MINIMAL SURFACES WITH SYMMETRIES

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Abstract

Objects with symmetries are of special interest in any mathematical theory.

In this talk, I shall discuss the existence of orientable minimal surfaces in Euclidean spaces $\mathbb{R}^n$, $n \geq 3$, with a given group of symmetries.

We show in particular that every finite group is a group of symmetries of a minimal surface.

F. Forstnerič: Minimal surfaces with symmetries. Preprint, August 2023.
`https://arxiv.org/abs/2308.12637`

An elementary introduction to minimal surfaces:

F. Forstnerič: Minimal surfaces in Euclidean spaces by way of complex analysis. European Congress of Mathematics, 9–43. EMS Press, Berlin, ©2023.

What is a minimal surface?

Euler 1744; Lagrange 1762 A smooth immersed surface $F : X \to \mathbb{R}^n$ $(n \geq 3)$ is a **minimal surface** if it is a **stationary point of the area functional**. Any small enough piece of such a surface has the smallest area among all surfaces with the same boundary.

Meusnier, 1776 A surface in $\mathbb{R}^n$ is a minimal surface if and only if its **mean curvature vector vanishes at every point**.

Let X be a smooth surface. An immersion $F : X \to \mathbb{R}^n$ determines on X a Riemannian metric $g = F^* ds^2$, which makes F an isometry, and hence a **conformal map**. By **Gauss**, there are local **isothermal coordinates** (x, y) at any point of X in which

$$g = \lambda(dx^2 + dy^2) \text{ for some function } \lambda > 0.$$

Transition maps between isothermal charts are conformal diffeomorphisms of plane domains, hence holomorphic or antiholomorphic. This endows X with the structure of a **conformal surface**, and of a **Riemann surface** if X is oriented.

If $F : X \to \mathbb{R}^n$ is a **conformal immersion**, then

$$F \text{ parameterizes a minimal surface} \iff F \text{ is a harmonic map}$$
$$\iff F \text{ is a stationary point of the energy functional.}$$

In any isothermal coordinate $z = x + \imath y$ on X, this is the **Laplace equation**

$$\Delta F = F_{xx} + F_{yy} = 4\frac{\partial^2 F}{\partial \bar{z}\, \partial z} = 0. \tag{1}$$

Write $\partial F = \frac{\partial F}{\partial z} dz = \frac{1}{2}\left(\frac{\partial F}{\partial x} - \imath \frac{\partial F}{\partial y}\right)(dx + \imath dy)$. Then, $\Re(2\partial F) = dF$ and

$$\Delta F = 0 \iff \partial F = (\partial F_1, \ldots, \partial F_n) \text{ is a holomorphic 1-form on } X.$$

It is elementary see that an immersion $F = (F_1, \ldots, F_n)$ is conformal iff

$$\sum_{i=1}^{n} (\partial F_i)^2 = 0. \tag{2}$$

Minimal surfaces are solutions of the nonlinear elliptic PDE (1), (2).

Let $A \subset \mathbb{C}^n$ denote the **null quadric**

$$A = \{z = (z_1, \ldots, z_n) : z_1^2 + z_2^2 + \cdots + z_n^2 = 0\},$$

and let $\overline{A} \subset \mathbb{CP}^n$ denote its projective closure. Pick a nontrivial holomorphic 1-form θ on X (possibly with zeros). If $F : X \to \mathbb{R}^n$ is a minimal surface then

$$2\partial F = f\theta,$$

where $f = (f_1, \ldots, f_n) : X \to \overline{A} \setminus \{0\}$ is a holomorphic map such that

$$\Re \oint_C f\theta = \oint_C dF = 0 \text{ for every closed curve } C \subset X. \tag{3}$$

Conversely, given f as above such that $f\theta$ is a nowhere vanishing holomorphic 1-form on X satisfying (3), the map $F : X \to \mathbb{R}^n$ given by

$$F(x) = \Re \int_*^x f\theta$$

is a conformal harmonic immersion.

A smooth map $T : \mathbb{R}^n \to \mathbb{R}^n$ maps minimal surfaces to minimal surfaces iff T is a **rigid map** — a composition of orthogonal maps, dilations, and translations.

Let G be a group acting on $\mathbb{R}^n$ by rigid transformations. A surface $S \subset \mathbb{R}^n$ is **G-invariant** if

$$g(S) = S \quad \text{for every } g \in G.$$

If $F : X \to S = F(X) \subset \mathbb{R}^n$ is an injective conformal immersion, then G also acts on X by conformal diffeomorphisms such that F is **G-equivariant**:

$$F \circ g = g \circ F \quad \text{for every } g \in G.$$

If X is a Riemann surface and every $g \in G$ preserves the orientation on $S = F(X)$, then G acts on X by holomorphic automorphisms.

