Runge and Mergelyan theorems for families of open Riemann surfaces

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Abstract

We develop the existence and approximation theory for holomorphic maps from families of complex structures on smooth open surfaces to any Oka manifold.

Along the way, we prove Runge and Mergelyan approximation theorems and Weierstrass interpolation theorem on families of open Riemann surfaces.

As an application, we construct families of directed holomorphic immersions and of conformal minimal immersions to Euclidean spaces.

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Preliminaries

Let X be a smooth open orientable surface.

A compact set K in X is said to be Runge if $X \setminus K$ has no holes. Such a set is holomorphically convex in the Riemann surface (X, J) for any complex structure J on X.

Let $l \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. We let B (the parameter space) be a locally compact and paracompact Hausdorff space if $l = 0$, and a manifold of class $\mathscr{C}^l$ (possibly with boundary) if $l > 0$.

We say that a function $f : B \times X \to \mathbb{C}$ is of class $\mathscr{C}^{l,k}$ if it has l derivatives in the B variable followed by k derivatives in the X variable, and they are continuous. The same definition applies to related objects such as complex structures, maps to a complex manifold, vector bundles, etc.

Almost complex structures

A complex structure on X is given by an endomorphism J of TX satisfying $J^2 = -Id$. Thus, J is a section of the vector bundle $T^*X \otimes TX \to X$ whose fibre over $x \in X$ is the space of $\mathbb{R}$-linear maps $T_xX \mapsto T_xX$.

A differentiable function $f : X \to \mathbb{C}$ is said to be J-holomorphic if the Cauchy–Riemann equation $df_x \circ J_x = \sqrt{-1}\, df_x$ holds for every $x \in X$.

Assuming that J is of Hölder class $\mathcal{C}^{(k,\alpha)}$ for some $k \in \mathbb{Z}_+$ and $0 < \alpha < 1$, there is an atlas $\{(U_i, \phi_i)\}$ of open sets $U_i \subset X$ with $\bigcup_i U_i = X$ and J-holomorphic charts $\phi_i : U_i \to \phi_i(U_i) \subset \mathbb{C}$ of class $\mathcal{C}^{(k+1,\alpha)}(U_i)$.

Hence, J determines on X the structure of a Riemann surface, denoted (X, J), which is $\mathcal{C}^{k+1}$ compatible with the smooth structure on X.

My first result extends the classical Runge–Behnke–Stein approximation theorem on open Riemann surfaces (Runge 1885, Behnke and Stein 1949), combined with the Weierstrass interpolation theorem (Weierstrass 1885, Florack 1948), to families of complex structures on a smooth open surface.

Runge theorem on families of open Riemann surfaces

Theorem

Assume that

- B *and* X *are as above,*
- $\{J_b\}_{b\in B}$ *is a family of complex structures on* X *of class* $\mathcal{C}^{l,(k,\alpha)}(B\times X)$ $(k\in\mathbb{Z}_+,\ 0\le l\le k+1,\ 0<\alpha<1)$,
- K *is a compact Runge subset of* X,
- A *is a closed discrete subset of* X,
- $U\subset B\times X$ *is an open set containing* $B\times(K\cup A)$, *and*
- $f:U\to\mathbb{C}$ *is a function of class* $\mathcal{C}^{l,0}(U)$ *such that* $f_b=f(b,\cdot\,)$ *is* J_b*-holomorphic on* $U_b=\{x\in X:(b,x)\in U\}$ *for every* $b\in B$.

Then, $f\in\mathcal{C}^{l,k+1}(U)$ *and there is a function* $F\in\mathcal{C}^{l,k+1}(B\times X)$ *satisfying:*

(a) *The function* $F_b=F(b,\cdot\,):X\to\mathbb{C}$ *is* J_b*-holomorphic for every* $b\in B$.

(b) F *approximates* f *in the fine* $\mathcal{C}^{l,k+1}$*-topology on* $B\times K$.

(c) F_b-f_b *vanishes to order* $r\le k+1$ *at every point* $a\in A$ *for every* $b\in B$.

