Every projective Oka manifold is elliptic

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Abstract

It was proved by Gromov 1989 that every elliptic complex manifold Y satisfies the parametric h-principle for holomorphic maps from any Stein manifolds X. A manifold Y satisfying the conclusion of his theorem is now called an

Oka manifold

Gromov asked whether the converse also holds. The first counterexamples among open (non-compact) complex manifolds were found only recently.

In this work, we show that the converse holds for every projective manifold.

F. F., F. Lárusson, Every projective Oka manifold is elliptic. March 2025.
https://arxiv.org/abs/2502.20028

Motivation: Flexibility versus rigidity in complex geometry

A central question of complex geometry is to understand the space $\mathcal{O}(X, Y)$ of holomorphic maps $X \to Y$ between a pair of complex manifolds. Are there many maps, or few maps? Which properties can they have?

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There are many holomorphic maps $\mathbb{C} \to \mathbb{C}$ and $\mathbb{C} \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, but there are no nonconstant algebraic maps $\mathbb{C} \to \mathbb{C}^*$ (fundamental theorem of algebra) or holomorphic maps $\mathbb{C} \to \mathbb{C} \setminus \{0, 1\}$ (Picard's theorem). Manifolds with the latter property are called hyperbolic. Similarly, the complement of five lines in general position in $\mathbb{CP}^2$ is hyperbolic. A very generic projective hypersurface of high enough degree is hyperbolic. Hyperbolic manifolds have been studied since 1967 when S. Kobayashi introduced his intrinsic pseudometric on complex manifold. A vast majority of complex manifolds are close to hyperbolic. Hyperbolicity is a major obstruction to solving global complex analytic problems.

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On the opposite side, Oka theory studies special complex manifolds, Oka manifolds, which admit many holomorphic maps from all Stein manifolds, i.e., closed complex submanifolds of affine spaces $\mathbb{C}^N$. Oka theory gives solutions to a variety of complex analytic problems in the absence of topological obstructions.

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OKA THEORY = h-PRINCIPLE IN COMPLEX GEOMETRY

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Every vector bundle on a Stein manifold X is the pullback of a universal bundle on a suitable Grassmann manifold Y (the classifying space) by a map $X \to Y$. Holomorphic maps give rise to holomorphic bundles on X, and homotopies of maps induce isomorphic bundles. Hence, Grauert's results essentially follow from:

Grauert 1958 Every complex Lie group and, more generally, every complex homogeneous manifold admits many holomorphic maps from any Stein space X.

What is the right way to interpret the phrase *many maps*?

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(b) If in addition f is holomorphic on a compact $\mathcal{O}(X)$-convex subset $K \subset X$ and on a closed complex subvariety X' of X, then the homotopy from f to a holomorphic map $F : X \to Y$ can be chosen to consist of maps with the same properties which approximate f on K and agree with f on X'.

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For $Y = \mathbb{C}$, these properties hold by the Oka–Weil–Cartan theorem.

F., 2005-9 A complex manifold Y is an Oka manifold iff it satisfies the Convex Approximation Property (CAP): Every holomorphic map $f : K \to Y$ from a compact convex set $K \subset \mathbb{C}^n$ is a uniform limit of entire maps $\mathbb{C}^n \to Y$.

Basic properties of Oka manifolds

- For every Stein manifold X and Oka manifold Y, the inclusion

$$\mathcal{O}(X, Y) \hookrightarrow \mathcal{C}(X, Y)$$

  is a weak homotopy equivalence. It is a homotopy equivalence if X of finite geometric type (Lárusson 2015).

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- An Oka manifold Y admits a dominating holomorphic map $f : \mathbb{C}^{\dim Y} \to Y$ with any given centre $f(0) = y \in Y$. Domination means that df_0 maps $\mathbb{C}^{\dim Y}$ onto $T_y Y$. Hence, f is locally biholomorphic at most points.

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- A Riemann surface is Oka iff it is not hyperbolic.
- Every Oka manifold is Liouville (every bounded psh function is constant).
- Kobayashi & Ochiai 1977 A compact complex manifold Y of Kodaira dimension $\kappa(Y) = \dim Y$ is not dominable by $\mathbb{C}^{\dim Y}$, so it is not Oka.

