Modelling families of complex curves and minimal surfaces

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Abstract

We develop the existence and approximation theory for holomorphic maps from families of complex structures on smooth open surfaces to Oka manifolds.

Along the way, we prove Runge and Mergelyan approximation theorems and Weierstrass interpolation theorem on families of open Riemann surfaces.

As an application, we construct families of directed holomorphic immersions and conformal minimal immersions to Euclidean spaces for a given family of conformal structures on a smooth oriented surface.

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Almost complex structures

An almost complex structure on a smooth orientable surface X is an endomorphism J of the tangent bundle TX satisfying $J^2 = -Id$. Every such J is determined by the choice of an orientation and a Riemannian metric on X.

A differentiable function $f : X \to \mathbb{C}$ is said to be J-holomorphic if the Cauchy–Riemann equation $df_x \circ J_x = \sqrt{-1}\, df_x$ holds for every $x \in X$.

Assuming that J is of Hölder class $\mathcal{C}^{(k,\alpha)}$ for some $k \in \mathbb{Z}_+$ and $0 < \alpha < 1$, there is an atlas $\{(U_i, \phi_i)\}$ of open sets $U_i \subset X$ with $\bigcup_i U_i = X$ and J-holomorphic charts $\phi_i : U_i \to \phi_i(U_i) \subset \mathbb{C}$ of class $\mathcal{C}^{(k+1,\alpha)}(U_i)$. Hence, J determines on X the structure of a Riemann surface, denoted (X, J), which is $\mathcal{C}^{k+1}$ compatible with the smooth structure on X.

On a smooth manifold X of dimension $2n \geq 4$, the same is true if J is formally integrable and of Hölder class $\mathcal{C}^{(k,\alpha)}$ for some $k = 1, 2, \ldots$ and $0 < \alpha < 1$ (Newlander and Nirenberg 1957, Nijenhuis and Woolf 1963, Kohn 1963, Malgrange 1969, Webster 1989,...).

Runge sets, parameter spaces

Let X be a smooth, connected, open, orientable surface.

A compact set K in X is said to be Runge if $X \setminus K$ has no holes.
Such a set is holomorphically convex in the Riemann surface (X, J) for any complex structure J on X.

Let $l \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$. We let T (the parameter space) be a paracompact Hausdorff space if $l = 0$, and a manifold of class $\mathcal{C}^l$ if $l > 0$.

We say that a function $f : T \times X \to \mathbb{C}$ is of class $\mathcal{C}^{l,k}$ if it has l derivatives in the T variable followed by k derivatives in the X variable, and they are continuous. The same definition applies to related objects.

Motivation and the main problem

Assume that X is a smooth orientable surface, T is a topological space, (Y, J_Y) is a complex manifold, and $f_t : X \to Y$ $(t \in T)$ is a continuous family of smooth immersions such that $f_t(X) \subset Y$ is a complex curve in Y for every $t \in T$.

Then, $J_t = f_t^* J_Y$ is a continuous family of complex structure on X such that $f_t : X \to Y$ is (J_t, J_Y)-holomorphic.

Problem

- *Can this be reversed? Given a continuous family $\{J_t\}_{t \in T}$ of complex structures on X, is there a continuous family $f_t : X \to Y$ $(t \in T)$ of nonconstant (J_t, J_Y)-holomorphic maps? Which other properties can be imposed?*
- *Is every family of continuous maps $f_t : X_t \to Y$ $(t \in T)$ homotopic to a family of nonconstant holomorphic maps?*
- *Assuming that f_t is (J_t, J_Y)-holomorphic on an open neighbourhood of a Runge compact $K \subset X$, can we approximate it on $T \times K$ by a family of holomorphic maps $F_t : X_t \to Y$? What about interpolation?*

Enter Oka manifolds

The answer is negative in general even for a single map $X \to Y$. For example, by Picard's theorem there is no nonconstant holomorphic map $\mathbb{C} \to \mathbb{C} \setminus \{0, 1\}$.

A complex manifold X is said to be a Stein manifold if X is biholomorphic to a closed complex submanifold of an affine space $\mathbb{C}^N$. By Behnke and Stein 1949, every open Riemann surface is a Stein manifold (a closed smooth complex curve in $\mathbb{C}^3$).