Mergelyan theorem

A more sophisticated approximation theorem was proved by Mergelyan 1951.

Its original version says that a continuous function on a compact Runge set $K \subset \mathbb{C}$, which is holomorphic in the interior of K, is a uniform limit on K of holomorphic polynomials.

Mergelyan's theorem was extended to open Riemann surfaces by Bishop 1958, with different proofs and generalizations by Sakai 1972, Scheinberg 1978, Gauthier 1979, and others. Results on approximation by rational function on the plane without poles in K were obtained by Vitushkin 1966, 1967.

Mergelyan theorem on families of open Riemann surfaces

Theorem

Assume that

- X *is a smooth oriented surface without boundary,*
- B *is a locally compact and paracompact Hausdorff space,*
- $\{J_b\}_{b\in B}$ *is a continuous family of complex structures of class* $\mathcal{C}^\alpha$ *on* X*,*
- K *is compact set in* X *whose holes (if any) have diameter* $\geq c > 0$*,*
- A *is a finite subset of* $\mathring{K}$*, and*
- $f : B \times K \to \mathbb{C}$ *is a continuous function such that* $f_b = f(b,\cdot) : K \to \mathbb{C}$ *is* J_b*-holomorphic on the interior* $\mathring{K}$ *for every* $b \in B$*.*

Given a continuous function $\epsilon : B \to (0,+\infty)$*, there is a continuous function* F *on a neighbourhood* $U \subset B \times X$ *of* $B \times K$ *such that for every* $b \in B$*:*

(a) *the function* $F_b = F(b,\cdot) : U_b \to \mathbb{C}$ *is* J_b*-holomorphic,*

(b) $\sup_{x\in K} |F_b(x) - f_b(x)| < \epsilon(b)$*, and*

(c) $F_b - f_b$ *vanishes to order* 1 *in every point* $a \in A$*.*

Oka manifolds

A complex manifold Y is called an Oka manifold if maps $X \to Y$ from any Stein space X satisfy all forms of the Oka principle:

(a) Every continuous map $f : X \to Y$ is homotopic to a holomorphic map.

(b) If in addition f is holomorphic on a compact $\mathcal{O}(X)$-convex subset $K \subset X$ and on a closed complex subvariety X' of X, then the homotopy from f to a holomorphic map $F : X \to Y$ can be chosen to consist of maps with the same properties which approximate f on K and agree with f on X'.

(c) A similar statement holds for families of maps depending continuously on a parameter in a compact Hausdorff space.

Theorem (F., 2005-9)

A complex manifold Y is an Oka manifold iff it satisfies the Convex Approximation Property (CAP): Every holomorphic map $f : K \to Y$ from a compact convex set $K \subset \mathbb{C}^n$ is a uniform limit of entire maps $\mathbb{C}^n \to Y$.

Methods used in the proof

Approximation and gluing techniques for sprays of holomorphic maps $X \to Y$ from a given Stein manifold X. They are inspired by the works of Grauert 1958 (when Y is homogeneous) and Gromov 1989 (when Y is elliptic).

The CAP property of Y ensures the existence of approximations on small convex bumps attached to a strongly pseudoconvex domain in X. This enlarges the domain of holomorphicity. The change of topology of the domain is effected by a Mergelyan-type theorem on special handlebodies with totally real core.

An important ingredient in the proof is the theorem of Siu 1976: every closed Stein subvariety X of a complex space Z admits a basis of open Stein neighbourhoods. We also use the following extension of Siu's theorem.

Theorem (F., 2005)

Let X be a closed Stein subvariety of a complex space Z and K be a compact subset of Z such that $K \cap X$ is $\mathcal{O}(X)$-convex and K is $\mathcal{O}(\Omega)$-convex in an open Stein neighbourhood $\Omega \subset Z$. Then, $K \cup X$ has a basis of open Stein neighborhoods $V \subset Z$ such that K is $\mathcal{O}(V)$-convex.

