Minimal metric on domains in real Euclidean spaces

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References

The talk is based on the following papers:

F. Forstnerič & D. Kalaj, Schwarz–Pick lemma for harmonic maps which are conformal at a point. Anal. PDE, **17(3)**:981–1003, 2024.

B. Drinovec Drnovšek and F. Forstnerič: Hyperbolic domains in real Euclidean spaces. Pure Appl. Math. Q., **19:6** (2023) 2689–2735.

Both papers were available as preprints in 2021.

Since 2023, **Gaussier and Sukhov** have been developing the subject of the minimal pseudometric on general Riemannian manifolds.

The minimal pseudometric

Let $\mathbb{D}$ denote the unit disc in $\mathbb{C}$ and Ω be a domain in $\mathbb{R}^n$, $n \geq 3$. We denote by $\mathrm{CH}(\mathbb{D}, \Omega)$ the space of (not necessarily immersed) conformal harmonic discs $\mathbb{D} \to \Omega$. Such a disc is a **minimal surface**. Let $z = x + \imath y$ be the complex coordinate on $\mathbb{D}$ and $\mathbf{x} = (x_1, \ldots, x_n)$ be the coordinate on $\mathbb{R}^n$.

We define a Finsler pseudometric (called the **minimal metric**) on the tangent bundle $T\Omega = \Omega \times \mathbb{R}^n$ by

$$g_\Omega(\mathbf{x}, \mathbf{v}) = \inf\{1/r > 0 : \exists f \in \mathrm{CH}(\mathbb{D}, \Omega),\ f(0) = \mathbf{x},\ f_x(0) = r\mathbf{v}\}.$$

Clearly, g_Ω is upper-semicontinuous and absolutely homogeneous:

$$g_\Omega(\mathbf{x}, t\mathbf{v}) = |t| g_\Omega(\mathbf{x}, \mathbf{v}) \quad \text{for} \quad t \in \mathbb{R}.$$

The **minimal pseudodistance** $\rho_\Omega : \Omega \times \Omega \to \mathbb{R}_+$ is defined by

$$\rho_\Omega(\mathbf{x}, \mathbf{y}) = \inf_\gamma \int_0^1 g_\Omega(\gamma(t), \dot\gamma(t))\, dt, \qquad \mathbf{x}, \mathbf{y} \in \Omega.$$

The infimum is over all piecewise smooth paths $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = \mathbf{x}$ and $\gamma(1) = \mathbf{y}$. Obviously, ρ_Ω satisfies the triangle inequality, but it need not be a distance function. In particular, $\rho_{\mathbb{R}^n}$ vanishes identically.

The distance decreasing property

For every conformal harmonic disc $f : \mathbb{D} \to \Omega$ we have

$$g_\Omega(f(z), df_z(\xi)) \le \frac{|\xi|}{1-|z|^2} = \mathscr{P}_{\mathbb{D}}(z,\xi), \quad z \in \mathbb{D},\ \xi \in \mathbb{R}^2,$$

and g_Ω is the biggest pseudometric on Ω with this property. For $z = 0$ this follows from the definition of g_Ω. For an arbitrary point $z \in \mathbb{D}$ it is obtained by replacing f by a conformal harmonic disc $f \circ \phi$, where $\phi \in \mathrm{Aut}(\mathbb{D})$ is a holomorphic automorphism of the disc interchanging 0 and z.

It follows that conformal harmonic maps $M \to \Omega$ (minimal surfaces) from any conformal surface M are distance-decreasing in the Poincaré (pseudo) metric on M and the minimal metric on Ω:

$$\rho_\Omega(f(x), f(x')) \le \mathrm{dist}_{\mathcal{P}_M}(x,x'), \qquad x, x' \in M;$$

furthermore, ρ_Ω is the biggest pseudodistance on Ω having this property.