Conversely, the image of a G-equivariant immersion is a G-invariant surface.

$* * *$ **Which groups arise in this way for minimal surfaces?** $* * *$

Euler 1744 The only minimal surfaces of rotation in $\mathbb{R}^3$ are planes and catenoids.

$$x^2 + y^2 = \cosh^2 z$$

$$(t, z) \mapsto (\cos t \cdot \cosh z, \sin t \cdot \cosh z, z)$$

The symmetry group consist of rotations in the (x, y)-plane and the reflection $z \mapsto -z$.

The helicoid (Archimedes' screw)

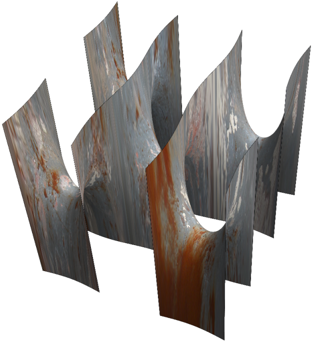
Meusnier 1776 The helicoid is a ruled minimal surface.
It is obtained by rotating a line and displacing it along the axis of rotation.

$$\begin{aligned} x &= \rho\cos(\alpha z), \\ y &= \rho\sin(\alpha z), \quad (z,\rho)\in\mathbb{R}^2. \end{aligned}$$

The group $\mathbb{Z}$ acts on the helicoid by translations $z \mapsto z + k2\pi/\alpha$, $k \in \mathbb{Z}$.

Also, $\mathbb{R}$ acts by translations and simultaneous rotations.

Scherk, 1835 The first Scherk's surface is doubly periodic, with the symmetry group $\mathbb{Z}^2$ of translations.

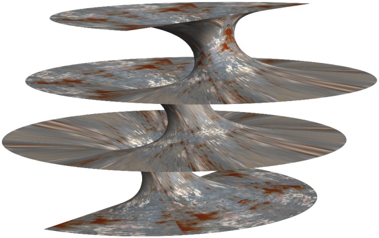


Its main branch is a graph over the square $P = (-\pi/2, \pi/2)^2$ given by

$$x_3 = \log \frac{\cos x_2}{\cos x_1}$$

Finn and Osserman, 1964
Sherk's surface S has the biggest absolute Gaussian curvature at $0 \in \mathbb{R}^3$ over all minimal graphs over P tangent to S at 0.

Bernhard Riemann 1867: A family R_λ, $\lambda > 0$, of periodic planar domains, properly embedded as minimal surfaces in $\mathbb{R}^3$ such that every horizontal plane intersects each R_λ in a circle or a line. As $\lambda \to 0$ his surfaces converge to a vertical catenoid, and as $\lambda \to \infty$ they converge to a vertical helicoid.



The symmetry group is $\mathbb{Z}$ acting by translations.

Let X be a connected open Riemann surface and $G \subset \mathrm{Aut}(X)$ be a **finite group of holomorphic automorphisms**. The **stabiliser** of $x \in X$ is

$$G_x = \{g \in G : gx = x\}.$$

This is a cyclic group of rotations in a local holomorphic coordinate at x.

Assume that G also acts on $\mathbb{R}^n$ by orthogonal transformations in $O(n, \mathbb{R})$.

Theorem

The following are equivalent:

- (a) *For every nontrivial stabiliser G_x $(x \in X)$ there is a G_x-invariant 2-plane $\Lambda_x \subset \mathbb{R}^n$ on which G_x acts effectively by rotations.*
- (b) *There exists a G-equivariant conformal minimal immersion $F : X \to \mathbb{R}^n$:*

$$F(gx) = gF(x), \quad x \in X, \ g \in G.$$

In particular, such F exists if the group G acts freely on X.

Two generalizations

The same holds if G is an infinite group which acts on a connected open Riemann surface X **properly discontinuously** by holomorphic automorphisms such that X/G is open, and G acts on $\mathbb{R}^n$ by **rigid transformations**, i.e., compositions of orthogonal maps, translations, and dilations.

The result also holds if the differential ∂F has poles. The poles of ∂F are **proper ends** of the minimal surface $F : X \to \mathbb{R}^n$ with **finite total curvature**:

$$\int K \cdot d\mathrm{Area} > -\infty,$$

where $K : X \to (-\infty, 0]$ is the Gaussian curvature function.

