group consist of rotations in the (x, y)-plane and the reflection $z \mapsto -z$.

The helicoid (Archimedes' screw)

Meusnier 1776 The helicoid is a ruled minimal surface.
It is obtained by rotating a line and displacing it along the axis of rotation.

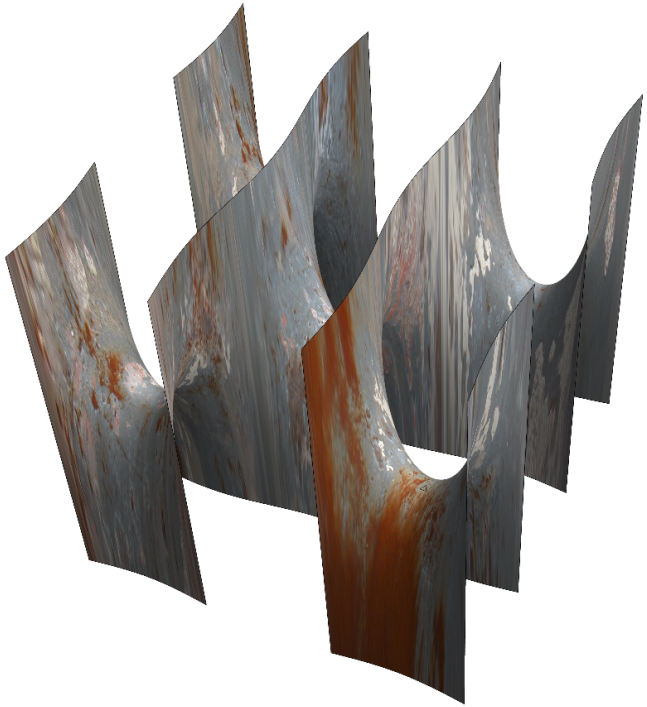
$$\begin{aligned} x &= \rho\cos(\alpha z), \\ y &= \rho\sin(\alpha z), \quad (z,\rho) \in \mathbb{R}^2. \end{aligned}$$

The group $\mathbb{Z}$ acts on the helicoid by translations $z \mapsto z + k2\pi/\alpha$, $k \in \mathbb{Z}$.

Also, the group $\mathbb{R}$ acts by translations and simultaneous rotations.

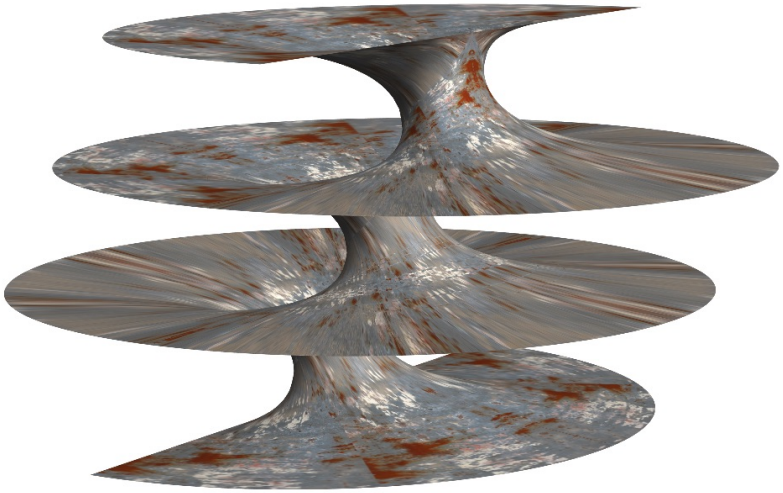
Scherk's first surface

Scherk, 1835 The first Scherk's surface is doubly periodic, with the group $\mathbb{Z}^2$ acting by translations.



Riemann's minimal examples

Bernhard Riemann 1867: A family R_λ, $\lambda > 0$, of periodic planar domains, properly embedded as minimal surfaces in $\mathbb{R}^3$ such that every horizontal plane intersects each R_λ in a circle or a line. As $\lambda \to 0$ his surfaces converge to a vertical catenoid, and as $\lambda \to \infty$ they converge to a vertical helicoid.



The group $\mathbb{Z}$ acts on R_λ by translations.

