Oka manifolds and their role in complex geometry

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Flexibility versus rigidity in complex geometry

A central question of complex geometry is to understand the space of holomorphic maps $X \to Y$ between a pair of complex manifolds. Are there many maps, or few maps? Which properties can such maps have?

There are many holomorphic maps $\mathbb{C} \to \mathbb{C}$ and $\mathbb{C} \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, but there are no nonconstant algebraic maps $\mathbb{C} \to \mathbb{C}^*$ or holomorphic maps $\mathbb{C} \to \mathbb{C} \setminus \{0, 1\}$. Manifolds with the latter property are called **hyperbolic**. Hyperbolicity is often an obstruction to solving complex analytic problems.

On the opposite side, **Oka theory** studies complex manifolds which admit many holomorphic maps from all **Stein manifolds**, i.e., closed complex submanifolds of affine spaces $\mathbb{C}^N$. It provides solutions to many complex-analytic problems in the absence of topological obstructions.

OKA THEORY = h-PRINCIPLE IN COMPLEX GEOMETRY

The h-principle

Gromov's general h-principle, 1970s The obstructions to solving certain partial differential relations are purely topological.

Basic h-principle: Every formal solution can be deformed to a genuine solution.

Parametric h-principle The inclusion

$$\{\text{genuine solutions}\} \hookrightarrow \{\text{formal solutions}\}$$

is a weak homotopy equivalence (often with additional features).

Examples of the h-principle

H. Whitney 1937 (attributed to **W. C. Graustein**): Smooth immersions of the circle into the plane are classified, up to isotopy, by the winding number.

$$\begin{aligned}\text{Formal solutions} &= \{\text{loops with a nowhere-vanishing vector field}\}\\ &\cong \{\text{loops in } \mathbb{R}^2 \setminus \{0\}\}\end{aligned}$$

This seems a first example of Gromov's h-principle for ample differential relations and the beginning of a major chapter in differential topology.

A few other examples:

- $\mathscr{C}^1$ isometric embeddings (Nash, Kuiper 1950s)
- Smale's sphere eversion (1950s)
- Classification of immersions by their tangent maps (Smale, Hirsch 1959)
- Classification of submersions by their tangent maps (Phillips 1966)
- Symplectic geometry, contact geometry

The first appearance of the h-principle in complex analysis

Kiyoshi Oka 1939; Hans Grauert 1958

For principal and their associate fibre bundles (e.g. vector bundles) over Stein spaces, the holomorphic classification agrees with the topological classification.

Equivalence between two such bundles is a section of an associated fibre bundle with homogeneous fibre.

Hence, the proof amounts to showing that every Stein space X admits many holomorphic maps $X \to Y$ to any complex homogeneous manifold Y, and many global holomorphic sections $X \to Z$ of any principal and related holomorphic fibre bundle $Z \to X$.

What is a good way to interpret 'many holomorphic maps'?

Look at properties of holomorphic functions $X \to \mathbb{C}$ on Stein spaces.

Oka–Weil and Oka–Cartan

Let X be a Stein space.

Oka–Weil 1936 If K is a compact holomorphically convex subset of X then every holomorphic function on (a neighbourhood of) K can be approximated uniformly on K by holomorphic functions on X.

Oka–Cartan 1951 If T is a closed complex subvariety of X then every holomorphic function on T extends to a holomorphic function on X.

These two results can be combined to approximation and (jet) interpolation. They extend to sections of coherent analytic sheaves on Stein spaces.

These are fundamental properties of Stein manifolds and Stein spaces.

A twist of philosophy: We can view them as properties of the target manifold, the complex number field $\mathbb{C}$. We now formulate them as properties of an arbitrary target manifold Y, taking into account topological obstructions.