Examples and properties of Oka manifolds

- Oka–Weil–Cartan $\mathbb{C}^n$ is Oka; Oka 1939 $\mathbb{C}^*$ is Oka.
- Grauert 1957 Every complex homogeneous manifold is Oka.
- Gromov 1989 Every elliptic complex manifold is Oka.
- Lárusson 2005 A model category for Oka theory.
- F. 2006 A complex manifold is Oka iff it satisfies CAP.

  The class of Oka manifolds is invariant under holomorphic fibre bundle projections with Oka fibres; in particular, under covering maps.
- F. 2017 An Oka manifold Y admits a strongly dominating surjective holomorphic map $f : \mathbb{C}^{\dim Y} \to Y$. If Y is compact algebraically elliptic then f can be chosen regular. Hence, no compact complex manifold of general type is Oka.
- Kusakabe 2021 If a complex manifold Y is a union of Zariski open Oka domains, then Y is Oka.
- Kusakabe 2024 The complement $\mathbb{C}^n \setminus K$ of any compact polynomialy convex set $K \subset \mathbb{C}^n$ for $n > 1$ is Oka. The same holds in any Stein manifold Y with Varolin's density property (complete holomorphic vector fields on Y densely generate the Lie algebra of holomorphic vector fields).

Oka manifolds and metric positivity

Campana and Winkelmann 2015 showed that every projective Oka manifold is special in the sense of Campana.

There is a close relationship between Kobayashi hyperbolicity and metric negativity. A hermitian manifold with negative holomorphic sectional curvature is hyperbolic (Grauert and Reckziegel 1965, Wu 1967,...).

Hence, it is natural to ask whether the Oka property is related to metric positivity. Evidence for this comes from the Frankel Conjecture, solved affirmatively by Mori 1979 and Siu and Yau 1980, saying that a compact Kähler manifold of positive holomorphic bisectional curvature is biholomorphic to a complex projective space, and hence is Oka. This was generalized by Mok 1988 who classified compact Kähler manifolds of nonnegative holomorphic bisectional curvature. Mok's theorem, together with results from Oka theory, implies:

Corollary (of Mok's theorem)

Every compact Kähler manifold of nonnegative holomorphic bisectional curvature is an Oka manifold.

Oka tubes in ample line bundles

Theorem (Kusakabe & F., 2024)

Let E be a holomorphic line bundle on a compact complex manifold X. Assume that for each point $x \in X$ there exists a divisor $D \in |E|$ whose complement $X \setminus D$ is a Stein neighbourhood of x with the density property.

Given a semipositive hermitian metric h on E, the disc bundle $\Delta_h(E) = \{e \in E : |e|_h < 1\}$ is an Oka manifold while $E \setminus \overline{\Delta_h(E)}$ is hyperbolic.

Hence, the zero section of E admits a basis of Oka neighbourhoods.

Every line bundle $E \to X$ as in the theorem is ample. We call this property the polarized density property (PDP) of the polarized manifold (X, E).

We show that every ample line bundle on a rational homogeneous manifold X of dimension > 1 satisfies PDP. By Borel–Remmert 1962, such X is a flag manifold. This class includes all projective spaces and Grassmannians.

The Oka principle for families of open Riemann surfaces

Assume the following:

- B is a finite CW complex if $l = 0$, or a manifold of class $\mathcal{C}^l$ if $l > 0$.
- X is a smooth open surface and $\pi : B \times X \to B$ is the projection.
- $\{J_b\}_{b \in B}$ is a family of complex structures on X of class $\mathcal{C}^{l,(k,\alpha)}(B \times X)$ for some $k \in \mathbb{Z}_+,\ l \le k+1,\ 0 < \alpha < 1$.
- $K \subset B \times X$ is a closed subset such that $\pi|_K : K \to B$ is proper, and for every $b \in B$ the fibre K_b is a compact Runge set in X, possibly empty.
- Y is an Oka manifold endowed with a distance function dist_Y.
- $f : B \times X \to Y$ is a continuous map, and there is an open set $U \subset B \times X$ containing K such that $f_b = f(b, \cdot) : X \to Y$ is J_b-holomorphic on U_b for every $b \in B$. (Such f is said to be X-holomorphic on U.)

