The nonhomogeneous Cauchy–Riemann equation on families of open Riemann surfaces

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Abstract

We show that on any smoothly bounded relatively compact domain Ω in a smooth open surface X, the nonhomogeneous Cauchy–Riemann equation (the $\bar{\partial}$-equation) can be solved for very general families of complex structures $\{J_b\}_{b\in B}$ on $\overline{\Omega}$ of some Hölder class, with a gain of one derivative in the space variable and without any loss of regularity in the parameter $b \in B$.

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We show that on any smoothly bounded relatively compact domain Ω in a smooth open surface X, the nonhomogeneous Cauchy–Riemann equation (the $\bar\partial$-equation) can be solved for very general families of complex structures $\{J_b\}_{b\in B}$ on $\overline{\Omega}$ of some Hölder class, with a gain of one derivative in the space variable and without any loss of regularity in the parameter $b \in B$.

An application is the Oka principle for complex line bundles on families of open Riemann surfaces.

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Assuming that X is a surface and J is of local Hölder class $\mathcal{C}^{(k,\alpha)}$ $(k \in \mathbb{Z}_+,\ 0 < \alpha < 1)$, there is an atlas $\{(U_i, \phi_i)\}$ of open sets $U_i \subset X$ with $\bigcup_i U_i = X$ and J-holomorphic charts $\phi_i : U_i \to \phi_i(U_i) \subset \mathbb{C}$ of class $\mathcal{C}^{(k+1,\alpha)}(U_i)$.

Hence, J determines on X the structure of a Riemann surface, denoted (X, J), whose underlying smooth structure is $\mathcal{C}^{(k+1,\alpha)}$ compatible with the given smooth structure on X.

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On a smooth manifold X of dimension $2n \geq 4$, the same is true if J is formally integrable and of Hölder class $\mathcal{C}^{(k,\alpha)}$ for some $k = 1, 2, \ldots$ and $0 < \alpha < 1$ (Newlander and Nirenberg 1957, Nijenhuis and Woolf 1963, Kohn 1963, Malgrange 1969, Webster 1989, Hörmander 1990...).

Complex structures and the Beltrami equation

A smooth Riemannian metric g on a surface X determines a unique conformal structure, and hence a complex structure $J = J_g$ if X is oriented. Conversely, every complex structure J arises in this way. Metrics g, g' determine the same conformal structure iff $g' = \lambda g$ for a positive function $\lambda : X \to (0, \infty)$.

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In a smooth local coordinate $z = x + \imath y$ ($\imath = \sqrt{-1}$) on $U \subset X$ we have

$$g = Edx^2 + 2Fdxdy + Gdy^2 = \lambda |dz + \mu d\bar{z}|^2$$

where $\lambda > 0$ and $\mu : U \to \mathbb{D} = \{|\zeta| < 1\}$ is the Beltrami coefficient.

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where $\lambda > 0$ and $\mu : U \to \mathbb{D} = \{|\zeta| < 1\}$ is the Beltrami coefficient. Then,

$$[J] = \frac{1}{\sqrt{EG - F^2}} \begin{pmatrix} -F & -G \\ E & F \end{pmatrix} = \begin{pmatrix} -b & -c \\ (b^2+1)/c & b \end{pmatrix}$$

where

$$\delta = EG - F^2 > 0, \quad b = F/\sqrt{\delta}, \quad c = G/\sqrt{\delta} > 0,$$

$$\mu = \frac{1 - c + \imath b}{1 + c + \imath b}.$$

Isothermal coordinates

Let $U \subset X$ be an open set. A local diffeomorphism $f : U \to \mathbb{C}$ is conformal from the g-structure on X to the standard conformal structure on $\mathbb{C}$ iff