Oka properties of a complex manifold Y

BOPA — the basic Oka property with approximation: For every compact $\mathscr{O}(X)$-convex subset K of a Stein space X and every continuous map $f_0 : X \to Y$ which is holomorphic on K there is a homotopy $f_t : X \to Y$ $(t \in [0,1])$ of maps of the same type such that $f_t|_K \approx f_0|_K$ for all t and f_1 is holomorphic on X.

BOPI — the basic Oka property with interpolation: For every closed complex subvariety T of a Stein space X and continuous map $f_0 : X \to Y$ such that $f_0|_T$ is holomorphic there is a homotopy $f_t : X \to Y$ $(t \in [0,1])$ of maps of the same type such that $f_t|_T = f_0|_T$ for all t and f_1 is holomorphic on X.

BOPAI = BOPA + BOPI.

POPA, POPI, POPAI — the parametric Oka properties: The analogous properties for families of maps $f_p : X \to Y$ depending continuously on a parameter $p \in P$ in a compact Hausdorff space, with f_p holomorphic for p in a compact subset $Q \subset P$. We ask for the existence of a homotopy $f^t : X \to Y$, $t \in [0,1]$, fixed for $p \in Q$, to a family of holomorphic maps $f^1_p : X \to Y$, $p \in P$.

Enter Oka manifolds — 2009

A complex manifold satisfying all these properties is called an

***** OKA MANIFOLD *****

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Complex Analysis

Oka manifolds

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Presented by Mikhaël Gromov

To Mikhaël Gromov on the occasion of his receiving the Abel Prize

MSC 2020: New subfield **32Q56 Oka principle and Oka manifolds**

Oka manifolds

Oka 1939, Grauert 1958 Every complex homogeneous manifold is Oka.

Gromov 1989 Every elliptic complex manifold is an Oka manifold.

F. 2005 A complex manifold Y enjoys the **convex approximation property (CAP)** if every holomorphic map $K \to Y$ from a compact convex set $K \subset \mathbb{C}^n$ is a limit of entire maps $\mathbb{C}^n \to Y$.

F. 2005–2009, 2017

- A complex manifold is an Oka manifold if and only if it enjoys CAP.
- The Oka properties described above are pairwise equivalent.
- If $Y \to Z$ is a holomorphic fibre bundle with Oka fibre, then Y is Oka iff Z is Oka. (This generalizes to **Oka maps**.)
- Every Oka manifold Y is the image of a strongly dominating holomorphic map $\mathbb{C}^{\dim Y} \to Y$.

Further properties of Oka manifolds

- **The weak homotopy equivalence principle:** For every Stein space X and Oka manifold Y, the inclusion

$$\mathscr{O}(X, Y) \hookrightarrow \mathscr{C}(X, Y)$$

is a weak homotopy equivalence.

- **Lárusson 2015** If X is a Stein manifold which admits a strongly plurisubharmonic exhaustion function with only finitely many critical points then the above inclusion is a genuine homotopy equivalence.

- A Riemann surface is Oka if and only if it is not hyperbolic.

- Every Oka manifold is Liouville, i.e., every bounded plurisubharmonic function on the manifold is constant.

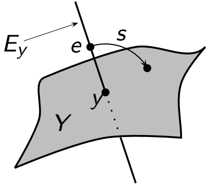
- **Kobayashi and Ochiai 1977** A compact complex manifold X of general type (i.e., of maximal Kodaira dimension $\kappa(X) = \dim X$) is not dominable by affine space. Hence, such a manifold is not Oka.

Gromov's ellipticity

Gromov 1989 A complex manifold Y is called **elliptic** if it admits a **dominating holomorphic spray**:

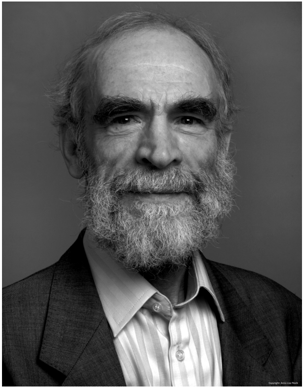
A holomorphic map $s \colon E \to Y$, defined on the total space of a holomorphic vector bundle E over Y, such that for all $y \in Y$ we have

$$s(0_y) = y \ \text{ and } \ s \colon E_y \to Y \text{ is a submersion at } 0_y \in E_y.$$



Sprays are used to linearize the approximation and gluing problems for manifold-valued holomorphic maps from Stein manifolds.