The main theorem

Theorem (main)

Given a continuous function $\epsilon : B \to (0, +\infty)$, there are a neighbourhood $U' \subset U$ of K and a homotopy $f_t : B \times X \to Y$ $(t \in I = [0,1])$ satisfying the following conditions.

(i) $f_0 = f$.

(ii) $f_{t,b} = f_t(b, \cdot\,) : X \to Y$ *is* J_b*-holomorphic on* $U'_b \supset K_b$ *for every* $b \in B$.

(iii) $\sup_{x \in K_b} \mathrm{dist}_Y(f_t(b,x), f(b,x)) < \epsilon(b)$ *for every* $b \in B$ *and* $t \in I$.

(iv) *The map* $F = f_1$ *is such that* $F_b = F(b, \cdot\,) : X \to Y$ *is* J_b*-holomorphic for every* $b \in B$.

(v) *If* Q *is a closed subset of* B *and* $U_b = X$ *for all* $b \in Q$*, then the homotopy* $f_{t,b}$ *can be chosen to be fixed for every* $b \in Q$*, so* $F = f$ *on* $Q \times X$.

The main theorem, the case $l > 0$

Theorem (continued)

Assume in addition that the following hold:

- *B is a manifold of class $\mathcal{C}^l$, $l > 0$.*
- *Q is a closed $\mathcal{C}^l$ submanifold of B.*
- *The family $\{J_b\}_{b \in B}$ is of class $\mathcal{C}^{l,(k,\alpha)}$ where $l \leq k+1$.*
- *The map $f : B \times X \to Y$ is X-holomorphic on a neighbourhood U of K and $f|_U \in \mathcal{C}^{l,0}(U, Y)$.*

Then $f|_U \in \mathcal{C}^{l,k+1}(U, Y)$, and there is a homotopy $f_t : B \times X \to Y$ $(t \in I)$ which is of class $\mathcal{C}^{l,k+1}$ on a neighbourhood of K, it satisfies conditions (i)–(v), and every f_t approximates f in the fine $\mathcal{C}^{l,k+1}$-topology on K.

The analogous results holds for the family $(X \times Z, J_b \times J_Z)$ where (X, J_b) are Riemann surfaces as above and (Z, J_Z) is a fixed Stein manifold.

Complex structures, and the Beltrami equation

The proof uses Oka theory and an extension of a theorem by Ahlfors and Bers 1960 (on quasiconformal maps $\mathbb{C} \to \mathbb{C}$) to domains in open Riemann surfaces.

A Riemannian metric g on a surface X determines a unique conformal structure, and hence a complex structure $J = J_g$ if X is oriented.
In a local coordinate $z = x + \imath y$ on $U \subset X$ we have

$$g = Edx^2 + 2Fdxdy + Gdy^2 = \lambda|dz + \mu d\bar{z}|^2,$$

where $\lambda > 0$ is a positive function and $\mu : U \to \mathbb{D} = \{|\zeta| < 1\}$. Then,

$$[J] = \frac{1}{\sqrt{EG - F^2}} \begin{pmatrix} -F & -G \\ E & F \end{pmatrix} = \begin{pmatrix} -b & -c \\ (b^2+1)/c & b \end{pmatrix}$$

where

$$\delta = EG - F^2 > 0, \quad b = F/\sqrt{\delta}, \quad c = G/\sqrt{\delta} > 0,$$
$$\mu = \frac{1 - c + \imath b}{1 + c + \imath b}, \quad \imath = \sqrt{-1}.$$

The function μ is called the Beltrami coefficient.

