# The Oka principle for tame families of Stein manifolds

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A complex bridge between analysis and geometry
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# Abstract

We develop the existence and approximation theory for continuous families of holomorphic maps from tame families of smooth integrable Stein structures on a smooth manifold $X$ to any Oka manifold.

Along the way, we prove the Oka–Weil approximation theorem for families of holomorphic functions, and also for sections of fibrewise holomorphic vector bundles on such families of Stein structures.

As applications, we obtain global solutions of the $\bar{\partial}$-equation on such families, and we prove the Oka principle for the classification of fibrewise holomorphic vector bundles on such families of Stein structures, generalising the classical results of Oka (1939) and Grauert (1958).

**F.F.: Runge and Mergelyan theorems on families of open Riemann surfaces. Preprint, January 2025.**

**F.F. and Edda Sigurðardóttir: The Oka principle for tame families of Stein manifolds. Preprint, October 2025.**

**F.F.: The universal family of punctured Riemann surfaces is Stein. Preprint, December 2025.**

# Almost complex structures and integrability

Let $X$ be a smooth orientable manifold of dimension $2n$. An almost complex structure on $X$ is an endomorphism $J$ of the tangent bundle $TX$ satisfying

$$J^2 = -Id.$$

A differentiable function $f : X \to \mathbb{C}$ is said to be $J$-holomorphic if it satisfies the Cauchy–Riemann equation

$$df_x \circ J_x = \sqrt{-1}\, df_x = J_{\mathrm{st}}\, df_x, \quad x \in X.$$

Newlander and Nirenberg 1957, Nijenhuis and Woolf 1963, Kohn 1963, Malgrange 1969, Webster 1989,... If $J$ is formally integrable and of local Hölder class $\mathcal{C}^{(k,\alpha)}$ for some $k = 1, 2, \ldots$ (or $k \geq 0$ if $n = 1$) and $0 < \alpha < 1$, there is an atlas $\{(U_i, \phi_i)\}$ of $J$-holomorphic charts $\phi_i : U_i \to \phi_i(U_i) \subset \mathbb{C}^n$ of class $\mathcal{C}^{(k+1,\alpha)}(U_i)$.

Note that the transition maps $\phi_{i,j} = \phi_i \circ \phi_j^{-1}$ are $J_{\mathrm{st}}$-holomorphic. Hence, such $J$ determines on $X$ the structure of a complex manifold $(X, J)$ whose underlying smooth structure is $\mathcal{C}^{(k+1,\alpha)}$ compatible with the given smooth structure on $X$.

# Families of Stein structures

In the sequel, we assume the following.

- $X$ is a smooth open orientable manifold of dimension $2n \geq 2$.
- $T$ is a locally compact and paracompact Hausdorff space.
  (In some results we shall impose a stronger assumption.)
- $\mathcal{J} = \{J_t\}_{t \in T}$ is a continuous family of smooth Stein structures on $X$.

A continuous map $f : Z = T \times X \to Y$ to a complex manifold $(Y, J_Y)$ is said to be $\mathcal{J}$-holomorphic if

$$f_t = f(t, \cdot\,) : X \to Y \quad \text{is } (J_t, J_Y)\text{-holomorphic for every } t \in T.$$

In particular, a function $f : Z \to \mathbb{C}$ is $\mathcal{J}$-holomorphic if the above holds for the standard complex structure $J_{\mathrm{st}}$ on $\mathbb{C}$.

The goal: develop the existence and approximation theory for $\mathcal{J}$-holomorphic functions $Z \to \mathbb{C}$ and, more generally, for $\mathcal{J}$-holomorphic maps $Z \to Y$ to Oka manifolds.

# $\mathcal{J}$-convex sets

Let $\pi : Z = T \times X \to T$ denote the projection. A closed subset $K \subset Z = T \times X$ is said to be proper if the restricted projection $\pi|_K : K \to T$ is proper. Equivalently, the fibres $K_t = \{x \in X : (t, x) \in K\}$ are compact and upper semicontinuous in $t \in T$.