Recall that $\kappa(Y) \in \{-\infty, 0, 1, \dots, \dim Y\}$ is the smallest integer k such that $\dim H^0(Y, K_Y^d) \le cd^k$ for some $c > 0$. Here, $K_Y = \wedge^{\dim Y} T^* Y$.

Examples of Oka manifolds

- Oka–Weil–Cartan $\mathbb{C}^n$ is Oka; Oka 1939 $\mathbb{C}^*$ is Oka.
- Grauert 1957 Every complex homogeneous manifold is Oka.
- Gromov 1989 Every elliptic complex manifold is Oka.
- F. 2002 Every subelliptic complex manifold is Oka.
- F. 2006 The class of Oka manifolds is invariant under holomorphic fibre bundle projections with Oka fibres.
- Kusakabe 2021 If a complex manifold Y is a union of Zariski open Oka domains, then Y is Oka.
- Kusakabe 2024 The complement $\mathbb{C}^n \setminus K$ of any compact polynomially convex set $K \subset \mathbb{C}^n$ for $n > 1$ is Oka. The same holds in any Stein manifold Y with Varolin's density property.
- Kaliman & Zaidenberg, 2024 Every smooth cubic projective hypersurface is A-elliptic, and hence Oka.
- Banecki 2024 Every rational projective manifold is A-elliptic, hence Oka.

Elliptic manifolds

Gromov 1989 A complex manifold Y is elliptic if it admits a dominating holomorphic spray, i.e., a holomorphic map $s : E \to Y$ from the total space of a holomorphic vector bundle $E \to Y$ such that for all $y \in Y$,

$$s(0_y) = y \quad \text{and} \quad s : E_y \to Y \text{ is a submersion at } 0_y \in E_y.$$

Gromov 1989: Every elliptic complex manifold is an Oka manifold.

A detailed proof was given by Jasna Prezelj & F., Math. Ann. 2000 & 2002.

The modern proof consists of two parts. The first one is the implication

$$\text{elliptic} \Longrightarrow \text{h-Runge approximation} \Longrightarrow \text{CAP}$$

This uses dominating sprays and the Oka–Weil theorem.

The second and main part is the implication CAP $\Longrightarrow$ OKA (F. 2006).

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Consider complements $\mathbb{C}^n \setminus K$ of compacts sets $K \subset \mathbb{C}^n$.

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Theorem (Lárusson & F., 2025)

Every projective Oka manifold is elliptic.

Scheme of proof

Let $Y \subset \mathbb{CP}^n$ be a projective manifold and $\pi : E \to Y$ an algebraic vector bundle. For $y \in Y$ let $E_y = \pi^{-1}(y)$. Along the zero section E_0 of E we have

$$TE|E_0 \cong E \oplus TY.$$

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Let V be an algebraic vector field on E which vanishes on E_0. There is a neighbourhood $\Omega \subset E$ of E_0 such that for any $e \in \Omega$, the flow $\phi_\tau(e)$ of V with $\phi_0(e) = e$ exists for all $0 \leq \tau \leq 1$. The holomorphic map

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Since E is a 1-convex manifold with the exceptional subvariety E_0, the Oka principle proved by Jasna Prezelj gives a global holomorphic spray $\tilde{s} : E \to Y$ which agrees with s to the second order along E_0. Hence, $\tilde{s}$ is a dominating holomorphic spray on Y.

Proof, 1

Denote by $z = [z_0 : z_1 : \cdots : z_n]$ the homogeneous coordinates on $\mathbb{CP}^n$.

Set $\Lambda_\alpha = \{z_\alpha = 0\}$ for $\alpha = 0, 1, \ldots, n$, and let $U_\alpha = \mathbb{CP}^n \setminus \Lambda_\alpha \cong \mathbb{C}^n$ be the affine chart with coordinates $(z_0/z_\alpha, \ldots, z_n/z_\alpha)$. Let $x = (x_1, \ldots, x_n)$ with $x_i = z_i/z_0$ be affine coordinates on U_0. Consider a polynomial vector field

$$W(x) = \sum_{i=1}^{n} V_i(x)\partial_{x_i}$$

on $U_0 \cong \mathbb{C}^n$ which is tangential to $Y \cap U_0$. By Serre's Theorem A, finitely many such vector fields span TY at every point $x \in Y \cap U_0$.