Isothermal coordinates

Let $U \subset X$ be an open set. A local diffeomorphism $f : U \to \mathbb{C}$ is conformal from the g-structure on X to the standard conformal structure on $\mathbb{C}$ iff

$$g = h|df|^2 \text{ holds for a positive function } h > 0.$$

A chart f with this property is said to be isothermal for g. Such f is J-holomorphic or J-antiholomorphic. Assume that f is orientation preserving, which amounts to $|f_z| > |f_{\bar{z}}|$. Note that

$$|df|^2 = |f_z dz + f_{\bar{z}} d\bar{z}|^2 = |f_z|^2 \cdot \left| dz + \frac{f_{\bar{z}}}{f_z} d\bar{z} \right|^2.$$

A comparison with

$$g = \lambda |dz + \mu d\bar{z}|^2$$

shows that f is isothermal iff it satisfies the Beltrami equation

$$f_{\bar{z}} = \mu f_z.$$

Semiglobal solutions of the Beltrami equation

Assume that X is an open Riemann surface and $z : X \to \mathbb{C}$ is a holomorphic immersion. Given a domain $\Omega \Subset X$ and a function $\mu \in \mathcal{C}^{(k,\alpha)}(\Omega, \mathbb{D})$, we denote by J_μ the associated complex structure on Ω, with J_0 the initial complex structure on X.

Theorem

Let Ω be a smoothly bounded relatively compact domain in X. For any $k \in \mathbb{Z}_+$ and $0 < \alpha < 1$ there is $c = c(k, \alpha) > 0$ such that for every $\mu \in \mathcal{C}^{(k,\alpha)}(\Omega, \mathbb{D})$ with $\|\mu\|_{k,\alpha} < c$ there is function $f = f(\mu) \in \mathcal{C}^{(k+1,\alpha)}(\Omega)$ solving the Beltrami equation $f_{\bar z} = \mu f_z$, depending smoothly on μ, with $f(0) = z|_\Omega$.

Furthermore, there is a (J_μ, J_0)-holomorphic diffeomorphism $\Phi_\mu : \Omega \to \Phi_\mu(\Omega) \subset X$ of class $\mathcal{C}^{(k+1,\alpha)}$, depending smoothly on μ near 0, such that Φ_0 is the identity map on Ω.

The outline of proof

The proof is inspired by Ahlfors and Bers 1960.

We look for a solution of the Beltrami equation $f_{\bar{z}} = \mu f_z$ on Ω in the form

$$f = f(\mu) = z|_{\Omega} + P(\phi), \quad \phi \in \mathcal{C}^{(k,\alpha)}(\Omega).$$

Here, $P : \mathcal{C}^{(k,\alpha)}(\Omega) \to \mathcal{C}^{(k+1,\alpha)}(\Omega)$ is the Cauchy operator associated to a Cauchy kernel on (X, J_0). Thus, $\phi = 0$ corresponds to $f(0) = z|_{\Omega}$. We have

$$f_{\bar{z}} = \partial_{\bar{z}} P(\phi) = \phi, \qquad f_z = 1 + \partial_z P(\phi) = 1 + S(\phi),$$

where S is the Beurling operator associated to P (a bounded linear operator on $\mathcal{C}^{(k,\alpha)}(\Omega)$). Inserting in the Beltrami equation $f_{\bar{z}} = \mu f_z$ gives

$$\phi = \mu(S(\phi) + 1) = \mu S(\phi) + \mu \iff (I - \mu S)\phi = \mu.$$

The outline of proof

Assuming that $\|\mu S\| \leq \|\mu\|_{(k,\alpha)}\|S\| < 1$, the operator $I - \mu S$ is invertible on $\mathcal{C}^{(k,\alpha)}(\Omega)$, with the bounded inverse

$$\Theta(\mu) = (I - \mu S)^{-1} = \sum_{j=0}^{\infty} (\mu S)^j.$$

For such μ, the equation for ϕ has the unique solution $\phi = \Theta(\mu)\mu$, and hence the Beltrami equation $f_{\bar{z}} = \mu f_z$ has the solution

$$f(\mu) = z|_\Omega + P(\Theta(\mu)\mu) \in \mathcal{C}^{(k+1,\alpha)}(\Omega).$$

It is not difficult to see that $f(\mu)$ is smooth (analytic) in μ.