The $\mathcal{J}$-convex hull of such $K$ is defined by

$$\widehat{K}_{\mathcal{J}} = \bigcup_{t \in T} \{t\} \times \widehat{K_t}_{J_t}.$$

## Definition

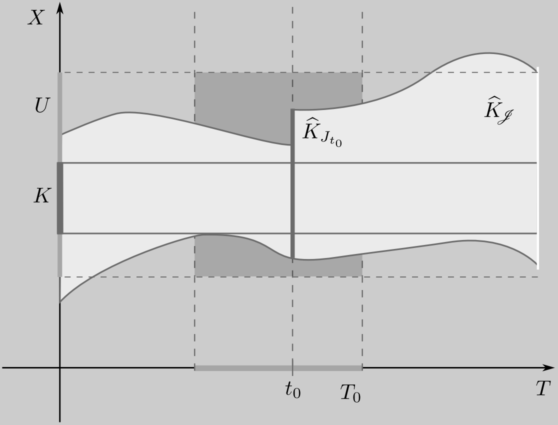
A family $\mathcal{J}$ is tame if for every compact subset $K \subset X$, $t_0 \in T$, and open set $U \subset X$ containing $\widehat{K}_{J_{t_0}}$ there is a neighbourhood $T_0 \subset T$ of $t_0$ such that

$$\widehat{K}_{J_t} \subset U \quad \text{for all } t \in T_0.$$

Equivalently, for any compact $K \subset X$, the hull $\widehat{T \times K}_{\mathcal{J}}$ is proper.

Note that any family of Riemann surfaces is tame.

# An upper semicontinuous family of hulls




An upper semicontinuous family of hulls $\widehat{K}_{J_t}$.

# Characterisation of tameness by psh functions

## Theorem

*Assume that $X$ is a smooth manifold, $T$ is a locally compact and paracompact Hausdorff space, and $\mathcal{J} = \{J_t\}_{t\in T}$ is a continuous family of integrable Stein structures on $X$ of local Hölder class $\mathcal{C}^{0,(k,\alpha)}(T \times X)$ for some $k \in \mathbb{N}$ and $0 < \alpha < 1$. The following conditions are equivalent.*

- (a) *The family $\mathcal{J}$ is tame.*
- (b) *There is a function $\rho : Z = T \times X \to \mathbb{R}$ of class $\mathcal{C}^{0,k+1}$ such that for every $t \in T$, $\rho_t = \rho(t, \cdot\,)$ is a strongly $J_t$-plurisubharmonic exhaustion function on $X$.*

*If $T$ is a $\mathcal{C}^l$ manifold and $\mathcal{J}$ is of class $\mathcal{C}^{l,(k,\alpha)}$ then $\rho$ can be chosen of class $\mathcal{C}^{l,k+1}$.*

If $\rho$ is as in part (b) then for any $c \in \mathbb{R}$ the set $\{\rho \leq c\}$ is proper and $\mathcal{J}$-convex.

Conversely, if $\mathcal{J}$ is tame and $K \subset Z$ is a proper $\mathcal{J}$-convex subset, then for every neighbourhood $U \subset Z$ of $K$ there is a function $\rho$ as in the theorem such that $\rho < 0$ on $K$ and $\rho > 0$ on $Z \setminus U$.

# A nontame family of Stein structures on $\mathbb{R}^4$

## Theorem

*Given a compact set $K \subset \mathbb{R}^{2n}$ $(n > 1)$ with nonempty interior, there is a family of smooth integrable complex structures $\mathcal{J} = \{J_t\}_{t\in\mathbb{R}}$ on $\mathbb{R}^{2n}$, depending smoothly on $t \in \mathbb{R}$, such that*

- *$J_0$ is the standard structure on $\mathbb{C}^n$,*
- *$(\mathbb{R}^{2n}, J_t)$ is biholomorphic to $\mathbb{C}^n$ for every $t \in \mathbb{R}$, and*
- *for any neighbourhood $T_0 \subset \mathbb{R}$ of $0 \in \mathbb{R}$ the set $\bigcup_{t\in T_0} \widehat{K}_{J_t} \subset \mathbb{R}^{2n}$ is unbounded.*

*In particular, the family $\mathcal{J}$ is not tame.*

Sketch of proof. It suffices to consider the case when $n = 2$ and $K$ is the closed ball in $\mathbb{R}^4 \cong \mathbb{C}^2$. By E. F. Wold, there is an injective holomorphic map $\Phi : \mathbb{C}^2 \hookrightarrow \mathbb{C}^2$ such that $\Phi(\mathbb{C}^2) \subset \mathbb{C}^* \times \mathbb{C}$ but the polynomial hull $\widehat{\Phi(K)}$ of $\Phi(K)$ contains the origin $0 \in \mathbb{C}^2$. In particular, $\Phi(\mathbb{C}^2)$ is not Runge in $\mathbb{C}^2$.