Every finite group is a symmetry group of a minimal surface

Greenberg 1960, 1974 Every countable group G is the automorphism group of a Riemann surface X. If G is finite then X can be taken compact.

It is elementary to see that every finite group G of order $n = |G|$ acts on $\mathbb{R}^{2n}$ by orthogonal maps such that for every $g \in G$ there is a 2-plane $\Lambda \subset \mathbb{R}^{2n}$ on which g acts by a rotation for the angle $2\pi/k$, where k is the order of g.

Corollary

For every finite group G of order $n > 1$ there exist an open connected Riemann surface X, effective actions of G by holomorphic automorphisms on X and by orthogonal transformations on $\mathbb{R}^{2n}$, and a G-equivariant conformal minimal immersion $X \to \mathbb{R}^{2n}$.

Hurwitz 1893, Maskit 1968 If X is a Riemann surface of genus $\mathfrak{g} \geq 2$ then

$$|\mathrm{Aut}(X)| \leq 84(\mathfrak{g} - 1).$$

Most such surfaces have no nontrivial automorphisms.

Proof of (b) $\implies$ (a)

Let $x \in X$ be a point with a nontrivial stabiliser G_x of order $k = |G_x| > 1$.

There is a local holomorphic coordinate z on X around x, with $z(x) = 0$, in which a generator of $G_x = \langle g \rangle$ is the rotation

$$gz = \mathrm{e}^{\mathrm{i}\phi} z, \quad \phi = 2\pi/k. \tag{4}$$

Assume that G acts on $\mathbb{R}^n$ by orthogonal maps and $F : X \to \mathbb{R}^n$ is a G-equivariant immersion. Differentiating $g \circ F = F \circ g$ gives

$$g \circ dF_x = dF_x \circ dg_x : T_x X \to \Lambda_x := dF_x(T_x X) \subset \mathbb{R}^n.$$

Since $dF_x : T_x X \to \Lambda_x$ is a linear isomorphism, we see that $\Lambda_x \subset \mathbb{R}^n$ is a G_x-invariant plane on which g acts as the rotation R_ϕ, so condition (a) holds.

Taking the $\mathbb{C}$-linear part of the above equation gives

$$g \circ \partial F_x = \partial F_x \circ dg_x. \tag{5}$$

Conversely, if condition (a) holds at $x \in X$ for the 2-plane $\Lambda_x \subset \mathbb{R}^n$ then the conformal linear map from the z-neighbourhood of $x \in X$ as in (4) to $\Lambda_x \subset \mathbb{R}^n$ is a G_x-equivariant conformal minimal immersion.

Corollary

Assume that G is a finite subgroup of the orthogonal group $O(n,\mathbb{R})$, $n \geq 3$.

Let $X \subset \mathbb{R}^n$ be a smoothly embedded, oriented, noncompact G-invariant surface such that every $g \in G$ preserves the orientation on X.

Then, X endowed with the complex structure induced by the embedding $X \hookrightarrow \mathbb{R}^n$ admits a G-equivariant conformal minimal immersion $F : X \to \mathbb{R}^n$.

Proof.

Condition (a) in the Theorem holds by the argument on the previous page. $\square$

The setup used in the proof of (a) $\Longrightarrow$ (b)

Let G be a finite group acting on an open Riemann surface X by holomorphic automorphisms. The set

$$X_0 = \{x \in X : G_x \neq \{1\}\}$$

is a closed, discrete, G-invariant subset of X, and G acts freely on

$$X_1 = X \setminus X_0 = \{x \in X : gx \neq x \text{ for all } g \in G \setminus \{1\}\}.$$

The orbit space X/G is an open Riemann surface,

$\pi : X \to X/G$ is a holomorphic map which branches precisely on X_0

$\pi : X_1 \to X_1/G$ is a holomorphic covering projection of degree $|G|$.

Choose a holomorphic immersion $\tilde{h} : X/G \to \mathbb{C}$. The holomorphic map

$$h = \tilde{h} \circ \pi : X \to \mathbb{C}$$

is G-invariant ($h \circ g = h$), and the holomorphic 1-form $\theta = dh$ satisfies

$$\theta_{gx} \circ dg_x = \theta_x \ \text{ for all } x \in X, \quad \{\theta = 0\} = X_0. \tag{6}$$