If $R : \mathbb{R}^n \to \mathbb{R}^m$ $(m \ge n \ge 3)$ is a *rigid transformation* (a composition of orthogonal maps, dilations, and translations) and $R(\Omega) \subset \Omega' \subset \mathbb{R}^m$, then

$$g_{\Omega'}(R(\mathbf{x}), dR_{\mathbf{x}}(\mathbf{u})) \le g_\Omega(\mathbf{x},\mathbf{u}), \quad \mathbf{x} \in \Omega,\ \mathbf{u} \in \mathbb{R}^n.$$

The minimal metric is defined by chains of conformal harmonic discs

Fix a pair of points $\mathbf{x}, \mathbf{y} \in \Omega \subset \mathbb{R}^n$. Given a chain of conformal harmonic discs $f_i : \mathbb{D} \to \Omega$ and points $a_i \in \mathbb{D}$ $(i = 1, \ldots, k)$ such that

$$f_1(0) = \mathbf{x}, \quad f_{i+1}(0) = f_i(a_i) \text{ for } i = 1, \ldots, k-1, \quad f_k(a_k) = \mathbf{y},$$

we define the length of the chain to be the number

$$\sum_{i=1}^{k} \mathrm{dist}_{\mathcal{P}_{\mathbb{D}}}(0, a_i) = \sum_{i=1}^{k} \frac{1}{2} \log \frac{1 + |a_i|}{1 - |a_i|} \geq 0.$$

The i-th summand on the right hand side is the Poincaré distance from 0 to a_i in $\mathbb{D}$. It is easily seen that $\rho_\Omega(\mathbf{x}, \mathbf{y})$ is the infimum of the lengths of such chains.

The Kobayashi pseudodistance on a complex manifold X is defined in the same way by using chains of holomorphic discs $\mathbb{D} \to X$.

Definition

A domain $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ is **hyperbolic** if ρ_Ω is a distance function on Ω, and is **complete hyperbolic** if (Ω, ρ_Ω) is a complete metric space.

Schwarz–Pick lemma for harmonic maps from the disc to the ball

Theorem (Kalaj & F., 2024)

Let $f : \mathbb{D} \to \mathbb{B}^n$ is a harmonic map for some $n \geq 2$ which is conformal at a point $z \in \mathbb{D}$. Denote by $\theta \in [0, \pi/2]$ the angle between the vector $f(z)$ and the plane $df_z(\mathbb{R}^2)$. Then:

$$\|df_z\| \;\leq\; \frac{1-|f(z)|^2}{1-|z|^2}\frac{1}{\sqrt{1-|f(z)|^2\sin^2\theta}}.$$

Equality holds if and only if f is a conformal diffeomorphism onto the affine disc

$$\Sigma = (f(z) + df_z(\mathbb{R}^2)) \cap \mathbb{B}^n.$$

The number $R = \sqrt{1-|f(z)|^2\sin^2\theta}$ is the radius of the disc Σ.
If $f(z) = 0$ or $df_z = 0$ then the angle θ is not defined, but it is irrelevant.

If $n = 2$ then $\theta = 0$, $R = 1$, so the result generalises the classical Schwarz–Pick lemma to harmonic maps $f : \mathbb{D} \to \mathbb{D}$ which are conformal only at the point z where we are estimating the differential df_z.

The extremal holomorphic discs in the complex ball $\mathbb{B}^n_{\mathbb{C}} \subset \mathbb{C}^n$ are the holomorphic parametrizations of complex affine discs in $\mathbb{B}^n_{\mathbb{C}}$. The proof uses the fact that the group of holomorphic automorphisms of $\mathbb{B}^n_{\mathbb{C}}$ acts transitively.

Comparison with our result shows that, up to the orientation:

The extremal holomorphic discs in $\mathbb{B}^n_{\mathbb{C}}$ are precisely those extremal conformal minimal discs whose images are complex.

The biggest group preserving the set of all conformal minimal discs in $\mathbb{B}^n$ under postcomposition is the orthogonal group, which does not act transitively. Our proof also gives a new proof of the complex Schwarz lemma without using the Möbius group $\mathrm{Aut}(\mathbb{B}^n_{\mathbb{C}})$.