A complex manifold Y is called an Oka manifold if maps $X \to Y$ from any Stein manifold X satisfy all forms of the Oka principle:

(a) Every continuous map $f : X \to Y$ is homotopic to a holomorphic map.

(b) If in addition f is holomorphic on a compact $\mathcal{O}(X)$-convex subset $K \subset X$ and on a closed complex subvariety X' of X, then the homotopy from f to a holomorphic map $F : X \to Y$ can be chosen to consist of maps with the same properties which approximate f on K and agree with f on X'.

(c) A similar statement holds for families of maps depending continuously on a parameter in a compact Hausdorff space.

Examples and properties of Oka manifolds

- Oka–Weil–Cartan, 1936–1951 $\mathbb{C}^n$ is Oka; Oka 1939 $\mathbb{C}^*$ is Oka.
- Grauert 1957 Every complex homogeneous manifold is Oka.
- Gromov 1989 Every elliptic complex manifold. Detailed proofs and extensions by J. Prezelj and F., 2000-2002.
- F., 2005-9 A complex manifold Y is an Oka manifold iff it satisfies the

  Convex Approximation Property (CAP): Every holomorphic map $f : K \to Y$ from a compact convex set $K \subset \mathbb{C}^n$ is a uniform limit of entire maps $\mathbb{C}^n \to Y$.

  The class of Oka manifolds is invariant under holomorphic fibre bundle projections with Oka fibres; in particular, under covering maps.
- Kusakabe 2021 If a complex manifold Y is a union of Zariski open Oka domains, then Y is Oka.
- Kusakabe 2024 The complement $\mathbb{C}^n \setminus K$ of any compact polynomialy convex set $K \subset \mathbb{C}^n$ for $n > 1$ is Oka. The same holds if $\mathbb{C}^n$ is replaced by any Stein manifold Y with Varolin's density property.

The Oka principle for families of open Riemann surfaces

Assume the following:

- T is a local Euclidean neighbourhood retract if $l = 0$ (this holds in particular if X is an at most countable locally compact CW-complex of finite dimension), and a manifold of class $\mathcal{C}^l$ if $l > 0$.
- X is a smooth open surface.
- $\{J_t\}_{t\in T}$ is a family of complex structures on X of class $\mathcal{C}^{l,(k,\alpha)}(T \times X)$ for some $k \in \mathbb{Z}_+$, $0 \le l \le k+1$, $0 < \alpha < 1$.
- $K \subset T \times X$ is a closed subset whose fibres $K_t = \{x \in X : (t,x) \in K\}$ are compact Runge sets which are upper semicontinuous in $t \in T$.
- Y is an Oka manifold endowed with a distance function dist_Y.
- $f : T \times X \to Y$ is a continuous map, and there is an open set $U \subset T \times X$ containing K such that $f_t = f(t, \cdot\,) : X \to Y$ is J_t-holomorphic on U_t for every $t \in T$. (Such f is said to be X-holomorphic on U.)

The main theorem

Theorem

Given a continuous function $\epsilon : T \to (0, +\infty)$*, there are a neighbourhood* $U' \subset U$ *of* K *and a homotopy* $f_s : T \times X \to Y$ $(s \in I = [0,1])$ *satisfying:*

(i) $f_0 = f$.

(ii) $f_s(t, \cdot\,) : X \to Y$ *is* J_t*-holomorphic on* $U'_t \supset K_t$ *for every* $t \in T$, $s \in I$.

(iii) $\sup_{x \in K_t} \operatorname{dist}_Y(f_s(t,x), f(t,x)) < \epsilon(t)$ *for every* $t \in T$ *and* $s \in I$.

(iv) *The map* $F = f_1 : T \times X \to Y$ *is* X*-holomorphic.*

(v) *If* Q *is a closed subset of* T *and* $U_t = X$ *for all* $t \in Q$*, then* $f_s(t, \cdot\,)$ *can be chosen independent of* $s \in I$ *for every* $t \in Q$*, so* $F = f$ *on* $Q \times X$.