If μ is close to 0 then $f(\mu) : \Omega \to \mathbb{C}$ is a J_μ-holomorphic immersion. Lifting $f(\mu)$ with respect to the immersion $z : X \to \mathbb{C}$ gives (J_μ, J_0)-biholomorphisms $\Phi_\mu : \Omega \to \Phi_\mu(\Omega)$ with $z \circ \Phi_\mu = f_\mu$ and Φ_0 the identity map on Ω.

The idea of proof of the main theorem

Assume that $\{J_b\}_{b\in B}$ is a family of complex structures on X as in the main theorem, $K \subset B \times X$ is a closed set with compact Runge fibres K_b, and $f : B \times X \to Y$ is a continuous map of class $\mathcal{C}^l(B)$ such that $f_b = f(b,\cdot)$ is J_b-holomorphic on a neighbourhood $U_b \supset K_b$ for every $b \in B$.

Fix $b_0 \in B$. Pick a smoothly bounded domain $\Omega \Subset X$ containing K_{b_0}. There is a neighbourhood $B_0 \subset B$ of b_0 and a family of (J_b, J_{b_0})-biholomorphic maps $\Phi_b : \Omega \to \Phi_b(\Omega)$ of class $\mathcal{C}^{l,(k+1,\alpha)}(B_0 \times \Omega)$. We may assume that $K_b \subset \Omega$ for all $b \in B_0$. Then, the map $h_b = f_b \circ \Phi_b^{-1} : \Phi_b(\Omega) \to Y$ is J_{b_0}-holomorphic on a neighbourhood of the compact Runge set $\widetilde{K}_b = \Phi_b(K_b)$ for $b \in B_0$.

We may assume that $B_0 \subset \mathbb{R}^n \subset \mathbb{C}^n$, so $B_0 \times X$ is a Levi-flat submanifold of $\mathbb{C}^n \times (X, J_{b_0})$ fibered over B_0. The compact subset $\widetilde{K}$ with fibres $\widetilde{K}_b$ $(b \in B_0)$ is $\mathcal{O}(\mathbb{C}^n \times X)$-convex. Since Y is an Oka manifold, we can approximate the maps h_b uniformly on $\widetilde{K}_b$ by J_{b_0}-holomorphic maps $\tilde{h}_b : X \to Y$ of class $\mathcal{C}^l$ in $b \in B_0$. Then, $\tilde{f}_b = \tilde{h}_b \circ \Phi_b : \Omega \to Y$ is J_b-holomorphic for every $b \in B_0$.

This is a step in an inductive construction which leads in the limit to an X-holomorphic map $F : B \times X \to Y$ of class $\mathcal{C}^{l,k+1}$.

The Gunning–Narasimhan theorem in families

By Gunning and Narasimhan 1967, every open Riemann surface X admits a holomorphic immersion $f : X \to \mathbb{C}$, i.e., it is a Riemann domain over $\mathbb{C}$. By using our main result, together with a Mergelyan theorem for families, we obtain the following generalization to families.

Theorem

Let B and X be as in the main theorem and $\{J_b\}_{b\in B}$ be a family of complex structures on X of class $\mathcal{C}^{l,(k,\alpha)}(B \times X)$ $(k \geq 1,\ 0 \leq l \leq k+1,\ 0 < \alpha < 1)$. Then there is a function $f : B \times X \to \mathbb{C}$ of class $\mathcal{C}^{l,k+1}$ such that

$$f_b = f(b, \cdot\,) : X \to \mathbb{C} \text{ is a } J_b\text{-holomorphic immersion for every } b \in B.$$

It follows that

- $\rho_b = |f_b|^2 : X \to \mathbb{R}_+$ is strongly subharmonic on $X_b = (X, J_b)$,
- $\theta_b = df_b$ is a nowhere vanishing J_b-holomorphic 1-form trivializing the canonical bundle K_{X_b} for every $b \in B$.

Directed holomorphic immersions

We describe a more general result concerning directed holomorphic immersions.