The minimal metric on the ball

Corollary

The minimal metric $g_{\mathbb{B}^n}$ on the ball $\mathbb{B}^n \subset \mathbb{R}^n$ is

$$\begin{aligned} g_{\mathbb{B}^n}(\mathbf{x}, \mathbf{v})^2 &= \frac{1 - |\mathbf{x}|^2 \sin^2 \phi}{(1 - |\mathbf{x}|^2)^2} \, |\mathbf{v}|^2 \\ &= \frac{(1 - |\mathbf{x}|^2)|\mathbf{v}|^2 + |\mathbf{x} \cdot \mathbf{v}|^2}{(1 - |\mathbf{x}|^2)^2} = \frac{|\mathbf{v}|^2}{1 - |\mathbf{x}|^2} + \frac{|\mathbf{x} \cdot \mathbf{v}|^2}{(1 - |\mathbf{x}|^2)^2}. \end{aligned}$$

where $\mathbf{x} \in \mathbb{B}^n$, $\mathbf{v} \in \mathbb{R}^n$, and ϕ is the angle between $\mathbf{v}$ and the line $\mathbb{R}\mathbf{x} \subset \mathbb{R}^n$.

This is the **Beltrami–Cayley–Klein metric** on $\mathbb{B}^n$.

The Beltrami–Cayley–Klein model of hyperbolic geometry was introduced by **Arthur Cayley (1859)** and **Eugenio Beltrami (1868)**, and it was developed by **Felix Klein (1871, 1873)**. The underlying space is the n-dimensional unit ball, geodesics are straight line segments with endpoints on the boundary sphere, and the distance between points on a geodesic is given by the cross ratio. This is a special case of the **Hilbert metric** on convex domains in $\mathbb{R}^n$, introduced by **David Hilbert** in 1895.

The Cayley–Klein metric is the restriction of the Kobayashi metric on the unit ball $\mathbb{B}^n_{\mathbb{C}} \subset \mathbb{C}^n$ to points $\mathbf{x} \in \mathbb{B}^n = \mathbb{B}^n_{\mathbb{C}} \cap \mathbb{R}^n$ and vectors in $T_{\mathbf{x}}\mathbb{R}^n \cong \mathbb{R}^n$.

It also equals $1/\sqrt{n+1}$ times the Bergman metric on $\mathbb{B}^n_{\mathbb{C}}$, restricted to $\mathbb{B}^n$ and real tangent vectors.

Lempert (1993) showed that on any convex domain $D \subset \mathbb{R}^n$ (or in $\mathbb{RP}^n$), the Hilbert metric is the restriction to D of the Kobayashi metric on the elliptic tube $D^* \subset \mathbb{C}^n$ over D. **However, the minimal metric on a convex domain does not equal the Hilbert metric in general, not even on ellipsoids.**

For $n \geq 3$, the Cayley–Klein metric is not conformally equivalent to the Euclidean metric on $\mathbb{B}^n$. We have that

$$\frac{|\mathbf{v}|}{\sqrt{1-|\mathbf{x}|^2}} \ \leq\ g_{\mathbb{B}^n}(\mathbf{x},\mathbf{v}) \ \leq\ \frac{|\mathbf{v}|}{1-|\mathbf{x}|^2}.$$

The upper bound is reached for $\measuredangle(\mathbf{x},\mathbf{v}) = 0$ (in the radial direction) and the lower bound for $\measuredangle(\mathbf{x},\mathbf{v}) = \pi/2$ (in the tangential direction).

Proof of the Corollary

Recall that

$$g_{\mathbb{B}^n}(\mathbf{x},\mathbf{v}) = \inf\{1/r > 0 : \exists f \in \mathrm{CH}(\mathbb{D},\mathbb{B}^n),\ f(0)=\mathbf{x},\ f_x(0) = r\mathbf{v}\}.$$

Let $\Lambda = \mathbf{x} + df_0(\mathbb{R}^2)$ and $\Delta = \Lambda \cap \mathbb{B}^n$. The Theorem implies

$$r|\mathbf{v}| = |f_x(0)| = |df_0| \le (1-|\mathbf{x}|^2)/R$$

where

$$R = \sqrt{1-|\mathbf{x}|^2\sin^2\theta} = \mathrm{radius}(\Delta), \quad \theta = \measuredangle(\mathbf{x},\Lambda) \in [0,\pi/2].$$

Equality is achieved when $f : \mathbb{D} \to \Delta$ is a conformal diffeomorphism. Thus:

$$\inf_f \frac{1}{r} = \frac{\inf_f R}{1-|\mathbf{x}|^2}|\mathbf{v}| = \frac{\sqrt{1-|x|^2\sin^2\phi}}{1-|\mathbf{x}|^2}|\mathbf{v}|$$

where $\phi = \sup_f \theta = \measuredangle(\mathbb{R}\mathbf{x},\mathbf{v})$.