Equivariant minimal surfaces of genus zero

Let $S \cong \mathbb{CP}^1$ be the unit sphere in $\mathbb{R}^3$. The group $SO(3,\mathbb{R})$ is a real 3-dimensional subgroup of the holomorphic automorphism group

$$\mathrm{Aut}(S) = \left\{ z \mapsto \frac{az+b}{cz+d} : \quad a,b,c,d \in \mathbb{C},\ ad-bc=1 \right\}.$$

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Finite subgroups of $SO(3,\mathbb{R})$ are called **spherical von Dyck groups**. Besides **the cyclic and the dihedral groups**, we have the **symmetry groups of Platonic solids**:

- the alternating group A_4 of order 12 is the group of symmetries of the tetrahedron,
- the symmetric group S_4 of order 24 is the group of symmetries of the cube and the octahedron, and
- the group A_5 of order 60 is the group of symmetries of the icosahedron and the dodecahedron.

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Xu 1995 Every closed subgroup $G \subsetneq SO(3,\mathbb{R})$ is a symmetry group of a proper minimal surface in $\mathbb{R}^3$ of genus zero with finite total curvature.

Examples by **Jorge and Meeks 1983**, **Rossman 1995**, **Small 1999**, and others.

The main theorem

Let X be a connected open Riemann surface and $G \subset \mathrm{Aut}(X)$ be a finite group of holomorphic automorphisms. The stabiliser of $x \in X$ is

$$G_x = \{g \in G : gx = x\}.$$

This is a cyclic group of rotations in a local holomorphic coordinate at x.

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Assume that G also acts on $\mathbb{R}^n$ by orthogonal transformations in $O(n, \mathbb{R})$.

Theorem

The following are equivalent:

(a) *For every nontrivial stabiliser G_x $(x \in X)$ there is a G_x-invariant 2-plane $\Lambda_x \subset \mathbb{R}^n$ on which G_x acts effectively by rotations.*

(b) *There exists a G-equivariant conformal minimal immersion $F : X \to \mathbb{R}^n$:*

$$F(gx) = gF(x), \quad x \in X,\ g \in G.$$

In particular, such F exists if the group G acts freely on X.

Two generalizations

The same holds if G is an infinite group acting on X **properly discontinuously** by holomorphic automorphisms such that X/G is noncompact (this implies that the genus of X is 0 or 1), and G acts on $\mathbb{R}^n$ by **rigid transformations**, i.e., compositions of orthogonal maps, translations, and dilations.

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The result also holds if the differential ∂F has poles. The poles of ∂F are **proper ends with finite total curvature** of the minimal surface $F : X \to \mathbb{R}^n$:

$$\int K \cdot d\mathrm{Area} > -\infty,$$

where $K : X \to (-\infty, 0]$ is the Gaussian curvature function.

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However, it remains an open problem whether there exist minimal surfaces of finite total curvature satisfying the theorem. In such case, X must be a compact Riemann surface punctured at finitely many points and ∂F must be a rational A-valued 1-form on X.

Every finite group is a symmetry group of a minimal surface

Greenberg 1960, 1974 Every countable group G is the automorphism group of a Riemann surface X. If G is finite then X can be taken compact.

Hurwitz 1893, Maskit 1968 If X is a Riemann surface of genus $\mathfrak{g} \geq 2$ then

$$|\operatorname{Aut}(X)| \leq 84(\mathfrak{g} - 1).$$

Most such surfaces have no nontrivial automorphisms.

It is elementary to see that every finite group G of order $n = |G|$ acts on $\mathbb{R}^{2n}$ by orthogonal maps such that for every $g \in G$ there is a 2-plane $\Lambda \subset \mathbb{R}^{2n}$ on which g acts by a rotation for the angle $2\pi/k$, where k is the order of g.

Corollary

For every finite group G of order $n > 1$ there exist an open connected Riemann surface X, effective actions of G by holomorphic automorphisms on X and by orthogonal transformations on $\mathbb{R}^{2n}$, and a G-equivariant conformal minimal immersion $X \to \mathbb{R}^{2n}$.

Proof of (b) $\implies$ (a)

Let $x \in X$ be a point with a nontrivial stabiliser G_x of order $k = |G_x| > 1$.

There is a local holomorphic coordinate z on X around x, with $z(x) = 0$, in which a generator of $G_x = \langle g \rangle$ is the rotation

$$gz = \mathrm{e}^{\mathrm{i}\phi} z, \quad \phi = 2\pi/k. \tag{4}$$

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Assume that G acts on $\mathbb{R}^n$ by orthogonal maps and $F : X \to \mathbb{R}^n$ is a G-equivariant immersion. Differentiating $g \circ F = F \circ g$ gives

$$g \circ dF_x = dF_x \circ dg_x : T_x X \to \Lambda_x := dF_x(T_x X) \subset \mathbb{R}^n.$$

Since $dF_x : T_x X \to \Lambda_x$ is a linear isomorphism, we see that $\Lambda_x \subset \mathbb{R}^n$ is a G_x-invariant plane on which g acts as the rotation R_ϕ, so condition (a) holds.