We construct a smooth family of smooth diffeomorphisms $\Psi_t : \mathbb{C}^2 \to \mathbb{C}^2$ such that $\Psi_t = \Phi$ holds on a neighbourhood of $t^{-1}K$ for every $t \neq 0$ and $\Psi_0 = \Phi$.

# A nontame family of Stein structures on $\mathbb{R}^4$

Let $J_t$ denote the complex structure on $\mathbb{R}^4 \cong \mathbb{C}^2$ obtained by pulling back by $\Psi_t$ the standard complex structure $J_{\mathrm{st}}$ on $\mathbb{C}^2$. Note that $J_t$ depends smoothly on $t$, it agrees with $J_{\mathrm{st}}$ on a neighbourhood of $t^{-1}K \supset K$ if $t \neq 0$, and $J_0 = J_{\mathrm{st}}$.

For $t \neq 0$, the $J_t$-convex hull of $K$ equals

$$\widehat{K}_{J_t} = \Psi_t^{-1}(\widehat{\Psi_t(K)}) = \Psi_t^{-1}(\widehat{\Phi(K)}), \tag{1}$$

where the second equality follows from the fact that $\Psi_t = \Phi$ on $t^{-1}K \supset K$.

We claim that $\widehat{K}_{J_t} \setminus t^{-1}K$ is nonempty for every $t \neq 0$. Indeed, if $\widehat{K}_{J_t} \subset t^{-1}K$ then, since $\Phi = \Psi_t$ on $t^{-1}K$, it follows from (1) that

$$\Phi(\widehat{K}_{J_t}) = \Psi_t(\widehat{K}_{J_t}) = \widehat{\Phi(K)},$$

a contradiction to $\widehat{\Phi(K)} \not\subset \Phi(\mathbb{C}^2)$.

As $t \to 0$, the sets $t^{-1}K$ increase to $\mathbb{C}^2$, and hence the hulls $\widehat{K}_{J_t}$ are not contained in any bounded subset of $\mathbb{C}^2$ for $t$ in a neighbourhood of 0.

# Examples of tame Stein families

1. If for every $t_0 \in T$ there are a compact set $K \subset X$ and a neighbourhood $T_0 \subset T$ of $t_0$ such that $J_t = J_{t_0}$ holds on $X \setminus K$ for all $t \in T_0$, then $\mathcal{J}$ is tame.

2. If $(X, J_0)$ is a Stein manifold and $\Phi_t : X \to \Phi_t(X) \subset X$ is a continuous family of diffeomorphisms onto Stein Runge domains in $X$, then the family of Stein structures $J_t = \Phi_t^* J_0$ on $X$ is tame.

3. Assume that $(Y, J_Y)$ is a Stein manifold and $F : T \times X \to Y$ is such that for every $t \in T$, $F_t = F(t, \cdot) : X \to Y$ is a proper immersion whose image $F_t(X)$ is an immersed complex submanifold of $Y$. Let $J_t = F_t^* J_Y$. Then, the Stein family $\mathcal{J} = \{J_t\}_{t \in T}$ is tame.

4. If $\{(X, J_t)\}_{t \in T}$ is a tame Stein family and $E \to T \times X$ is a fibrewise holomorphic vector bundle, then the induced family of Stein structures on the fibres $E_t = E|(\{t\} \times X)$ is tame.

5. If $\{(X, J_t)\}_{t \in T}$ is tame and $H_t \subset X$ is a continuous family of closed complex hypersurfaces then the Stein family $\{(X \setminus H_t, J_t)\}_{t \in T}$ is tame.