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We associate to W the horizontal vector field

$$V(x,t) = \sum_{i=1}^{n} t\, V_i(x)\partial_{x_i}$$

on the trivial line bundle $U_0 \times \mathbb{C} \cong \mathbb{C}^{n+1}$, where $t \in \mathbb{C}$ is the fibre coordinate.

Proof, 2

Let $\mathbb{U} \to \mathbb{CP}^n$ be the universal bundle. We claim that, for all sufficiently large $k > 0$, V extends to an algebraic vector field on the line bundle $L = \mathbb{U}^k$.

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For every $\alpha = 0, 1, \ldots, n$, we have a line bundle trivialisation $\theta_\alpha : L|U_\alpha \overset{\cong}{\to} U_\alpha \times \mathbb{C}$ with transition maps $\theta_{\alpha,\beta} = \theta_\alpha \circ \theta_\beta^{-1}$ on $(U_\alpha \cap U_\beta) \times \mathbb{C}$ given by

$$\theta_{\alpha,\beta}([z], t) = \big([z], (z_\alpha / z_\beta)^k t\big), \quad 0 \le \alpha, \beta \le n.$$

In particular, $\theta_{\alpha,0}([z], t) = \big([z], (z_\alpha / z_0)^k t\big).$

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We analyse the behaviour of V near the hyperplane $\Lambda_0 \setminus \Lambda_\alpha$ for $\alpha = 1, \ldots, n$. Replacing the coordinate $x_\alpha = z_\alpha / z_0$ by $1/x_\alpha = z_0/z_\alpha$, the vector field V has the same form with rational coefficient $V_i(x)$ having poles along the hyperplane $\{x_\alpha = 0\} = \{z_0 = 0\}$. In these coordinates, $\theta_{\alpha,0}(x, t) = \left(x, x_\alpha^{-k} t\right)$ and

$$D\,\theta_{\alpha,0}(x, t) = \begin{pmatrix} I_n & 0 \\ b & x_\alpha^{-k} \end{pmatrix},$$

where I_n is the identity $n \times n$ matrix and $b = (0, \ldots, -k x_\alpha^{-k-1} t, \ldots, 0)$.

Proof, 3

Hence, the vector field $V' = (\theta_{\alpha,0})_* V$ on the chart $L|U_\alpha$ for $x \in U_0 \cap U_\alpha$ equals

$$V'(x,t) = D\,\theta_{\alpha,0}(x,t)V(x,t) = \sum_{i=1}^{n} tV_i(x)\partial_{x_i} + (-k)t^2 x_\alpha^{-k-1} V_\alpha(x)\partial_{t'}.$$

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In terms of the new fibre variable $t' = x_\alpha^{-k}t$ (so $t = x_\alpha^{k}t'$) we have

$$V'(x,t') = \sum_{i=1}^{n} t' x_\alpha^{k} V_i(x)\partial_{x_i} - k(t')^2 x_\alpha^{k-1} V_\alpha(x)\partial_{t'}.$$

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Applying this argument for every $\alpha = 1, \ldots, n$, we see that for $k > 0$ big enough the vector field V extends to the line bundle $L = \mathbb{U}^k$ and it vanishes on $L_0 \cup (L|\Lambda_0)$.

Proof, 4

Since V vanishes on the zero section L_0 of L, there is a neighbourhood $\Omega \subset L$ of L_0 such that the flow $\phi_\tau(e)$ of V with $\phi_0(e) = e \in \Omega$ exists for all $\tau \in [0, 1]$. The map

$$s = \pi \circ \phi_1 : \Omega \to \mathbb{CP}^n$$

is a local holomorphic spray on $\mathbb{CP}^n$.

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Set $L|Y = \pi^{-1}(Y)$. For every $e = (x, t) \in L|Y$ we have $d\pi_e V(e) = tW(x) \in T_x Y$. Hence, $s : \Omega \cap L|Y \to Y$ is a local holomorphic spray on Y.