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Conversely, if condition (a) holds at $x \in X$ for the 2-plane $\Lambda_x \subset \mathbb{R}^n$, then the conformal linear map from the z-neighbourhood of $x \in X$ as in (4) to $\Lambda_x \subset \mathbb{R}^n$ is a G_x-equivariant conformal minimal immersion.

Proof of (a) $\implies$ (b)

Let G be a finite group acting on an open Riemann surface X by holomorphic automorphisms. The set

$$X_0 = \{x \in X : G_x \neq \{1\}\}$$

is a closed, discrete, G-invariant subset of X, and G acts freely on

$$X_1 = X \setminus X_0 = \{x \in X : gx \neq x \text{ for all } g \in G \setminus \{1\}\}.$$

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The orbit space X/G is an open Riemann surface,

$\pi : X \to X/G$ is a holomorphic map which branches precisely on X_0,

$\pi : X_1 \to X_1/G$ is a holomorphic covering projection of degree $|G|$.

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Choose a holomorphic immersion $\tilde{h} : X/G \to \mathbb{C}$. The holomorphic map

$$h = \tilde{h} \circ \pi : X \to \mathbb{C}$$

is G-invariant, it branches precisely on X_0, and the holomorphic 1-form $\theta = dh$ on X satisfies the following condition for every $g \in G$:

$$\theta_{gx} \circ dg_x = \theta_x \quad \text{for all } x \in X, \quad \{\theta = 0\} = X_0. \tag{5}$$

Notation

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\begin{aligned}
A &= \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n_* : z_1^2 + z_2^2 + \cdots + z_n^2 = 0\} \\
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Y &= \overline{A} \setminus \{0\} = A \cup Y_\infty \\
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Thus, A is the affine cone determined by the projective manifold Y_∞.

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To any oriented 2-plane $0 \in \Lambda \subset \mathbb{R}^n$ we associate a complex null line $L \subset A \cup \{0\} \subset \mathbb{C}^n$ by choosing an oriented basis (u, v) of Λ such that $\|u\| = \|v\| \neq 0$ and $u \cdot v = 0$ (a conformal frame) and setting

$$L = L(\Lambda) = \mathbb{C}(u - \mathrm{i}v).$$

A rotation R_ϕ on Λ corresponds to the multiplication by $\mathrm{e}^{\mathrm{i}\phi}$ on $L(\Lambda)$.

Weierstrass representation of a G-equivariant minimal surface

Fix a point x_0 in the Riemann surface X.

Every G-equivariant conformal minimal immersion $F : X \to \mathbb{R}^n$ is of the form

$$F(x) = v + \Re \int_{x_0}^{x} f\theta \quad \text{for } x \in X \text{ and } v = F(x_0), \tag{6}$$

where

$$f = 2\partial F/\theta : X \to Y = A \cup Y_\infty$$

is a G-equivariant holomorphic map satisfying $f^{-1}(Y_\infty) = X_0$, $f\theta = 2\partial F$ has no zeros or poles, and it satisfies the period conditions

$$\Re \int_C f\theta = 0 \quad \text{for every } [C] \in H_1(X, \mathbb{Z}), \tag{7}$$

$$gv = v + \Re \int_{x_0}^{gx_0} f\theta \quad \text{for all } g \in G. \tag{8}$$

Proof of (a) $\implies$ (b), part 1

Step 1: We find a G-equivariant conformal minimal immersion $F_0 : V \to \mathbb{R}^n$ from a neighbourhood of the closed discrete subset $X_0 \subset X$.

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Fix $x_0 \in X_0$ and set $k = |G_{x_0}| > 1$. Let $G_{x_0} = \langle g_0 \rangle$. There is a holomorphic coordinate z on a disc $x_0 \in \Delta \subset X$, with $z(x_0) = 0$, such that

$$g_0 z = \mathrm{e}^{\mathrm{i}\phi} z, \qquad \phi = 2\pi/k.$$

We have $\theta(z) = a(z) z^{k-1} dz$ where $a(z)$ is a nonvanishing g_0-invariant holomorphic function on Δ.