## The Oka–Weil theorem on tame families of Stein manifolds

### Theorem (1)

*Assume that*

- $X$ *is a smooth manifold of dimension* $2n$*,*
- $T$ *is a locally compact and paracompact Hausdorff space,*
- $k \geq 1$ *and* $r \geq 2k + 2n + 9$ *are integers or* $k = r = +\infty$*,*
- $\mathcal{J} = \{J_t\}_{t \in T}$ *is a tame family of Stein structures of class* $\mathcal{C}^{0,r}$ *on* $X$*,*
- $K \subset T \times X$ *is a proper* $\mathcal{J}$*-convex subset,*
- $U \subset T \times X$ *is an open set containing* $K$*,*
- $Q$ *is a closed subset of* $T$*, and*
- $f : U \cup (Q \times X) \to \mathbb{C}$ *is a* $\mathcal{J}$*-holomorphic function.*

*Then,* $f \in \mathcal{C}^{0,k}$*, and it can be approximated in the fine* $\mathcal{C}^{0,k}$ *topology on* $K$ *by* $\mathcal{J}$*-holomorphic functions* $F : T \times X \to \mathbb{C}$ *such that* $F = f$ *on* $Q \times X$*.*

*Conversely, if for every* $f \in \mathcal{O}_{J_{t_0}}(X)$ *there are a neighbourhood* $T_0 \subset T$ *of* $t_0$ *and a* $\mathcal{J}$*-holomorphic function* $F : T_0 \times X \to \mathbb{C}$ *such that* $F(t_0, \cdot) = f$*, then* $\mathcal{J}$ *is tame.*

# Enter Oka manifolds

A complex manifold $Y$ is called an Oka manifold if maps $X \to Y$ from any Stein manifold $X$ satisfy all forms of the Oka principle:

(a) Every continuous map $f : X \to Y$ is homotopic to a holomorphic map.

(b) If in addition $f$ is holomorphic on a compact $\mathcal{O}(X)$-convex subset $K \subset X$ and on a closed complex subvariety $X'$ of $X$, then the homotopy from $f$ to a holomorphic map $F : X \to Y$ can be chosen to consist of maps with the same properties which approximate $f$ on $K$ and agree with $f$ on $X'$.

(c) A similar statement holds for families of maps depending continuously on a parameter in a compact Hausdorff space.

**Theorem (F., 2005-9)**

*A complex manifold $Y$ is Oka iff every holomorphic map $K \to Y$ from a compact convex set $K \subset \mathbb{C}^n$ is a uniform limit of entire maps $\mathbb{C}^n \to Y$.*

# The Oka principle for tame families of Stein manifolds

## Theorem (main)

*Assume that*

- *$T$ is a finite CW complex or a countable locally compact CW-complex of finite dimension,*
- *$X$ is a smooth manifold,*
- *$\mathcal{J} = \{J_t\}_{t \in T}$ is a tame family of smooth Stein structures on $X$, and*
- *$Y$ is an Oka manifold.*

*Then, any continuous map $f : Z = T \times X \to Y$ is homotopic to a $\mathcal{J}$-holomorphic map $F : Z \to Y$.*

*If in addition $f$ is $\mathcal{J}$-holomorphic on a neighbourhood of a $\mathcal{J}$-convex subset $K \subset Z$ and on $Q \times X$, where $Q$ is a closed subset of $T$, then $F$ can be chosen to approximate $f$ in the fine topology on $K$ and to agree with $f$ on $Q \times X$.*

# A theorem of Hamilton with parameters

In the proof, we need the following result.

## Theorem

*Assume that $(X, J)$ be a Stein manifold, $\Omega \Subset X$ is a relatively compact, smoothly bounded, strongly pseudoconvex domain, $T$ is a topological space, and $\mathcal{J} = \{J_t\}_{t \in T}$ is a continuous family of smooth integrable complex structures on $\overline{\Omega}$ such that for some $t_0 \in T$, $J_{t_0}$ is the restriction of $J$ to $\overline{\Omega}$.*

*Then there exist a neighbourhood $T_0 \subset T$ of $t_0$ and a continuous family of smooth diffeomorphisms*

$$\Phi_t : \overline{\Omega} \to \overline{\Omega}_t = \Phi_t(\overline{\Omega}) \subset X, \quad t \in T_0,$$