Gromov's Oka principle



Gromov 1989: Every elliptic manifold is an Oka manifold.

The Oka principle holds for sections of elliptic submersions $Z \to X$ over a Stein base X.

Detailed proofs were given by **Prezelj & F., 2000–2002.**

A homotopy approximation theorem

Given the equivalence CAP $\iff$ Oka (F. 2009), the main point is that

$$\text{ellipticity} \implies \text{CAP}.$$

This is a special case of Gromov's

Homotopy approximation theorem: Let $f\colon X \to Y$ be a holomorphic map to a complex manifold Y. Assume that $K \subset X$ is a compact $\mathscr{O}(X)$-convex set and $f_t : U \to Y$ $(t \in [0,1])$ is a homotopy of holomorphic maps on an open set $K \subset U \subset X$ such that $f_0 = f|_U$.

If Y is elliptic, then we can approximate f_t uniformly on K by a homotopy $\tilde{f}_t : K \to Y$ of holomorphic maps, with $\tilde{f}_0 = f$.

By using a dominating spray $s\colon E \to Y$, the proof reduces to the Oka–Weil approximation theorem for sections of vector bundles over X.

Subelliptic manifolds

A complex manifold Y is **subelliptic** if there exist finitely many sprays $s_j : E_j \to Y$ which together dominate:

$$\sum_j ds_j(E_{j,y}) = T_y Y \quad \text{for all } y \in Y.$$

F. 2002 Every subelliptic manifold is Oka. Sections of a subelliptic holomorphic submersion over a Stein base satisfy the Oka principle.

Gromov 1989 An algebraic manifold Y is **algebraically (sub) elliptic** if it admits an algebraic dominating spray (or a family of sprays).
Algebraic subellipticity is a Zariski open property.

Kaliman and Zaidenberg 2024 Every algebraically subelliptic manifold is algebraically elliptic. (This is not known in the holomorphic category.)

Examples of (sub) elliptic manifolds

- **Every $\mathbb{C}$-homogeneous Y is elliptic:** $Y \times \mathfrak{g} \xrightarrow{s} Y$, $(y, v) \mapsto e^v \cdot y$.
- **A flexible manifold Y is elliptic.** Such Y admits $\mathbb{C}$-complete holomorphic vector fields $V_1, \ldots, V_k$ spanning the tangent space $T_y Y$ at every point. Let ϕ^j_t denote the flow of V_j for time $t \in \mathbb{C}$. The map $s : Y \times \mathbb{C}^k \to Y$,

$$s(y, t_1, \ldots, t_k) = \phi^1_{t_1} \circ \cdots \circ \phi^k_{t_k}(y)$$

  is then a dominating spray on Y.
- **An algebraically flexible manifold is algebraically elliptic.** The tangent bundle of such a manifold is pointwise spanned by LND's.
- A spray of this type exists on $\mathbb{C}^n \setminus A$, where A is algebraic subvariety with $\dim A \leq n-2$. We can use complete vector fields $f(\pi(z))v$ where $v \in \mathbb{C}^n$, $\pi : \mathbb{C}^n \to \mathbb{C}^{n-1}$ is a linear projection with $\pi(v) = 0$, and f is a polynomial on $\mathbb{C}^{n-1}$ that vanishes on the subvariety $\pi(A) \subset \mathbb{C}^{n-1}$.
- **Arzhantsev, Kaliman and Zaidenberg 2024** Every uniformly rational projective manifold $Y \subset \mathbb{P}^n$ is algebraically elliptic.
  The affine cone $C(Y) \subset \mathbb{C}^{n+1} \setminus \{0\}$ is also algebraically elliptic.