(vi) *If* $Y = \mathbb{C}$ *then* T *can be any Hausdorff paracompact space, so the conclusion holds for the universal family of complex structures on* X.

The main theorem, the case $l > 0$

Theorem (continued)

Assume in addition that the following hold:

- *T is a manifold of class $\mathscr{C}^l$, $l > 0$, and Q is a closed $\mathscr{C}^l$ submanifold of T.*
- *The family $\{J_t\}_{t \in T}$ is of class $\mathscr{C}^{l,(k,\alpha)}$ with $l \leq k+1$.*
- *The map $f : T \times X \to Y$ is X-holomorphic on a neighbourhood U of K and $f|_U \in \mathscr{C}^{l,0}(U, Y)$.*

Then, $f|_U \in \mathscr{C}^{l,k+1}(U, Y)$ and there is a homotopy $f_s : T \times X \to Y$ $(s \in I)$ satisfying conditions (i)–(iv) such that f_s approximates f in the fine $\mathscr{C}^{l,k+1}$-topology on K for every $s \in I$, and $F = f_1 : T \times X \to Y$ is X-holomorphic and of class $\mathscr{C}^{l,k+1}$.

The analogous results holds for the family $(X \times Z, J_t \times J_Z)$ where (X, J_t) are Riemann surfaces as above and (Z, J_Z) is a Stein manifold.

Complex structures and the Beltrami equation

The proof uses Oka theory and an extension of a theorem by Ahlfors and Bers 1960 on quasiconformal maps $\mathbb{C} \to \mathbb{C}$ to domains in open Riemann surfaces.

A Riemannian metric g on a surface X determines a unique conformal structure, and hence a complex structure $J = J_g$ if X is oriented.

In a local coordinate $z = x + \imath y$ ($\imath = \sqrt{-1}$) on $U \subset X$ we have

$$g = Edx^2 + 2Fdxdy + Gdy^2 = \lambda |dz + \mu d\bar{z}|^2$$

where $\lambda > 0$ and $\mu : U \to \mathbb{D} = \{|\zeta| < 1\}$ is the Beltrami coefficient. Then,

$$[J] = \frac{1}{\sqrt{EG - F^2}} \begin{pmatrix} -F & -G \\ E & F \end{pmatrix} = \begin{pmatrix} -b & -c \\ (b^2+1)/c & b \end{pmatrix}$$

where

$$\delta = EG - F^2 > 0, \quad b = F/\sqrt{\delta}, \quad c = G/\sqrt{\delta} > 0,$$
$$\mu = \frac{1 - c + \imath b}{1 + c + \imath b}.$$

Isothermal coordinates

Let $U \subset X$ be an open set. A local diffeomorphism $f : U \to \mathbb{C}$ is conformal from the g-structure on X to the standard conformal structure on $\mathbb{C}$ iff

$$g = h|df|^2 \text{ holds for a positive function } h > 0.$$

A chart f with this property is said to be isothermal for g. Such f is J-holomorphic or J-antiholomorphic. Assume that f is orientation preserving, which amounts to $|f_z| > |f_{\bar{z}}|$. Note that

$$|df|^2 = |f_z dz + f_{\bar{z}} d\bar{z}|^2 = |f_z|^2 \cdot \left| dz + \frac{f_{\bar{z}}}{f_z} d\bar{z} \right|^2.$$

A comparison with

$$g = \lambda |dz + \mu d\bar{z}|^2$$

shows that f is isothermal iff it satisfies the Beltrami equation

$$f_{\bar{z}} = \mu f_z.$$

A version of Ahlfors–Bers–Hamilton theorem

Assume that (X, J_0) is an open Riemann surface and $z : X \to \mathbb{C}$ is a holomorphic immersion. Given a domain $\Omega \Subset X$ and a function $\mu \in \mathcal{C}^{(k,\alpha)}(\Omega, \mathbb{D})$, we denote by J_μ the associated complex structure on Ω, with J_0 the initial complex structure on X.