*such that $\Phi_t$ is a biholomorphic map from $(\Omega, J_t)$ onto $(\Omega_t, J_{t_0})$ for every $t \in T_0$ and $\Phi_{t_0} = \mathrm{Id}_\Omega$.*

# Sketch of proof

Choose a holomorphic embedding $f : X \hookrightarrow \mathbb{C}^N$ and a holomorphic retraction $\tau : O \to f(X)$ from a neighbourhood of $f(X)$. Then,

$$\alpha_t := \bar{\partial}_{J_t}(f|_{\overline{\Omega}})$$

is a continuous family of $\bar{\partial}_{J_t}$-closed $(0,1)$-forms on $\overline{\Omega}$, with $\alpha_{t_0} = 0$.

By Greene and Krantz 1982, for every $t \in T$ close to $t_0$ we can solve

$$\bar{\partial}_{J_t}\phi_t = \alpha_t \quad \text{on } \overline{\Omega}$$

with continuous dependence on $t$ such that $\phi_{t_0} = 0$. The map

$$f_t = f - \alpha_t : \overline{\Omega} \to \mathbb{C}^N$$

is then $J_t$-holomorphic on $\Omega$. Hence, the map

$$\Phi_t = f^{-1} \circ \tau \circ f_t : \overline{\Omega} \to X$$

is $(J_t, J)$-holomorphic, continuous in $t$, and $\Phi_{t_0} = \mathrm{Id}_\Omega$. It follows that $F_t$ is $(J_t, J)$-biholomorphic on $D$ for $t$ close to $t_0$.

# Proof of the main theorem

Assume that $\{J_t\}_{t\in T}$ is a family of Stein structures on $X$ as in the main theorem, $K \subset Z = T \times X$ is a $\mathcal{J}$-convex subset, and $f : Z \to Y$ is a continuous map which is $\mathcal{J}$-holomorphic on a neighbourhood $U \supset K$.

Fix $t_0 \in T$. Pick a smoothly bounded domain $\Omega \Subset X$ containing $K_{t_0}$. There is a neighbourhood $T_0 \subset T$ of $t_0$ and a family of $(J_t, J_{t_0})$-biholomorphic maps

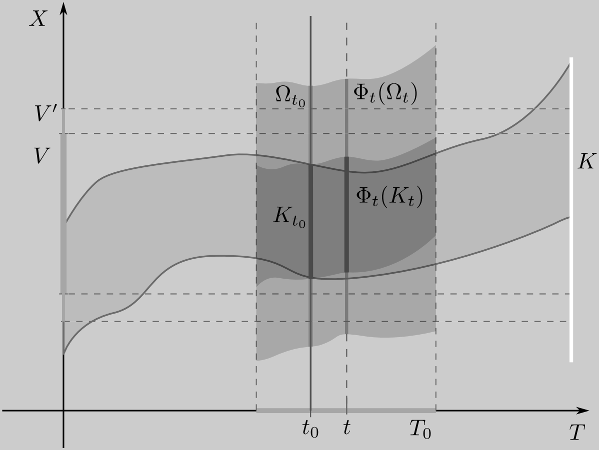
$$\Phi_t : \Omega \to \Phi_t(\Omega), \quad t \in T_0.$$

We may assume that $K_t \subset \Omega$ for all $t \in T_0$. Then, for every $t \in T_0$ the map

$$h_t = f_t \circ \Phi_t^{-1} : \Phi_t(\Omega) \to Y$$

is $J_{t_0}$-holomorphic on a neighbourhood of the compact set $\widetilde{K}_t = \Phi_t(K_t)$.

# The sets $\Phi_t(K_t) \subset \Phi_t(\Omega)$




The shaded areas depict the sets $\widetilde{K}_t = \Phi_t(K_t) \subset \Phi_t(\Omega)$ for $t \in T_0$.

# Proof of the main theorem, 2

We may assume that $T_0$ is compact and contained in $\mathbb{R}^n \subset \mathbb{C}^n$.