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A chart f with this property is said to be isothermal for g. Such f is J-holomorphic or J-antiholomorphic. Assume that f is in the same orientation class as $z : U \to \mathbb{C}$, which amounts to $|f_z| > |f_{\bar{z}}|$. Note that

$$|df|^2 = |f_z dz + f_{\bar{z}} d\bar{z}|^2 = |f_z|^2 \cdot \left| dz + \frac{f_{\bar{z}}}{f_z} d\bar{z} \right|^2.$$

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A comparison with

$$g = \lambda |dz + \mu d\bar{z}|^2$$

shows that f is isothermal iff it satisfies the Beltrami equation

$$f_{\bar{z}} = \mu f_z .$$

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By Gunning and Narasimhan 1967, X admits a J_0-holomorphic immersion $z = u + \imath v : X \to \mathbb{C}$. Its differential $dz = du + \imath dv$ is a nowhere vanishing holomorphic 1-form on X trivialising the canonical bundle $T^*X = K_X$, $|dz|^2 = du^2 + dv^2$ is a Riemannian metric on X determining J_0, $\frac{\imath}{2} dz \wedge d\bar{z} = du \wedge dv$ is the associated area form, and $d\sigma = du\, dv$ is the surface measure on X.

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The function z provides a local holomorphic coordinate on X at every point. Any Riemannian metric g on X is globally of the form

$$g = \lambda |dz + \mu d\bar{z}|^2$$

for some functions $\lambda : X \to (0, \infty)$ and $\mu : X \to \mathbb{D}$. Conversely, any function $\mu : X \to \mathbb{D}$ determines a Riemannian metric $g_\mu = |dz + \mu d\bar{z}|^2$, and hence a complex structure J_μ on X, with J_0 the given reference structure.

A version of Ahlfors–Bers–Hamilton theorem

Assume that (X, J_0) is an open Riemann surface and $z : X \to \mathbb{C}$ is a holomorphic immersion. Given a domain $\Omega \Subset X$ and a function $\mu \in \mathcal{C}^{(k,\alpha)}(\Omega, \mathbb{D})$, we denote by J_μ the associated complex structure on Ω, with J_0 the initial structure.

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Theorem (the mapping theorem)

Let Ω be a relatively compact domain in X with $\mathcal{C}^{(k+1,\alpha)}$ boundary $(k \in \mathbb{Z}_+,\ 0 < \alpha < 1)$. There exists $c = c(k, \alpha) > 0$ such that for every $\mu \in \mathcal{C}^{(k,\alpha)}(\Omega, \mathbb{D})$ with $\|\mu\|_{k,\alpha} < c$ there is function $f = f(\mu) \in \mathcal{C}^{(k+1,\alpha)}(\Omega)$ solving the Beltrami equation $f_{\bar z} = \mu f_z$, depending analytically on μ, with $f(0) = z|_\Omega$.

Note that $f(\mu) : \Omega \to \mathbb{C}$ is J_μ-holomorphic, and an immersion if μ is small enough.

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Corollary

For every complex structure J of class $\mathcal{C}^{(k,\alpha)}$ on Ω which is sufficiently close to J_0 there is a (J, J_0)-biholomorphic map $\Phi_J : \Omega \to \Phi_J(\Omega) \subset X$ of class $\mathcal{C}^{(k+1,\alpha)}(\Omega)$ depending analytically on J, with $\Phi_{J_0} = \mathrm{Id}_\Omega$.

The tools: Cauchy kernel and Cauchy–Green formula

Let X be an open Riemann surface and $z = u + \imath v : X \to \mathbb{C}$ a holomorphic immersion. Set $D_X = \{(x,x) : x \in X\} \subset X \times X$.

Scheinberg 1978 There is a meromorphic 1-form on $X \times X$ of the form

$$\omega(q,x) = \xi(q,x)dz(x) \quad \text{for } q, x \in X$$

which is holomorphic on $X \times X \setminus D_X$ and for each $q \in X$, $\omega(q,\cdot\,)$ has a simple pole at q with residue 1.