Notation

$$
\begin{aligned}
A &= \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : z_1^2 + z_2^2 + \cdots + z_n^2 = 0\} \text{ the null quadric} \\
A_* &= A \setminus \{0\} \text{ the punctured null quadric} \\
\overline{A} &= \text{the closure of } A \text{ in } \mathbb{CP}^n = \mathbb{C}^n \cup \mathbb{CP}^{n-1} \\
Y &= \overline{A} \setminus \{0\} = A_* \cup Y_0 \\
Y_0 &= Y \setminus A_* = \{[z_1 : \cdots : z_n] \in \mathbb{CP}^{n-1} : z_1^2 + z_2^2 + \cdots + z_n^2 = 0\}.
\end{aligned}
$$

The actions of $O(n,\mathbb{R}) \subset O(n,\mathbb{C})$ on $\mathbb{C}^n$ extends to $\mathbb{CP}^n$, with $A_* \subset Y$ and the hyperplane at infinity $\mathbb{CP}^n \setminus \mathbb{C}^n \cong \mathbb{CP}^{n-1}$ being invariant submanifolds.

To any oriented 2-plane $0 \in \Lambda \subset \mathbb{R}^n$ we associate a complex line $L \subset A \subset \mathbb{C}^n$ by choosing an oriented basis (u, v) of Λ such that $\|u\| = \|v\| \neq 0$ and $u \cdot v = 0$ (a **conformal frame**) and setting

$$L = L(\Lambda) = \mathbb{C}(u - \mathrm{i}v) \subset A \subset \mathbb{C}^n.$$

A rotation R_ϕ on Λ corresponds to the multiplication by $\mathrm{e}^{\mathrm{i}\phi}$ on $L(\Lambda)$.

Assume that X and G are as in the Theorem.

Every G-equivariant conformal minimal immersion $F : X \to \mathbb{R}^n$ is of the form

$$F(x) = v + \Re \int_{x_0}^{x} f\theta \quad \text{for } x_0, x \in X \text{ and } v = F(x_0), \tag{7}$$

where

$$f = 2\partial F/\theta : X \to Y = A_* \cup Y_0$$

is a G-equivariant holomorphic map satisfying $f^{-1}(Y_0) = X_0$ such that $f\theta$ has no zeros or poles, and it satisfies the period conditions

$$\begin{aligned} \Re \int_C f\theta &= 0 \quad \text{for every } [C] \in H_1(X, \mathbb{Z}), & (8) \\ gv &= v + \Re \int_{x_0}^{gx_0} f\theta \quad \text{for all } g \in G. & (9) \end{aligned}$$

Proof

Suppose that $F : X \to \mathbb{R}^n$ (7) is a G-equivariant conformal minimal immersion. Then, $2\partial F = f\theta$ is holomorphic, conditions (8) hold, and $f : X \to Y$ is G-equivariant:

$$f(gx) = \frac{2\partial F_{gx}}{\theta_{gx}} = \frac{2\partial F_{gx} \circ dg_x}{\theta_{gx} \circ dg_x} = \frac{g\, 2\partial F_x}{\theta_x} = g\, f(x). \tag{10}$$

The G-equivariance condition on F at x_0 gives (9) with $v = F(x_0)$:

$$gv = gF(x_0) = F(gx_0) = v + \Re \int_{x_0}^{gx_0} f\theta \quad \text{for all } g \in G.$$

Conversely, assume that $f : X \to Y$ is a G-equivariant holomorphic map satisfying the stated conditions. Then, the map F given by (7) is a conformal minimal immersion. Given a path $\gamma : [0,1] \to X$, we have for any $g \in G$:

$$\int_{g\gamma} f\theta = \int_0^1 f(g\gamma(t))\, \theta_{g\gamma(t)}(dg_{\gamma(t)}\dot\gamma(t))\, dt \overset{(6)}{=} \int_0^1 gf(\gamma(t))\, \theta_{\gamma(t)}(\dot\gamma(t))\, dt = g\int_\gamma f\theta,$$

$$\begin{aligned} F(gx) &= v + \Re \int_{x_0}^{gx} f\theta = \left(v + \Re \int_{x_0}^{gx_0} f\theta\right) + \Re \int_{gx_0}^{gx} f\theta \\ &\overset{(9)}{=} gv + \Re g \int_{x_0}^{x} f\theta = gF(x). \end{aligned}$$

Proof of the Theorem, 1

Step 1: We find a G-equivariant conformal minimal immersion $F_0 : V \to \mathbb{R}^n$ from a neighbourhood of the closed discrete subset $X_0 \subset X$.