The maximum is reached when $\Lambda = \mathrm{Span}(\mathbf{v},\mathbf{w})$ where $\mathbf{w} \perp \mathrm{Span}(\mathbf{x},\mathbf{v})$.

Proof of the Theorem, 1

It suffices to consider the case $z = 0$. Indeed, with f and z as in the theorem, let $\phi_z \in \mathrm{Aut}(\mathbb{D})$ be such that $\phi_z(0) = z$. The harmonic map $\tilde{f} = f \circ \phi_z : \mathbb{D} \to \mathbb{B}^n$ is then conformal at the origin. Since $|\phi_z'(0)| = 1 - |z|^2$, the estimate for f at z follows from the estimate for $\tilde{f}$ at $z = 0$.

We find an explicit conformal parameterization of affine discs in $\mathbb{B}^n$. Fix a point $\mathbf{q} \in \mathbb{B}^n$ and a 2-plane $0 \in \Lambda \subset \mathbb{R}^n$, and consider the affine disc $\Sigma = (\mathbf{q} + \Lambda) \cap \mathbb{B}^n$. Let $\mathbf{p} \in \Sigma$ be the closest point to the origin.

If $n = 2$ then $\mathbf{p} = 0$ and $\Sigma = \mathbb{D}$. Suppose now that $n = 3$; the case $n > 3$ will be the same. By an orthogonal rotation on $\mathbb{R}^3$ we may assume that

$$\mathbf{p} = (0, 0, p) \quad \text{and} \quad \Sigma = \left\{ (x, y, p) : x^2 + y^2 < 1 - p^2 \right\}.$$

Let $\mathbf{q} = (b_1, b_2, p) \in \Sigma$, and let θ denote the angle between $\mathbf{q}$ and Σ. Set

$$R = \sqrt{1 - p^2} = \sqrt{1 - |\mathbf{q}|^2 \sin^2 \theta}, \qquad a = \frac{b_1 + \mathrm{i} b_2}{R} \in \mathbb{D}, \qquad |a| = \frac{|\mathbf{q}| \cos \theta}{R}.$$

We orient Σ by the pair of tangent vectors ∂_x, ∂_y.

Proof, 2

Every orientation preserving conformal parameterization $f : \mathbb{D} \to \Sigma$ with $f(0) = \mathbf{q}$ is then of the form

$$f(z) = \left(R \cdot \Re \frac{e^{it}z + a}{1 + \bar{a}e^{it}z}, R \cdot \Im \frac{e^{it}z + a}{1 + \bar{a}e^{it}z}, p \right) = \left(R \, \frac{e^{it}z + a}{1 + \bar{a}e^{it}z}, p \right)$$

for $z \in \mathbb{D}$ and some $t \in \mathbb{R}$. (If $n = 2$ then $p = 0$ and $R = 1$.)

Recall that $R^2 = 1 - |\mathbf{q}|^2 \sin^2 \theta$ and $R^2 |a|^2 = |\mathbf{q}|^2 \cos^2 \theta$. Hence:

$$\begin{aligned} \|df_0\| &= R\,(1 - |a|^2) = \frac{R^2 - R^2|a|^2}{R} = \frac{(1 - |\mathbf{q}|^2 \sin^2 \theta) - |\mathbf{q}|^2 \cos^2 \theta}{R} \\ &= \frac{1 - |\mathbf{q}|^2}{\sqrt{1 - |\mathbf{q}|^2 \sin^2 \theta}} = \frac{1 - |f(0)|^2}{\sqrt{1 - |f(0)|^2 \sin^2 \theta}}. \end{aligned}$$

This shows that the conformal parameterizations of the proper affine discs in the ball satisfy the equality in the theorem at every point.