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$$\xi(q, x) = \frac{1}{z(x) - z(q)} + h(q, x), \quad h \text{ holomorphic on } U.$$

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Given a relatively compact smoothly bounded domain $\Omega \Subset X$, $f \in \mathcal{C}^1(\overline{\Omega})$ and $q \in \Omega$, we have the Cauchy–Green formula

$$\begin{aligned} f(q) &= \frac{1}{2\pi\imath} \int_{x \in b\Omega} f(x)\, \omega(q,x) - \frac{1}{2\pi\imath} \int_{x \in \Omega} \overline{\partial} f(x) \wedge \omega(q,x) \\ &= \frac{1}{2\pi\imath} \int_{x \in b\Omega} f(x)\, \xi(q,x) dz(x) - \frac{1}{\pi} \int_{x \in \Omega} f_{\bar z}(x) \xi(q,x) d\sigma(x). \end{aligned}$$

The Cauchy–Green and the Beurling operator

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$$P(\phi)(q) = -\frac{1}{\pi} \int_{x \in \Omega} \phi(x) \xi(q, x) d\sigma(x)$$

solving the $\bar{\partial}$-equation

$$\partial_{\bar{z}} P(\phi) = \phi$$

and the Beurling operator (a Calderón–Zygmund type singular integral operator)

$$S(\phi)(q) = \partial_z P(\phi)(q) = -\frac{1}{\pi} P.V. \int_{\Omega} \phi(x) \partial_{z(q)} \xi(q, x) d\sigma(x).$$

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For every $k \in \mathbb{Z}_+$ and $0 < \alpha < 1$,

$$P : \mathcal{C}^{(k,\alpha)}(\Omega) \to \mathcal{C}^{(k+1,\alpha)}(\Omega) \quad \text{and} \quad S : \mathcal{C}^{(k,\alpha)}(\Omega) \to \mathcal{C}^{(k,\alpha)}(\Omega)$$

are bounded linear operators. (In S, we use a bounded linear extension operator $\mathcal{C}^{(k,\alpha)}(\Omega) \to \mathcal{C}_0^{(k,\alpha)}(\Omega')$ with $\Omega \Subset \Omega' \Subset X$.) In any local holomorphic coordinate, they differ from the standard operators on $\mathbb{C}$ by a smoothing operator.

Proof of the mapping theorem

The proof is inspired by Ahlfors and Bers 1960 who considered the planar case.

We look for a solution of the Beltrami equation $f_{\bar{z}} = \mu f_z$ on Ω in the form

$$f = f(\mu) = z|_{\Omega} + P(\phi), \quad \phi \in \mathcal{C}^{(k,\alpha)}(\Omega).$$

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Inserting in the Beltrami equation $f_{\bar{z}} = \mu f_z$ gives

$$\phi = \mu(S(\phi) + 1) = \mu S(\phi) + \mu \iff (I - \mu S)\phi = \mu.$$

Proof of the mapping theorem, 2

Assuming that $\|\mu S\| \leq \|\mu\|_{(k,\alpha)}\|S\| < 1$, the operator $I - \mu S$ is invertible on $\mathcal{C}^{(k,\alpha)}(\Omega)$, with the bounded inverse

$$\Theta(\mu) = (I - \mu S)^{-1} = \sum_{j=0}^{\infty} (\mu S)^j.$$

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For such μ, the equation $(I - \mu S)\phi = \mu$ has the unique solution $\phi = \Theta(\mu)\mu$, and hence the Beltrami equation $f_{\bar{z}} = \mu f_z$ has the solution

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It follows that $f(\mu)$ is analytic in μ.

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It follows that $f(\mu)$ is analytic in μ.

If μ is close to 0 then $f(\mu) : \Omega \to \mathbb{C}$ is a J_μ-holomorphic immersion. Lifting $f(\mu)$ with respect to the immersion $z : X \to \mathbb{C}$ gives (J_μ, J_0)-biholomorphisms $\Phi_\mu : \Omega \to \Phi_\mu(\Omega)$ with $z \circ \Phi_\mu = f_\mu$ and Φ_0 the identity map on Ω.