Notable applications of modern Oka theory

- **Eliashberg & Gromov 1992; Schürmann 1996**
  Every Stein manifold of dimension $n > 1$ admits a proper holomorphic embedding in $\mathbb{C}^{[3n/2]+1}$ and immersion in $\mathbb{C}^{[(3n+1)/2]}$. These dimensions are optimal, and they solve a conjecture of **Forster 1970**.

- **Eliashberg & Gromov 1985** Holomorphic immersions $X^n \to \mathbb{C}^N$ for $N > n \geq 1$ satisfy the basic h-principle with respect to their tangent maps.

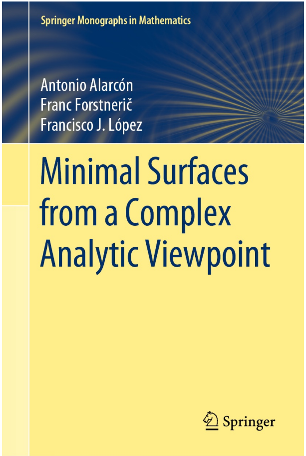
- **F. 2003** Holomorphic subersions $X^n \to \mathbb{C}^m$ for $n > m \geq 1$ satisfy the basic h-principle with respect to their tangent maps. In particular, every Stein manifold admits many holomorphic functions without critical points.

- **Wold & F. 2009, 2013** Proper holomorphic embeddings of bordered Riemann surfaces in $\mathbb{C}^2$.

- **Ivarsson & Kutzschebauch 2012** Null homotopic holomorphic maps $X \to SL_m(\mathbb{C})$ satisfy the Vaserstein factorisation theorem into upper and lower triangular matrix-valued maps.

- **Alarcón & F. 2014, Lárusson & F. 2019** Immersions $X \to \mathbb{C}^n$ from open Riemann surfaces which are directed by a nondegenerate Oka cone $A \subset \mathbb{C}^n \setminus \{0\}$ satisfy the parametric h-principle.

Applications to minimal surfaces (2021)



A minimal surface in $\mathbb{R}^n$, $n \geq 3$, is given by a conformal harmonic immersion $F : X \to \mathbb{R}^n$ from an open Riemann surface.

The $(1,0)$-differential $\Phi = \partial F$ is a holomorphic 1-form with exact real part $2\Re\Phi = dF$ and values in the cone

$$A = \{z_1^2 + z_2^2 + \cdots + z_n^2 = 0\} \setminus \{0\},$$

and vice versa.

The cone A is algebraically flexible.

Applications of Oka theory yield a variety of new results on minimal surfaces in Euclidean spaces.

Oka principle in the algebraic category

Loday 1972 The Oka principle fails in general for algebraic (regular) maps from an affine variety X to an algebraically elliptic manifold Y. For example, let

$$\Sigma^n = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} : z_0^2 + z_1^2 + \cdots + z_n^2 = 1\}.$$

Every algebraic map $\Sigma^p \times \Sigma^q \to \Sigma^{p+q}$ is null-homotopic when p and q are odd, but there always exists a nontrivial continuous (and holomorphic) map.

Lárusson & Truong 2019 Algebraic analogues of properties CAP, BOPI, BOPA fail for morphisms from affine varieties to any compact algebraic manifold, and to any algebraic manifold with a nontrivial rational curve.

The following is the closest algebraic analogue of the basic Oka principle.

F. 2006 Let X be an affine variety and Y be an algebraically elliptic manifold. Every holomorphic map $X \to Y$ which is homotopic to an algebraic map can be approximated, uniformly on compacts in X, by algebraic maps.

In particular, a compact algebraically elliptic manifold Y admits a surjective strongly dominating morphism $\mathbb{C}^n \to Y$ with $n = \dim Y$.