Let $f : \mathbb{D} \to \mathbb{B}^3$ be as above, where we may assume that $t = 0$.

Suppose that $g : \mathbb{D} \to \mathbb{B}^3$ is a harmonic map such that $g(0) = f(0)$, g is conformal at 0, and $dg_0(\mathbb{R}^2) = df_0(\mathbb{R}^2)$. Up to replacing g by $g(e^{\imath t} z)$ or $g(e^{\imath t} \bar{z})$ for some $t \in \mathbb{R}$, we may assume that

$$dg_0 = r\, df_0 \quad \text{for some } r > 0.$$

We must prove that $r \le 1$, and that $r = 1$ if and only if $g = f$.

Consider the holomorphic map $F : \mathbb{D} \to \Omega = \mathbb{B}^3 \times \imath\mathbb{R}^3$ with $f = \Re F$, given by

$$F(z) = \left(R \cdot \frac{z+a}{1+\bar{a}z}, -R \cdot \imath \frac{z+a}{1+\bar{a}z}, p \right), \quad z \in \mathbb{D}.$$

Let $G : \mathbb{D} \to \Omega$ be the holomorphic map with $\Re G = g$ and $G(0) = F(0)$.

By the Cauchy–Riemann equations, the condition $dg_0 = r\, df_0$ implies

$$G'(0) = r\, F'(0).$$

Proof, 4

It follows that the map $(F(z) - G(z))/z$ is holomorphic on $\mathbb{D}$ and

$$\lim_{z \to 0} \frac{F(z) - G(z)}{z} = F'(0) - G'(0) = (1 - r)F'(0).$$

Since $g : \mathbb{D} \to \mathbb{B}^3$ is a bounded harmonic map, it has a nontangential boundary value at almost every point of the circle $\mathbb{T} = b\mathbb{D}$. Since the Hilbert transform is an isometry on the Hilbert space $L^2(\mathbb{T})$, the same is true for G.

Denote by $\langle \cdot, \cdot \rangle$ the complex bilinear form on $\mathbb{C}^n$ given by

$$\langle z, w \rangle = \sum_{i=1}^{n} z_i w_i$$

for $z, w \in \mathbb{C}^n$.

Proof, 5

For each $z = e^{\mathrm{i}t} \in b\mathbb{D}$ the vector $f(z) \in b\mathbb{B}^3$ is the unit normal vector to the sphere $b\mathbb{B}^3$ at the point $f(z)$. Since $\mathbb{B}^3$ is strongly convex, we have that

$$\Re \left\langle F(z) - G(z), f(z) \right\rangle = \left\langle f(z) - g(z), f(z) \right\rangle \geq 0 \quad a.e.\ z \in b\mathbb{D},$$

and the value is positive for almost every $z \in b\mathbb{D}$ if and only if $g \neq f$.

Consider the map $\tilde{f} : b\mathbb{D} \to \mathbb{C}^3$ given by

$$\tilde{f}(z) = z|1 + \bar{a}z|^2 f(z), \qquad |z| = 1.$$

A calculation, taking into account $z\bar{z} = 1$ on $b\mathbb{D}$, gives

$$\tilde{f}(z) = \begin{pmatrix} \frac{c}{2}\left(1 + a^2 + 4(\Re a)z + (1 + \bar{a}^2)z^2\right) \\ \frac{c}{2}\left(\mathrm{i}(1 - a^2) + 4(\Im a)z + \mathrm{i}(\bar{a}^2 - 1)z^2\right) \\ p\,(z + a)(1 + \bar{a}z) \end{pmatrix}, \qquad |z| = 1.$$

Conclusion of the proof

We extend $\tilde{f}$ to all $z \in \mathbb{C}$ by letting it equal the holomorphic polynomial map on the right hand side above. Since $|1 + \bar{a}z|^2 > 0$ for $z \in \overline{\mathbb{D}}$, we have

$$\begin{aligned} h(z) &:= \Re \left\langle F(z) - G(z), |1 + \bar{a}z|^2 f(z) \right\rangle \\ &= \left\langle f(z) - g(z), |1 + \bar{a}z|^2 f(z) \right\rangle \geq 0 \quad a.e.\ z \in b\mathbb{D}, \end{aligned}$$

and $h > 0$ almost everywhere on $b\mathbb{D}$ if and only if $g \neq f$.