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Since V is linear in the fibre variable t, we have

$$s(x, \delta t) = \pi \circ \phi_1(x, \delta t) = \pi \circ \phi_\delta(x,t)$$

for every $(x,t) \in \Omega$ and $0 \le \delta \le 1$. Differentiating on δ at $\delta = 0$ and noting that $\frac{d}{d\delta}\Big|_{\delta=0} \phi_\delta(x,t) = V(x,t)$ and $d\pi_{(x,t)} V(x,t) = tW(x)$ gives

$$Vds_x(t) := \frac{d}{d\delta}\Big|_{\delta=0} s(x, \delta t) = tW(x) \in T_x Y, \quad x \in Y \cap U_0, \ t \in \mathbb{C}.$$

Proof, 5

In order to find a local dominating spray on Y, we proceed as follows. For $\alpha \in \{0, 1, \dots, n\}$, set $Y_\alpha = Y \cap U_\alpha$. Choose $m \in \mathbb{N}$ such that the tangent bundle TY_α is pointwise generated by m polynomial vector fields

$$W_j^\alpha(x) = \sum_{i=1}^{n} V_{i,j}^\alpha(x) \partial_{x_i}, \quad j = 1, \dots, m$$

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In order to find a local dominating spray on Y, we proceed as follows. For $\alpha \in \{0, 1, \ldots, n\}$, set $Y_\alpha = Y \cap U_\alpha$. Choose $m \in \mathbb{N}$ such that the tangent bundle TY_α is pointwise generated by m polynomial vector fields

$$W_j^\alpha(x) = \sum_{i=1}^{n} V_{i,j}^\alpha(x) \partial_{x_i}, \quad j = 1, \ldots, m$$

on U_α for every $\alpha \in \{0, 1, \ldots, n\}$. For $k \in \mathbb{N}$ big enough, this gives an algebraic vector field V^α on the vector bundle $E^\alpha = m\mathbb{U}^k$ of the form

$$V^\alpha(x, t) = \sum_{i=1}^{n} \sum_{j=1}^{m} t_j \, V_{i,j}^\alpha(x) \partial_{x_i},$$

where $x = (z_0/z_\alpha, \ldots, z_n/z_\alpha)$ and $t = (t_1, \ldots, t_m)$ are coordinates on the chart $E^\alpha | U_\alpha \cong \mathbb{C}^n \times \mathbb{C}^m$.

In other charts $E^\alpha | U_\beta$ for $\beta \neq \alpha$, V^α is of the same form but also has a vertical component of size $|t|^2$.

Proof, 6

Consider the vector bundle

$$\pi : E = E^0 \oplus E^1 \oplus \cdots \oplus E^n = (n+1)m\mathbb{U}^k \to \mathbb{CP}^n.$$

The polynomial vector field V^α on E^α extends to a regular algebraic vector field on E which vanishes on E_0. Indeed, we first extend it horizontally to each summand $E^\beta|U_\alpha$ of $E|U_\alpha$ for $\beta \neq \alpha$. As seen before, for k big enough the resulting vector field on $E|U_\alpha$ extends to an algebraic vector field V^α on E.

Consider the algebraic vector field $V = \sum_{\alpha=0}^n V^\alpha$ on E. Since V vanishes on E_0, there is a neighbourhood $\Omega \subset E$ of E_0 with convex fibres such that the flow $\phi_\tau(e)$ of V exists for any $e \in \Omega$ and $\tau \in [0,1]$. Consider the holomorphic spray

$$s = \pi \circ \phi_1 : \Omega \to \mathbb{CP}^n.$$

We claim that $s : \Omega \cap \pi^{-1}(Y) \to Y$ is dominating. To see this, consider V on a chart $E|U_\alpha$. Let $\alpha = 0$ for simplicity of notation.

Proof, 7

In the affine coordinates $x = (z_1/z_0, \dots, z_n/z_0)$ on U_0 and fibre coordinates $t = (t^0, t^1, \dots, t^n)$ on $E|U_0$, where $t^\alpha = (t_1^\alpha, \dots, t_m^\alpha)$ are fibre coordinates on the direct summand $E^\alpha|U_0$ of $E|U_0$, we have

$$V(x,t) = \sum_{\alpha=0}^{n} \sum_{i=1}^{n} \sum_{j=1}^{m} t_j^\alpha \, V_{i,j}^\alpha(x) \partial_{x_i} + \widetilde{V}(x,t) = \Theta(x,t) + \widetilde{V}(x,t)$$

where $|\widetilde{V}(x,t)| = O(|t|^2)$. Since $\Theta(x,t)$ is linear in t, its flow ψ_τ satisfies

$$\pi \circ \psi_1(x, \delta t) = \pi \circ \psi_\delta(x,t),$$

and hence the vertical derivative of the spray $\tilde{s} = \pi \circ \psi_1 : \Omega \to \mathbb{CP}^n$ equals

$$Vd\tilde{s}_x(t) = \sum_{\alpha=0}^{n} \sum_{j=1}^{m} t_j^\alpha \, W_j^\alpha(x).$$