The compact subset $\widetilde{K} \subset T_0 \times X$ with fibres $\widetilde{K}_t = \Phi_t(K_t)$ $(t \in T_0)$ is $\mathcal{O}(\mathbb{C}^n \times X)$-convex.

Since $Y$ is an Oka manifold, we can approximate the maps $h_t = f_t \circ \Phi_t^{-1} : \Phi_t(\Omega) \to Y$ uniformly on $\widetilde{K}_t$ by $J_{t_0}$-holomorphic maps $\tilde{h}_t : X \to Y$ depending continuously on $t \in T_0$.

Then, $\tilde{f}_t = \tilde{h}_t \circ \Phi_t : \Omega \to Y$ is $J_t$-holomorphic for every $t \in T_0$.

This is a step in an inductive construction which leads in the limit to an $\mathcal{J}$-holomorphic map $F : T \times X \to Y$.

# The Oka principle for vector bundles on tame families

Assume that $T$, $X$, and $\mathcal{J} = \{J_t\}_{t \in T}$ are as in the main theorem (so $\mathcal{J}$ is tame).

A complex vector bundle $E \to Z = T \times X$ or rank $r$ is said to be $\mathcal{J}$-holomorphic is it has a vector bundle atlas whose transition maps are $\mathcal{J}$-holomorphic maps to $GL_r(\mathbb{C})$.

## Theorem

*Under the above assumptions, the following hold.*

- *Every topological vector bundle on $Z = T \times X$ is isomorphic to a $\mathcal{J}$-holomorphic vector bundle.*
- *Every pair of $\mathcal{J}$-holomorphic vector bundles which are topologically isomorphic are also isomorphic as $\mathcal{J}$-holomorphic vector bundles.*
- *The isomorphism classes of $\mathcal{J}$-holomorphic line bundles are in bijective correspondence with $H^2(Z, \mathbb{Z})$.*

# The Teichmüller family $V(g,n) \to T(g,n)$

The Teichmüller space $T(X)$ of a Riemann surface $X$ is finite dimensional iff

$$X = \widehat{X} \setminus \{x_1, \ldots, x_n\}$$

is a compact Riemann surface $\widehat{X}$ of genus $g$ with $n$ punctures. Such $X$ is said to be of finite conformal type and we write $T(X) = T(g,n)$. The universal family

$$\pi : \widehat{V}(g,n) \to T(g,n)$$

is a holomorphic submersion whose fibre over $t \in T(g,n)$ is the compact Riemann surface $(\widehat{X}, J_t)$ with the complex structure $J_t$ determined by $t$, and with $n$ distinguished holomorphic sections

$$s_1, \ldots, s_n : T(g,n) \to \widehat{V}(g,n)$$

with pairwise disjoint images representing the punctures. The open subset

$$V(g,n) = \widehat{V}(g,n) \setminus \bigcup_{i=1}^{n} s_i(T(g,n)) \xrightarrow{\pi} T(g,n)$$

is the universal family of $n$-punctured compact Riemann surfaces of genus $g$.

# The universal family $V(g,n)$ for $n \geq 1$ is Stein

Behrs and Ehrenpreis 1964 $T(g,n)$ is a Stein manifold.

Bers embedding theorem $T(g,n)$ can be holomorphically realised as a bounded contractible domain in some $\mathbb{C}^N$. If $2g+n \geq 3$ then $N = 3g-3+n$.

Marković 2018 If $g \geq 2$ then $T(g,n)$ is not a convex domain in $\mathbb{C}^{3g-3+n}$.

## Theorem (F. 2025)

*Let $\pi : V(g,n) \to T(g,n)$ denote the Teichmüller submersion and $n \geq 1$.*

- *$V(g,n)$ is a Stein manifold.*
- *Holomorphic fibrewise algebraic functions are dense in $\mathcal{O}(V(g,n))$, and they provide a proper embedding of $V(g,n)$ in a Euclidean space over any relatively compact open subset $U \Subset T(g,n)$.*
- *A relative Oka principle holds for fibrewise algebraic maps from $V(g,n)$ to any flexible algebraic manifold.*

https://arxiv.org/abs/2511.07963, December 2025

THANK YOU FOR YOUR ATTENTION

and

ALL THE BEST TO GRAZIANO!