From the definition of $\tilde{f}$ we see that

$$h(z) = \Re \left\langle \frac{F(z) - G(z)}{z}, \tilde{f}(z) \right\rangle \quad a.e.\ z \in b\mathbb{D}$$

Since the maps $(F(z) - G(z))/z$ and $\tilde{f}(z)$ are holomorphic on $\mathbb{D}$, h extends to a nonnegative harmonic function on $\mathbb{D}$ which is positive on $\mathbb{D}$ unless $f = g$.

At $z = 0$ we have

$$h(0) = \Re \left\langle F'(0) - G'(0), \tilde{f}(0) \right\rangle = (1 - r) \Re \left\langle F'(0), \tilde{f}(0) \right\rangle \geq 0,$$

with equality if and only if $g = f$. Applying this to the constant map $g(z) = f(0)$ (so $r = 0$) gives $\Re \langle F'(0), \tilde{f}(0) \rangle > 0$. It follows that $r \leq 1$, with equality if and only if $g = f$. This completes the proof.

Our proof is motivated by the seminal work of **László Lempert (1981)** on Kobayashi extremal holomorphic discs in bounded strongly convex domains $\Omega \subset \mathbb{C}^n$ with smooth boundaries.

In Lempert's terminology, a proper holomorphic disc $F : \mathbb{D} \to \Omega$ extending continuously to $\overline{\mathbb{D}}$ is said to be a **stationary disc** if, denoting by $\nu(z)$ the unit normal to $b\Omega$ along the circle $F(b\mathbb{D})$, there is a positive function $q > 0$ on $b\mathbb{D}$ such that the map

$$b\mathbb{D} \ni z \mapsto \; z\, q(z)\overline{\nu(z)} \in \mathbb{C}^n$$

extends to a holomorphic map $\tilde{f} : \mathbb{D} \to \mathbb{C}^n$. The use of such a map, along with the convexity of the domain, enables the arguments used above to show that a stationary disc F is the unique Kobayashi extremal disc in Ω through the point $F(a)$ in the tangent direction $F'(a)$ for every $a \in \mathbb{D}$.

In our case, $\nu(z) = f(z)$ is real-valued, and a suitable map is

$$\tilde{f}(z) = z\,|1 + \bar{a}z|^2 f(z), \qquad |z| = 1.$$

The fact that $\Omega = \mathbb{B}^n \times i\mathbb{R}^n$ is unbounded does not matter.

Strongly 2-convex domains are complete hyperbolic

The following result is an analogue of Graham's theorem on complete Kobayashi hyperbolicity of bounded strongly pseudoconvex domains in $\mathbb{C}^n$.

Theorem (B. Drinovec Drnovšek & F. 2023)

Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 3)$ with smooth boundary $b\Omega$ whose principal curvatures $\nu_1 \leq \nu_2 \leq \ldots \leq \nu_{n-1}$ at any point $p \in b\Omega$ satisfy $\nu_1 + \nu_2 > 0$. Then, Ω is complete hyperbolic.

The proof relies on the localization principle for the minimal metric and on a lower bound in terms of a **Sibony-type metric**, defined in terms of minimal plurisubharmonic functions.

Theorem (Fiacchi 2023)

Every domain in the previous theorem is Gromov hyperbolic.

Problem

Is the minimal distance to an embedded minimal surface in $\mathbb{R}^3$ infinite?

Complement of an embedded FTC minimal surface is weakly hyperbolic

Theorem (F. 2023)

Let $M \subset \mathbb{R}^3$ be an embedded minimal surface of finite total Gaussian curvature. Then, there are no parabolic minimal surfaces in $\mathbb{R}^3 \setminus M$.