Since the vectors $W_j^0(x)$ for $j = 1, \dots, m$ span $T_x Y$ for every $x \in Y_0$, $\tilde{s}$ is dominating over Y_0. Since $|\widetilde{V}(x,t)| = O(|t|^2)$, the flow ϕ_τ of V satisfies

$$\phi_\tau(x,t) = \psi_\tau(x,t) + O(|t|^2) \text{ as } |t| \to 0 \text{ and } \tau \in [0,1].$$

Proof, 8

Hence, the spray $s = \pi \circ \phi_1 : \Omega \to \mathbb{CP}^n$ satisfies

$$Vds_x(x,t) = Vd\tilde{s}_x(x,t) \quad \text{for } x \in U_0,$$

so it is dominating on $Y_0 = Y \cap U_0$. The same argument holds on every chart $E|U_\alpha$, which proves that the spray $s : \Omega \cap \pi^{-1}(Y) \to Y$ is dominating.

We now replace the bundle $E \to \mathbb{CP}^n$ by $E|Y = \pi^{-1}(Y) \to Y$, and we replace the spray s by its restriction $s : E|Y \cap \Omega \to Y$.

Note that $E = (n+1)m\mathbb{U}^k|Y$ is a Griffiths negative bundle, hence a 1-convex manifold with the exceptional subset $E_0 \cong Y$ (Grauert 1962).

If Y is an Oka manifold, the Oka principle of Prezelj 2010, 2016 gives a holomorphic map $\tilde{s} : E \to Y$ which agrees with the spray s to the second order along the zero section E_0 of E. Hence, $\tilde{s}$ is dominating, so Y is elliptic.

Remark. Our proof gives holomorphic sprays. This is the best possible since there exist projective Oka manifolds which are not algebraically elliptic.

Local dominating sprays on positive vector bundles

Proposition

Assume that Y is a compact complex manifold and the vector bundle $\pi : E \to Y$ is generated by global holomorphic sections (this holds if E is sufficiently positive). If there is a local dominating holomorphic spray $s : U \to Y$ defined on a neighbourhood $U \subset E$ of the zero section E_0, then Y is a complex homogeneous manifold.

Proof. The vertical derivative $Vds|E_0 : E \to TY$ is a vector bundle epimorphism. Given a holomorphic section $\xi : Y \to E$, the map

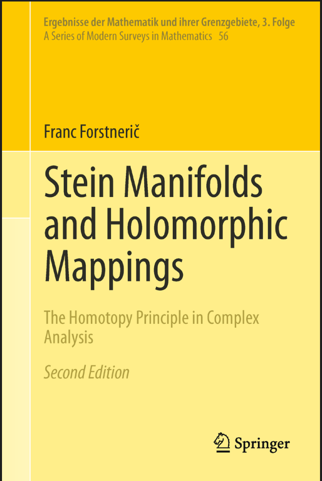
$$Y \ni y \mapsto V_\xi(y) := Vds_y(\xi(y)) \in T_y Y$$

is a holomorphic vector field on Y.

Applying this argument to sections $\xi_1, \ldots, \xi_m : Y \to E$ generating E gives holomorphic vector fields $V_1, \ldots, V_m$ on Y spanning the tangent bundle TY.

Since Y is compact, their flows are complex 1-parameter subgroups of the holomorphic automorphism group $\mathrm{Aut}(Y)$, a finite-dimensional complex Lie group. The spanning property implies that $\mathrm{Aut}(Y)$ acts transitively on Y.

A commercial



The main results on the Oka–Grauert–Gromov theory up to 2017 are presented in my monograph.

Subsequent developments up to 2023 are summarised in my survey

Recent developments on Oka manifolds. Indag. Math., 34(2) (2023) 367–417.

There have been many developments since 2023, the story is ongoing...

THANK YOU FOR YOUR ATTENTION