Recent developments and open problems

Kusakabe 2021 Assume that a complex manifold Y satisfies the following condition $\mathrm{C\text{-}Ell}_1$ (a special case of condition Ell_1 considered by Gromov):

For every holomorphic map $f\colon U \to Y$ from a bounded open convex set $U \subset \mathbb{C}^n$ there is a holomorphic $F\colon U \times \mathbb{C}^N \to Y$ such that $F(\cdot\,, 0) = f$ and

$$\frac{\partial}{\partial \zeta}\Big|_{\zeta=0} F(z,\zeta) : \mathbb{C}^N \to T_{f(z)}Y \ \text{ is surjective for every } z \in U.$$

Then, Y is an Oka manifold. Hence, the following conditions are equivalent:

$$\mathrm{C\text{-}Ell}_1 \iff \mathrm{CAP} \iff \mathrm{OKA}.$$

Corollary (Kusakabe 2021)

If a complex manifold Y is a union of Zariski open Oka domains, then Y is Oka.

Complements of polynomially convex sets are Oka

Kusakabe 2024 If K is a compact polynomially convex set in $\mathbb{C}^n$ $(n > 1)$ then $\mathbb{C}^n \setminus K$ is Oka. The same holds if K is an closed unbounded polynomially convex set in $\mathbb{C}^n$ with a proper projection to $\mathbb{C}^{n-2}$.

The analogous result holds in any Stein manifold X with Varolin's density property: every holomorphic vector field on X can be approximated on compacts by sums and commutators of complete holomorphic vector fields.

Andrist, Shcherbina & Wold 2016 If K is a compact set in $\mathbb{C}^n$ $(n \geq 3)$ with infinitely many limit points then $\mathbb{C}^n \setminus K$ is not (sub) elliptic.

Hence, Kusakabe's theorem gives a huge number of nonelliptic Oka manifolds, thereby solving a longstanding open problem.

Problem: is there a non-elliptic compact (or even projective) Oka manifold?

Complements of polynomially convex sets are Oka

To see this, we show that $\mathbb{C}^n \setminus K$ enjoys condition $\mathrm{C\text{-}Ell}_1$.

Let $L \subset \mathbb{C}^N$ be a compact convex set and $f\colon U \to \mathbb{C}^n \setminus K$ be a holomorphic map from a convex open neighbourhood $U \subset \mathbb{C}^N$ of L. Let $\Gamma = \{(z, f(z)) : z \in L\}$. The set

$$(L \times K) \cup \Gamma$$

is then polynomialy convex in $\mathbb{C}^N \times \mathbb{C}^n$. Let

$$G(z, \zeta) = (z, \psi(z, \zeta))$$

be the identity map on a neighborhood of $U \times K$ and the contraction

$$\psi(z, \zeta) = \frac{1}{2}\zeta + \frac{1}{2}f(z)$$

to the point $f(z) \in \mathbb{C}^n$ for each (z, ζ) in a neighbourhood of Γ.

Complements of polynomially convex sets are Oka, 2

Rosay & F. 1993 We can approximate the biholomorphic map G uniformly on $(L \times K) \cup \Gamma$ by a holomorphic automorphism $\Phi \in \mathrm{Aut}(U \times \mathbb{C}^n)$ of the form

$$\Phi(z, \zeta) = \big(z, \phi(z, \zeta)\big), \quad z \in U,\ \zeta \in \mathbb{C}^n.$$

Iteration of this procedure leads to a holomorphic maps $F\colon U \times \mathbb{C}^n \to \mathbb{C}^n$ such that for all $z \in U$ we have $F(z, 0) = f(z)$ and

$$F(z, \cdot\,) : \mathbb{C}^n \to \mathbb{C}^n \setminus K \text{ is a Fatou–Bieberbach map.}$$

Thus, $\mathbb{C}^n \setminus K$ satisfies condition $\mathrm{C\text{-}Ell}_1$, so it is Oka.