The $\bar{\partial}$-equation for a family of complex structures

Theorem (main)

Assume that (X, J) is an open Riemann surface and $\Omega \Subset X$ is a relatively compact domain with $\mathcal{C}^{(k+1,\alpha)}$ boundary for some $k \in \mathbb{Z}_+$ and $0 < \alpha < 1$.

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$$B_c = \{\mu \in \mathcal{C}^{(k,\alpha)}(\overline{\Omega}) : \|\mu\|_{k,\alpha} < c\}.$$

There exists $c > 0$ such that for any map

$$B_c \ni \mu \mapsto \beta_\mu \in \Gamma^{(k,\alpha)}(\overline{\Omega}, T^{*(0,1)}_{J_\mu}\overline{\Omega})$$

of class $\mathcal{C}^l$, $l \in \{0, 1, \ldots, \infty, \omega\}$, there is a function $f \in \mathcal{C}^{l,(k+1,\alpha)}(B_c \times \overline{\Omega})$ such that for every $\mu \in B_c$ the function $f_\mu = f(\mu, \cdot\,) : \overline{\Omega} \to \mathbb{C}$ satisfies

$$\bar{\partial}_{J_\mu} f_\mu = \beta_\mu.$$

Proof, 1

Let $z : X \to \mathbb{C}$ be a J-holomorphic immersion. By the mapping theorem, there is $c > 0$ and a function $h : B_c \times \overline{\Omega} \to \mathbb{C}$ such that for every $\mu \in B_c$,

$$h_\mu = h(\mu, \cdot\,) : \overline{\Omega} \to \mathbb{C}$$

is a J_μ-holomorphic immersion of class $\mathcal{C}^{(k+1,\alpha)}(\overline{\Omega})$, analytic in μ, with $h_0 = z$. The differentials dh_μ and $\overline{dh_\mu} = d\overline{h_\mu}$ span the bundles $T^{*(1,0)}_{J_\mu}(X)$ and $T^{*(0,1)}_{J_\mu}(X)$.

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We shall express the equation $\bar\partial_{J_\mu} f_\mu = \beta_\mu$ as a nonhomogeneous Beltrami equation.

For $\mu \in B_c$ we can uniquely express any complex 1-form β on $\overline{\Omega}$ as

$$\beta = A dz + B d\bar z = A_\mu dh_\mu + B_\mu d\overline{h_\mu}.$$

We now express A_μ and B_μ in terms of the functions A, B, μ, and

$$g_\mu := (h_\mu)_z \in \mathcal{C}^{(k,\alpha)}(\overline{\Omega}), \quad g_\mu \neq 0.$$

Proof, 2

We have that

$$dh_\mu = (h_\mu)_z\, dz + (h_\mu)_{\bar{z}}\, d\bar{z} = g_\mu\, dz + \mu g_\mu\, d\bar{z},$$

where the second identity follows from $(h_\mu)_{\bar{z}} = \mu (h_\mu)_z$.

Proof, 2

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$$\begin{aligned}
A dz + B d\bar z &= A_\mu dh_\mu + B_\mu d\overline{h_\mu} \\
&= A_\mu (g_\mu dz + \mu g_\mu d\bar z) + B_\mu (\overline{\mu g_\mu}\, dz + \overline{g_\mu}\, d\bar z) \\
&= (A_\mu g_\mu + B_\mu \overline{\mu g_\mu}) dz + (A_\mu \mu g_\mu + B_\mu \overline{g_\mu}) d\bar z.
\end{aligned}$$

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Equating the coefficients gives

$$A = A_\mu g_\mu + B_\mu \overline{\mu g_\mu}, \qquad B = A_\mu \mu g_\mu + B_\mu \overline{g_\mu}.$$