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Fix $x_0 \in X_0$ and set $k = |G_{x_0}| > 1$. Let $G_{x_0} = \langle g_0 \rangle$. There is a holomorphic coordinate z on a disc $x_0 \in \Delta \subset X$, with $z(x_0) = 0$, such that

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We extend F_0 and f_0 by G-equivariance to the orbit $G \cdot \Delta$ and perform the same construction on all G-orbits of X_0. This defines a G-equivariant map $f_0 : V \to Y$ on a G-invariant neighbourhood $V \subset X$ of X_0, with

$$f_0^{-1}(Y_\infty) = X_0.$$

Proof of (a) $\implies$ (b), part 2

Step 2: We find a G-equivariant holomorphic map $f : X \to Y$ which agrees with f_0 on X_0, it satisfies $f(X_1) \subset A$, and the period conditions (7) and (8) hold. The map $F : X \to \mathbb{R}^n$ given by (6) then solves the problem.

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Consider the diagonal action of G on $X \times Y$ by

$$g(x, y) = (gx, gy), \quad x \in X, \ g \in G.$$

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$$\Omega := (X_1 \times A)/G \subset Z_1 \subset Z$$

is a G-invariant Zariski open domain without singularities.

Proof of (a) $\implies$ (b), part 3

1. The restricted projection

$$\rho : \Omega = (X_1 \times A)/G \to X_1/G \tag{9}$$

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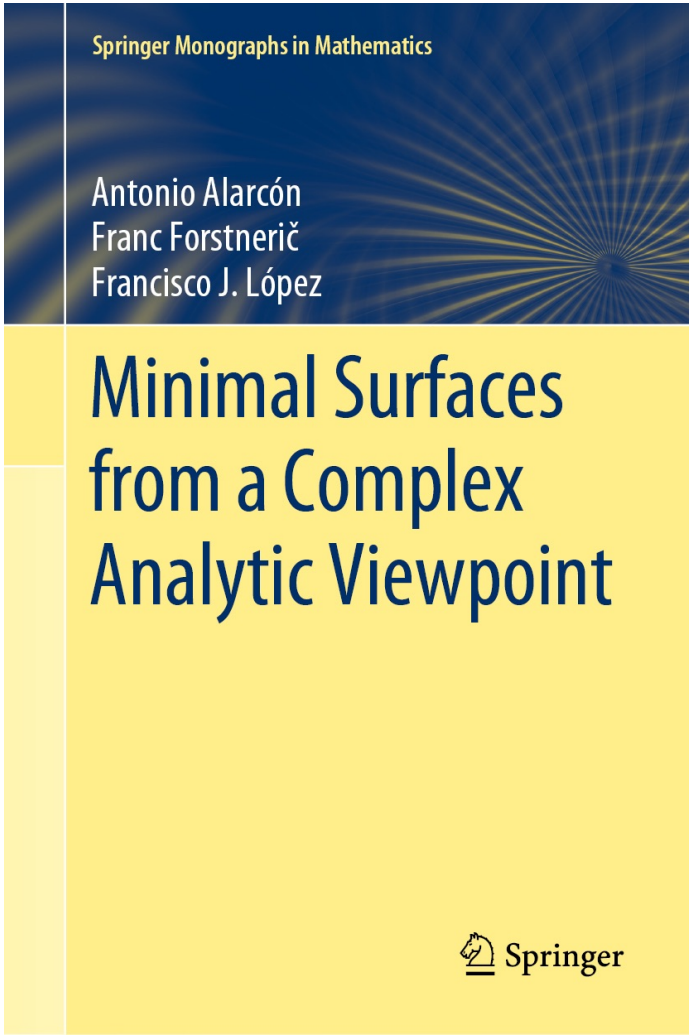
4. The fibre A of (9) is $O(n, \mathbb{C})$-homogeneous, hence an **Oka manifold**. Therefore, sections of $\rho : Z = (X \times Y)/G \to X/G$ mapping X_1/G to Ω satisfy the Oka principle (**F. 2003**). This gives a holomorphic section
$$\tilde{f} : X/G \to Z, \quad \tilde{f}(X_1/G) \subset \Omega, \quad f = \tilde{f}_0 \text{ on } X_0/G.$$

Completion of the proof



The convex hull of A equals $\mathbb{C}^n$.

Combining **Oka theory** with the **convex integration theory**, the holomorphic section $\tilde{f} : X/G \to Z$ can be chosen such that the corresponding G-equivariant map $f : X \to Y$ satisfies the period conditions (7), (8).

Hence, f integrates to a G-equivariant conformal minimal immersion $X \to \mathbb{R}^n$.

Thank your for your attention

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