Theorem

Let Ω be a smoothly bounded relatively compact domain in X. For any $k \in \mathbb{Z}_+$ and $0 < \alpha < 1$ there is a constant $c = c(k, \alpha) > 0$ such that for every $\mu \in \mathcal{C}^{(k,\alpha)}(\Omega, \mathbb{D})$ with $\|\mu\|_{k,\alpha} < c$ there is function $f = f(\mu) \in \mathcal{C}^{(k+1,\alpha)}(\Omega)$ solving the Beltrami equation $f_{\bar{z}} = \mu f_z$, depending smoothly on μ, with $f(0) = z|_\Omega$.

Corollary

For every complex structure J of class $\mathcal{C}^{k,\alpha}$ on Ω ($k \in \mathbb{Z}_+$, $0 < \alpha < 1$) which is sufficiently close to J_0 there is a (J, J_0)-biholomorphic map $\Phi_J : \Omega \to \Phi_J(\Omega) \subset X$ of class $\mathcal{C}^{k+1,\alpha}$ depending smoothly on J, with Φ_{J_0} the identity on Ω.

The Cauchy–Green formula

Let X be an open Riemann surface and $z = u + \imath v : X \to \mathbb{C}$ a holomorphic immersion. There is a meromorphic 1-form on $X \times X$ of the form

$$\omega(q,x) = \xi(q,x)dz(x) \quad \text{for } q,x \in X$$

which is holomorphic on $X \times X \setminus D_X$ and for each $q \in X$, $\omega(q,\cdot\,)$ has a simple pole at q with residue 1. In a neighbourhood $U \subset X \times X$ of D_X,

$$\xi(q,x) = \frac{1}{z(x) - z(q)} + h(q,x), \quad h \text{ holomorphic on } U.$$

Given a relatively compact smoothly bounded domain $\Omega \Subset X$, $f \in \mathcal{C}^1(\overline{\Omega})$ and $q \in \Omega$, we have the Cauchy–Green formula

$$\begin{aligned} f(q) &= \frac{1}{2\pi\imath}\int_{x \in b\Omega} f(x)\,\omega(q,x) - \frac{1}{2\pi\imath}\int_{x\in\Omega} \overline{\partial} f(x) \wedge \omega(q,x) \\ &= \frac{1}{2\pi\imath}\int_{x \in b\Omega} f(x)\,\xi(q,x)dz(x) - \frac{1}{\pi}\int_{x\in\Omega} f_{\bar z}(x)\xi(q,x)d\sigma(x). \end{aligned}$$

The Cauchy–Green and Beurling operators

We have the Cauchy–Green operator

$$P(\phi)(q) = -\frac{1}{\pi}\int_{x\in\Omega}\phi(x)\xi(q,x)d\sigma(x)$$

satisfying

$$\partial_{\bar{z}}P(\phi) = \phi$$

and the Beurling operator

$$S(\phi)(q) = \partial_z P(\phi)(q) = -\frac{1}{\pi}\int_{\Omega}\phi(x)\partial_{z(q)}\xi(q,x)d\sigma(x).$$

For every $k \in \mathbb{Z}_+$ and $0 < \alpha < 1$,

$$P : \mathcal{C}^{k,\alpha}(\Omega) \to \mathcal{C}^{k+1,\alpha}(\Omega) \quad \text{and} \quad S : \mathcal{C}^{k,\alpha}(\Omega) \to \mathcal{C}^{k,\alpha}(\Omega)$$

are bounded linear operators.

The outline of proof

The proof is inspired by Ahlfors and Bers 1960.

We look for a solution of the Beltrami equation $f_{\bar{z}} = \mu f_z$ on Ω in the form

$$f = f(\mu) = z|_\Omega + P(\phi), \quad \phi \in \mathcal{C}^{(k,\alpha)}(\Omega).$$

Here, P is the Cauchy operator associated to a Cauchy kernel on (X, J_0). Thus, $\phi = 0$ corresponds to $f(0) = z|_\Omega$. We have

$$f_{\bar{z}} = \partial_{\bar{z}} P(\phi) = \phi, \qquad f_z = 1 + \partial_z P(\phi) = 1 + S(\phi),$$

where S is the Beurling operator associated to P.