Fix $x_0 \in X_0$ and set $k = |G_{x_0}| > 1$. Let $G_{x_0} = \langle g_0 \rangle$. There is a holomorphic coordinate z on a disc $x_0 \in \Delta \subset X$, with $z(x_0) = 0$, such that

$$g_0 z = \mathrm{e}^{\mathrm{i}\phi} z, \qquad \phi = 2\pi/k.$$

Let $\Lambda \subset \mathbb{R}^n$ be a G_{x_0}-invariant plane on which g_0 acts as the rotation R_ϕ. Then, g_0 acts on the null line $L = L(\Lambda)$ as multiplication by $\mathrm{e}^{\mathrm{i}\phi}$.

The conformal linear map $F_0 : \Delta \to \Lambda$ is G_{x_0}-equivariant, and $2\partial F_0 = f_0 \theta$ where

$$f_0(z) = \frac{y_0}{z^{k-1}} \quad \text{for some } y_0 \in L \text{ and all } z \in \Delta.$$

We extend F_0 and f_0 by G-equivariance to the orbit $G \cdot \Delta$ and perform the same construction on all G-orbits of X_0.

This defines a G-equivariant map $f_0 : V \to Y$ on a G-invariant neighbourhood $V \subset X$ of X_0, with $f_0^{-1}(Y_0) = X_0$.

Step 2: We find a G-equivariant holomorphic map $f : X \to Y$ which agrees with f_0 on X_0, it satisfies $f(X_1) \subset A_*$, and the period conditions (8) and (9) hold. The map $F : X \to \mathbb{R}^n$ given by (7) then solves the problem.

Consider the action of G on $X \times Y$ by

$$g(x,y) = (gx, gy), \quad x \in X, \ g \in G.$$

The projection $X \times Y \to X$ is then G-equivariant, so it induces a projection

$$\rho : Z = (X \times Y)/G \to X/G.$$

Note that Z is a reduced complex space, the map ρ is holomorphic, it is branched over the closed discrete subset X_0/G of X/G, and the restriction

$$\rho : Z_1 = \rho^{-1}(X_1/G) \to X_1/G$$

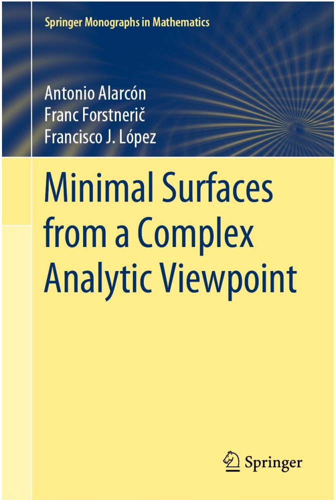
is a holomorphic G-bundle with fibre $Y = A_* \cup Y_0$. The subset

$$\Omega := (X_1 \times A_*)/G \subset Z_1 \subset Z$$

is a G-invariant Zariski open domain without singularities.

1. The restricted projection
$$\rho : \Omega = (X_1 \times A_*)/G \to X_1/G \tag{11}$$
is a holomorphic G-bundle with fibre A_*.
2. A G-equivariant map $f : X \to Y$ is the same thing as a section $\tilde{f} : X/G \to Z$ of $\rho : Z \to X/G$.
3. The map f_0 from step 1 gives a local holomorphic section $\tilde{f}_0$ of $Z \to X/G$ on a neighbourhood $V/G \subset X/G$ of X_0/G such that
$$\tilde{f}_0\big((V \setminus X_0)/G)\big) \subset \Omega.$$
4. The fibre A_* of (11) is $O(n, \mathbb{C})$-homogeneous, hence an **Oka manifold**. Therefore, sections of $\rho : Z = (X \times Y)/G \to X/G$ mapping X_1/G to Ω satisfy the Oka principle (**F. 2003**). This gives a global holomorphic section $\tilde{f} : X/G \to Z$ with $\tilde{f}(X_1/G) \subset \Omega$ which agrees with $\tilde{f}_0$ on X_0/G.
5. $\tilde{f}$ can be chosen such that the corresponding G-equivariant map $f : X \to Y$ integrates to a G-equivariant conformal minimal immersion.

The main reference



Our book (2021) includes proofs of the Runge/Mergelyan approximation theorem, the Weierstrass interpolation theorem, and related results in the classical theory of minimal surfaces in Euclidean spaces. They are obtained by combining **Oka-theoretic methods** with **convex integration theory.**

Since the convex hull of the null quadric $A \subset \mathbb{C}^n$ equals $\mathbb{C}^n$, the holomorphic map $f : X \to A_* \cup Y_0$ can be chosen such that the value of the integral $\int_\gamma f\theta$ on any given curve $\gamma \subset X$ assumes an arbitrary value in $\mathbb{C}^n$. Hence, we can arrange the desired period conditions.

In a galaxy of minimal surfaces

Thank your for your attention

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