A connected compact projective manifold $Y \subset \mathbb{CP}^{n-1}$ determines the punctured complex cone

$$A = \{(z_1, \dots, z_n) \in \mathbb{C}^n_* : [z_1 : \cdots : z_n] \in Y\}$$

which is smooth and connected, its closure $\overline{A} = A \cup \{0\} \subset \mathbb{C}^n$ is an algebraic subvariety of $\mathbb{C}^n$, and A is an Oka manifold if and only if Y is an Oka manifold.

Let X be a connected open Riemann surface and θ be a nowhere vanishing holomorphic 1-form on X. A holomorphic immersion $h : X \to \mathbb{C}^n$ is said to be an A-immersion if its complex derivative with respect to any local holomorphic coordinate on X takes its values in A. Clearly, this holds iff the holomorphic map $f = dh/\theta : X \to \mathbb{C}^n$ assume values in A.

Directed holomorphic immersions

Conversely, a holomorphic map $f : X \to A$ satisfying the period vanishing conditions

$$\int_C f\theta = 0 \quad \text{for all closed curves } C \subset X$$

integrates to a holomorphic A-immersion $h : X \to \mathbb{C}^n$ by setting

$$h(x) = v + \int_{x_0}^{x} f\theta, \quad x \in X$$

for any $x_0 \in X$ and $v \in \mathbb{C}^n$.

Since $f\theta$ is a holomorphic 1-form, it suffices to verify the above period vanishing conditions on a basis of the homology group $H_1(X, \mathbb{Z}) \cong \mathbb{Z}^r$, a free abelian group of some rank $r \in \mathbb{Z}_+ \cup \{\infty\}$.

Families of directed holomorphic immersions

Theorem

Assume that

- $A \subset \mathbb{C}^n_*$ *is a smooth Oka cone,*
- X, B, *and* $\{J_b\}_{b\in B}$ *are as in the main theorem, and*
- $\{\theta_b\}_{b\in B}$ *is a family of nowhere vanishing* J_b*-holomorphic* 1*-forms on* X.

Given a continuous map $f_0 : B \times X \to A$, *there is map* $h : B \times X \to \mathbb{C}^n$ *of class* $\mathcal{C}^{l,k+1}$ *such that* $h_b = h(b, \cdot)$ *a* J_b*-holomorphic* A*-immersion for every* $b \in B$, *and the map* $f : B \times X \to A$ *defined by* $f(b, \cdot) = dh_b/\theta_b$ *is homotopic to* f_0.

By taking $A = \mathbb{C}^n_*$, we obtain an h-principle for families of J_b-holomorphic immersions $h_b : X \to \mathbb{C}^n$. The special case $n = 1$ is the generalized Gunning–Narasimhan theorem.

Holomorphic null curves and conformal minimal immersions

Another case of major interest is the null quadric

$$\mathbb{A} = \big\{(z_1, \ldots, z_n) \in \mathbb{C}^n_* : z_1^2 + z_2^2 + \cdots + z_n^2 = 0\big\}, \quad n \geq 3.$$

Holomorphic $\mathbb{A}$-immersions are called holomorphic null curves in $\mathbb{C}^n$.

The real and imaginary part of a holomorphic null immersion $X \to \mathbb{C}^n$ are conformal harmonic (minimal) immersions $X \to \mathbb{R}^n$. Conversely, a conformal minimal immersion $X \to \mathbb{R}^n$ is locally the real part of a holomorphic null curve.

Corollary

Let X, $\{J_b\}_{b\in B}$, and $\{\theta_b\}_{b\in B}$ be as in the previous theorem. Given a continuous map $f_0 : B \times X \to \mathbb{A}$, there is a map $u : B \times X \to \mathbb{R}^n$ of class $\mathcal{C}^{l,k+1}$ such that $u_b = u(b, \cdot\,) : (X, J_b) \to \mathbb{R}^n$ is a nonflat conformal minimal immersion for every $b \in B$, and the X-holomorphic map $f : B \times X \to \mathbb{A}$, defined by $f(b, \cdot\,) = \partial_{J_b} u_b / \theta_b$ for all $b \in B$, is homotopic to f_0.

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