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$$dh_\mu = (h_\mu)_z\,dz + (h_\mu)_{\bar z}\,d\bar z = g_\mu\,dz + \mu g_\mu\,d\bar z,$$

where the second identity follows from $(h_\mu)_{\bar z} = \mu (h_\mu)_z$. Hence

$$\begin{aligned} A dz + B d\bar z &= A_\mu dh_\mu + B_\mu d\overline{h_\mu} \\ &= A_\mu (g_\mu dz + \mu g_\mu d\bar z) + B_\mu (\overline{\mu g_\mu}\,dz + \overline{g_\mu}\,d\bar z) \\ &= (A_\mu g_\mu + B_\mu \overline{\mu g_\mu}) dz + (A_\mu \mu g_\mu + B_\mu \overline{g_\mu}) d\bar z. \end{aligned}$$

Equating the coefficients gives

$$A = A_\mu g_\mu + B_\mu \overline{\mu g_\mu}, \qquad B = A_\mu \mu g_\mu + B_\mu \overline{g_\mu}.$$

Solving these equations on A_μ and B_μ we obtain

$$A_\mu = \frac{A - \bar\mu B}{(1 - |\mu|^2) g_\mu}, \qquad B_\mu = \frac{B - \mu A}{(1 - |\mu|^2)\overline{g_\mu}}.$$

Proof, 3

For the 1-form $df = f_z dz + f_{\bar{z}} d\bar{z} = f_{h_\mu} dh_\mu + f_{\overline{h_\mu}} d\overline{h_\mu}$ we have

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Inserting these quantities in the above expression for B_μ shows that the nonhomogeneous Cauchy–Riemann equation

$$\bar\partial_{J_\mu} f = u_\mu d\overline{h_\mu} \iff f_{\overline{h_\mu}} = \frac{f_{\bar z} - \mu f_z}{(1-|\mu|^2)\overline{g_\mu}} = u_\mu$$

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$$f_{\bar z} - \mu f_z = (1-|\mu|^2)\overline{g_\mu}\, u_\mu. \tag{1}$$

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Let P and S be the Cauchy–Green and the Beurling operator associated to the immersion $z : X \to \mathbb{C}$.

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Since $(I - \mu S)^{-1} \in \mathrm{Lin}(\mathcal{C}^{k,\alpha}(\overline{\Omega}))$ is analytic in μ and $(1 - |\mu|^2)\overline{g_\mu}\, u_\mu \in \mathcal{C}^{l,(k,\alpha)}(B_c \times \overline{\Omega})$, the map $(\mu, x) \mapsto \phi_\mu(x)$ belongs to $\mathcal{C}^{l,(k,\alpha)}(B_c \times \overline{\Omega})$. Finally, the solution of (1) is

$$f_\mu = P(\phi_\mu),$$

and the map $(\mu, x) \to f_\mu(x)$ belongs to $\mathcal{C}^{l,(k+1,\alpha)}(B_c \times \overline{\Omega})$.

Solution of the global $\bar{\partial}$-equation in families

Corollary

Assume that

- *B is a paracompact Hausdorff space if $l = 0$ and a $\mathcal{C}^l$ manifold if $l \in \mathbb{N}$,*
- *X is a smooth open orientable surface, and*
- *$\{J_b\}_{b \in B}$ is a family of complex structures of class $\mathcal{C}^{l,(k,\alpha)}$ on a smooth open orientable surface X, where $l, k \in \mathbb{Z}_+$, $l \leq k+1$, $0 < \alpha < 1$.*

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Given a family $\{\beta_b\}_{b\in B}$ *of* $(0,1)$*-forms* $\beta_b \in \Gamma(X, T^{*(0,1)}_{J_b}X)$ *of class* $\mathcal{C}^{l,(k,\alpha)}$, *there is a function* $f : B \times X \to \mathbb{C}$ *of class* $\mathcal{C}^{l,(k+1,\alpha)}$ *satisfying*

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$$\bar{\partial}_{J_b} f(b, \cdot) = \beta_b \text{ on } X \text{ for every } b \in B.$$

The condition $l \leq k+1$ in the corollary is due to the use of the Runge approximation theorem for fibrewise holomorphic functions on families of open Riemann surfaces, which was proved in my paper *Runge and Mergelyan theorems on families of open Riemann surfaces (2024)*.