A minimal surface is called **parabolic** if it is the image $M = f(R)$ of a conformal harmonic map from a Riemann surface of the form $R = \overline{R} \setminus P$, where P is a finite set in a compact Riemann surface $\overline{M}$.

Idea of proof. There is an $\epsilon > 0$ such that every point $\mathbf{x}$ in the ϵ-tube $M(\epsilon)$ around M has a unique nearest point $\pi(\mathbf{x}) \in M$.

It follows that the signed distance function $\delta : M(\epsilon) \to \mathbb{R}$ is smooth and minimal plurisubharmonic on $M_-(\epsilon) = \{\delta < 0\}$.

Let Ω be the connected component of $\mathbb{R}^3 \setminus M$ containing $M_-(\epsilon)$. We can find a negative minimal psh function $\rho : \Omega \to \mathbb{R}_-$ which agrees with δ on $M_-(\epsilon')$ for some $0 < \epsilon' < \epsilon$.

The restriction of ρ to any minimal surface $\Sigma \subset \Omega$ is subharmonic. If Σ is parabolic then $\rho|_\Sigma$ is constant. Applying this to translates of Σ, we see that such a surface cannot exist.

Theorem (B. Drinovec Drnovšek & F. 2023)

The following are equivalent for a convex domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$.

1. (i) Ω *is complete hyperbolic.*
2. (ii) Ω *is hyperbolic.*
3. (iii) Ω *does not contain any 2-dimensional affine subspaces.*
4. (iv) Ω *is contained in the intersection of $n-1$ halfspaces determined by independent linear functionals.*

Note: A convex domain in $\mathbb{C}^n$ is Kobayashi hyperbolic iff if it does not contain any affine complex line (Barth 1980, Harris 1979, Bracci and Saracco 2009).

The implications (i)⇒(ii)⇒(iii) are obvious, and (iii)⇒(iv) is seen by elementary convexity theory. We now explain the proof of (iv)⇒(i).

We first show that **the minimal distance to an affine hyperplane is infinite.** The Schwarz lemma for positive harmonic functions $f : \mathbb{D} \to (0, +\infty)$ gives

$$|\nabla f(0)| \leq 2f(0).$$

Hyperbolicity of convex domains, 2

For $\mathbb{H}^n = \{x_1 > 0\}$ this implies

$$g_{\mathbb{H}^n}((x_1, \ldots, x_n), (v_1, \ldots, v_n)) \geq |v_1|/2x_1.$$

For any path $\gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t)) \in \mathbb{H}^n,\ t \in [0, 1)$ it follows that

$$\int_0^1 g_{\mathbb{H}^n}(\gamma(t), \dot\gamma(t))\, dt \geq \int_0^1 \frac{|\dot\gamma_1(t)|}{2\gamma_1(t)}\, dt.$$

If $\gamma(t) \to 0$ or $\gamma(t) \to +\infty$ as $t \to 1$ then the integral equals $+\infty$.

At any boundary point $\mathbf{p} \in b\Omega$ of a convex domain there is a supporting affine hyperplane $\Sigma \subset \mathbb{R}^n$ passing through $\mathbf{p}$ such that Ω is contained in a half-space of $\mathbb{R}^n \setminus \Sigma$. Hence, any path $\mathbf{x}(t) \in \Omega$ $(t \in [0, 1))$ which clusters at some point $\mathbf{p} \in \mathbf{b\Omega}$ as $t \to 1$ has infinite g_Ω-length. This gives:

Corollary

A convex domain $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ is locally complete hyperbolic at any boundary point $\mathbf{p} \in b\Omega$.

Hyperbolicity of convex domains, 3

Assume now that Ω satisfies condition (iv). For simplicity, assume that

$$\Omega \subset \{x_1 \geq 0, \dots, x_{n-1} \geq 0\}.$$

Let $\mathbf{x}(t) = (x_1(t), \dots, x_n(t))$, $t \in [0, 1)$, be a divergent path in Ω.