Furthermore, $\mathbb{CP}^n \setminus K$ is Oka for every such K by the localization theorem for Oka manifolds.

Further applications

F. & Wold, 2023 Complements of most closed convex sets $E \subset \mathbb{C}^n$ for $n > 1$ are Oka. This holds in particular if E does not contain any affine real line.

Drinovec Drnovšek & F. 2023 For most closed convex sets $E \subset \mathbb{C}^n$, any Stein manifold X with $2 \dim X < n$ admits a proper holomorphic embedding $f \colon X \hookrightarrow \mathbb{C}^n$ with $f(X) \cap E = \varnothing$.

Kusakabe & F. 2024 Let $L \to X$ be a holomorphic line bundle on a compact complex manifold X. Assume that for each $x \in X$ there exists a divisor $D \in |L|$ whose complement $X \setminus D$ is a Stein neighbourhood of x with the density property. Then, for any semipositive hermitian metric h on L the disc bundle $\Delta_h(L) = \{h < 1\}$ is an Oka manifold.

Examples: ample line bundles on projective spaces, Grassmannians, flag manifolds,.... This can be contrasted with the following classical result:

Grauert 1961 If (L, h) is a negative holomorphic hermitian line bundle on a compact complex manifold X then $\{0 < h < 1\}$ is Kobayashi hyperbolic.

Oka properties and metric positivity

A challenging open problem is to understand the relationship between Oka properties and metric positivity of compact hermitian or Kähler manifolds.

The specialness of varieties of low Kodaira dimension is analogous to the specialness of Riemannian manifolds of positive curvature, and general type corresponds to the genericity of non-positive curvature.

We mention a few known results in this direction, beginning with the following.

Grauert & Reckziegel 1965 A compact hermitian manifold with negative holomorphic sectional curvature is Kobayashi hyperbolic.

This is a generalization of the Ahlfors–Schwarz lemma. There are many further results on this subject (**Wu 1967, Kobayashi 1970, Greene and Wu 1979**).

Oka properties and metric positivity

Mori 1979, Siu and Yau 1980, Mok 1988 The universal cover of a compact Kähler manifold with nonnegative holomorphic bisectional curvature is biholomorphic to

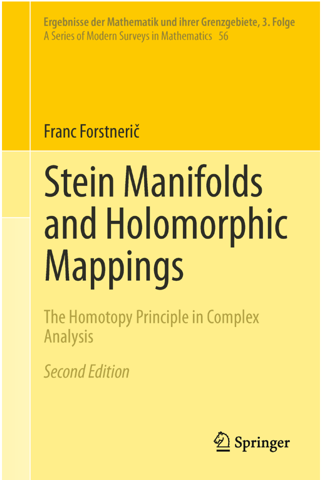
$$\mathbb{C}^k \times \mathbb{CP}^{n_1} \times \cdots \times \mathbb{CP}^{n_l} \times M_1 \times \cdots \times M_p$$

where each M_j is a compact hermitian symmetric space with its canonical complex structure and Kähler metric. **Every such manifold Y is Oka.**

Campana & Peternell 1991 A compact projective manifold with dim ≤ 3 with nef tangent bundle is an Oka manifold.

Lárusson and F. 2024 A projective manifold that is birationally equivalent to an algebraically elliptic projective manifold is an aOka-1 manifold, i.e., it has the Oka properties for regular maps $X \to Y$ from any affine algebraic curve X. This holds in particular for all rational manifolds.

Conjecture Every compact rationally connected manifold is an Oka-1 and aOka-1 manifold. (If the holomorphic sectional curvature of a compact Kähler manifold is positive then, by Yau's conjecture solved by **X. Yang 2018**, the manifold is rationally connected.)



The main results on this subject up to 2017, discussed in this talk, are presented in my Ergebnisse monograph.

Developments after 2017 are summarised in my survey

Recent developments on Oka manifolds.
Indag. Math., 34(2) (2023) 367–417.

Thank you for your attention.