Vanishing of Dolbeault cohomology

Assume that B, X, and $\mathcal{J} = \{J_b\}_{b \in B}$ are as above, where $\mathcal{J}$ is of class $\mathcal{C}^{l,(k,\alpha)}$ for some $0 \leq l \leq k+1$ and $0 < \alpha < 1$. Denote by $\mathcal{O}$ the sheaf of germs of $\mathcal{J}$-holomorphic functions f of class $\mathcal{C}^l$ on $Z = B \times X$.

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Theorem

$H^q(Z, \mathcal{O}) = 0$ for all $q = 1, 2, \ldots$.

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Proof. Consider the sequence of homomorphisms of sheaves of abelian groups

$$0 \longrightarrow \mathcal{O} \hookrightarrow \mathcal{C}^{l,(k+1,\alpha)} \xrightarrow{\bar\partial} \mathcal{C}^{l,(k,\alpha)}_{(0,1)} \longrightarrow 0$$

where $\bar\partial$ equals $\bar\partial_{J_b}$ on $Z_b = (X, J_b)$ for every $b \in B$. By the main theorem, the sequence is exact. The second and the third sheaf are fine sheaves, so their cohomology groups of order ≥ 1 vanish. It follows that

$$H^1(Z, \mathcal{O}) = \Gamma\big(Z, \mathcal{C}^{l,(k,\alpha)}_{(0,1)}\big)/\bar\partial\,\Gamma\big(Z, \mathcal{C}^{l,(k+1,\alpha)}\big) = 0$$

by the Corollary, and $H^q(Z, \mathcal{O}) = 0$ for $q \ge 2$.

The Oka principle for line bundles on families

Let B, X and $\mathcal{J} = \{J_b\}_{b \in B}$ be as above. Denote by

$$Pic(Z) \cong H^1(Z, \mathcal{O}^*)$$

the set of isomorphism classes of fibrewise holomorphic line bundles on $Z = B \times X$. We have the following Oka principle.

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Theorem

Every topological complex line bundle on $Z = B \times X$ is isomorphic to a fibrewise holomorphic line bundle, and any two fibrewise holomorphic line bundles on Z which are topologically isomorphic are also isomorphic as fibrewise holomorphic line bundles. Furthermore,

$$Pic(Z) \cong H^2(Z, \mathbb{Z}).$$

Proof

Let $\sigma(f) = e^{2\pi \imath f}$. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \hookrightarrow & \mathcal{O} & \xrightarrow{\sigma} & \mathcal{O}^* & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
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Since $\mathcal{C}$ is a fine sheaf, we have $H^q(Z, \mathcal{C}) = 0$ for all $q \in \mathbb{N}$. We proved that $H^q(Z, \mathcal{O}) = 0$ for all $q \in \mathbb{N}$. Hence, the relevant part of the associated long exact sequence of cohomology groups gives

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^1(Z, \mathcal{O}^*) & \longrightarrow & H^2(Z; \mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \\
0 & \longrightarrow & H^1(Z, \mathcal{C}^*) & \longrightarrow & H^2(Z; \mathbb{Z}) & \longrightarrow & 0
\end{array}$$

Thus, all arrows in the central square are isomorphisms. Since $Pic(Z) \cong H^1(Z, \mathcal{O}^*)$ and $H^1(Z, \mathcal{C}^*)$ is the set of isomorphisms classes of topological line bundles on Z, the theorem follows.

THANK YOU FOR YOUR ATTENTION