If $\mathbf{x}(t)$ clusters at some point $\mathbf{p} \in b\Omega$ as $t \to 1$, then $\mathbf{x}(t)$ has infinite length. Likewise, if one of the functions $x_i(t)$ $(i = 1, \dots, n-1)$ clusters at $+\infty$, then $\mathbf{x}(t)$ has infinite minimal length in the halfspace $\mathbb{H}_i = \{x_i \geq 0\}$, and hence also in $\Omega \subset \mathbb{H}_i$.

It remains to consider the case when

$$x_i(t) \leq c_1, \qquad t \in [0, 1),\ i = 1, \dots, n-1,$$

and $\mathbf{x}(t)$ does not cluster anywhere on $b\Omega$. In this case, $x_n(t) \in \mathbb{R}$ clusters at $\pm\infty$, and hence $\int_0^1 |\dot{x}_n(t)| dt = +\infty$. It remains to show that

$$g_\Omega(\mathbf{x}(t), \dot{\mathbf{x}}(t)) \geq c_2 |\dot{x}_n(t)|$$

for some $c_2 > 0$ depending only on c_1.

Hyperbolicity of convex domains, 4

Fix a point $\mathbf{x} = \mathbf{x}(t)$ and a unit vector $\mathbf{v} = (\mathbf{v}', v_n) \in \mathbb{R}^n$, and consider a conformal harmonic map $f = (f_1, f_2, \ldots, f_n) : \mathbb{D} \to \Omega$ such that

$$f(0) = \mathbf{x} \ \text{ and } \ f_x(0) = r\mathbf{v} \ \text{ for some } r > 0.$$

Then, $f_y(0) = r\mathbf{w} = r(\mathbf{w}', w_n)$ where $(\mathbf{v}, \mathbf{w})$ is an orthonormal frame:

$$0 = \mathbf{v} \cdot \mathbf{w} = \mathbf{v}' \cdot \mathbf{w}' + v_n w_n, \qquad |\mathbf{v}| = |\mathbf{w}| = 1.$$

From this and the Cauchy–Schwarz inequality it follows that

$$v_n^2(1 - |\mathbf{w}'|^2) = v_n^2 w_n^2 = |\mathbf{v}' \cdot \mathbf{w}'|^2 \le |\mathbf{v}'|^2 |\mathbf{w}'|^2 = (1 - v_n^2)|\mathbf{w}'|^2,$$

and hence

$$|v_n| \le |\mathbf{w}'| \le \sqrt{n-1} \max_{i=1,\ldots,n-1} |w_i|.$$

Therefore,

$$r|v_n| \le \sqrt{n-1} \max_{i=1,\ldots,n-1} r|w_i| \le 2\sqrt{n-1} \max_{i=1,\ldots,n-1} x_i.$$

The last inequality is seen by applying the Schwarz lemma to the conformal harmonic disc $z \mapsto \tilde{f}(z) = f(\imath z)$ in each of the half-spaces $\mathbb{H}_i = \{x_i > 0\}$. Note that $\tilde{f}(0) = f(0) = \mathbf{x}$ and $\tilde{f}_x(0) = f_y(0) = r\mathbf{w}$.

This gives

$$\frac{1}{r} \geq \frac{|v_n|}{2\sqrt{n-1}\max_{i=1,\ldots,n-1} x_i} \geq \frac{|v_n|}{2c_1\sqrt{n-1}} = c_2|v_n|$$

for any $r > 0$ as above, with $c_2 = 1/4c_1\sqrt{N-1} > 0$.

Taking the infimum of the left hand side gives

$$g_\Omega(\mathbf{x}, \mathbf{v}) \geq c_2|v_n|.$$

Applying this with $\mathbf{x} = \mathbf{x}(t)$ and $\mathbf{v} = \dot{\mathbf{x}}(t)$ yields

$$g_\Omega(\mathbf{x}(t), \dot{\mathbf{x}}(t)) \geq c_2|\dot{x}_n(t)|.$$

This proves that $\int_0^1 g_\Omega(\mathbf{x}(t), \dot{\mathbf{x}}(t))\, dt = +\infty$.

Hence, every divergent path in Ω has infinite g_Ω-length, so Ω is complete hyperbolic.

$\sim$ **Thank you for your attention** $\sim$