Inserting in the Beltrami equation $f_{\bar{z}} = \mu f_z$ gives

$$\phi = \mu(S(\phi) + 1) = \mu S(\phi) + \mu \iff (I - \mu S)\phi = \mu.$$

The outline of proof

Assuming that $\|\mu S\| \leq \|\mu\|_{(k,\alpha)}\|S\| < 1$, the operator $I - \mu S$ is invertible on $\mathcal{C}^{(k,\alpha)}(\Omega)$, with the bounded inverse

$$\Theta(\mu) = (I - \mu S)^{-1} = \sum_{j=0}^{\infty} (\mu S)^j.$$

For such μ, the equation for ϕ has the unique solution $\phi = \Theta(\mu)\mu$, and hence the Beltrami equation $f_{\bar{z}} = \mu f_z$ has the solution

$$f(\mu) = z|_\Omega + P(\Theta(\mu)\mu) \in \mathcal{C}^{(k+1,\alpha)}(\Omega).$$

It follows that $f(\mu)$ is smooth (analytic) in μ.

If μ is close to 0 then $f(\mu) : \Omega \to \mathbb{C}$ is a J_μ-holomorphic immersion. Lifting $f(\mu)$ with respect to the immersion $z : X \to \mathbb{C}$ gives (J_μ, J_0)-biholomorphisms $\Phi_\mu : \Omega \to \Phi_\mu(\Omega)$ with $z \circ \Phi_\mu = f_\mu$ and Φ_0 the identity map on Ω.

The idea of proof of the main theorem

Assume that $\{J_t\}_{t\in T}$ is a family of complex structures on X as in the main theorem, $K \subset T \times X$ is a closed set with compact Runge fibres K_t, and $f : T \times X \to Y$ is a map of class $\mathcal{C}^{l,0}(T \times X)$ such that $f_t = f(t, \cdot)$ is J_t-holomorphic on a neighbourhood $U_t \supset K_t$ for every $t \in T$.

Fix $t_0 \in T$. Pick a smoothly bounded domain $\Omega \Subset X$ containing K_{t_0}. There is a neighbourhood $T_0 \subset T$ of t_0 and a family of (J_t, J_{t_0})-biholomorphic maps $\Phi_t : \Omega \to \Phi_t(\Omega)$ of class $\mathcal{C}^{l,(k+1,\alpha)}(T_0 \times \Omega)$. We may assume that $K_t \subset \Omega$ for all $t \in T_0$. Then, the map $h_t = f_t \circ \Phi_t^{-1} : \Phi_t(\Omega) \to Y$ is J_{t_0}-holomorphic on a neighbourhood of the compact Runge set $\widetilde{K}_t = \Phi_t(K_t)$ for $t \in T_0$.

We may assume that $T_0 \subset \mathbb{R}^n \subset \mathbb{C}^n$, so $T_0 \times X$ is a Levi-flat submanifold of $\mathbb{C}^n \times (X, J_{t_0})$ fibered over T_0. The compact subset $\widetilde{K}$ with fibres $\widetilde{K}_t$ $(t \in T_0)$ is $\mathcal{O}(\mathbb{C}^n \times X)$-convex. Since Y is an Oka manifold, we can approximate the maps h_t uniformly on $\widetilde{K}_t$ by J_{t_0}-holomorphic maps $\tilde{h}_t : X \to Y$ of class $\mathcal{C}^l$ in $t \in T_0$. Then, $\tilde{f}_t = \tilde{h}_t \circ \Phi_t : \Omega \to Y$ is J_t-holomorphic for every $t \in T_0$.

This is a step in an inductive construction which leads in the limit to an X-holomorphic map $F : T \times X \to Y$ of class $\mathcal{C}^{l,k+1}$.

Directed holomorphic immersions

A connected projective manifold $Y \subset \mathbb{CP}^{n-1}$ determines the punctured complex cone

$$A = A(Y) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n_* : [z_1 : \cdots : z_n] \in Y\}.$$

A is smooth and connected, $\overline{A} = A \cup \{0\} \subset \mathbb{C}^n$ is an algebraic subvariety of $\mathbb{C}^n$, and A is an Oka manifold if and only if $Y = \mathbb{P}(A)$ is an Oka manifold.

This holds in particular if $Y \subset \mathbb{P}^{n-1}$ is a smooth uniformly rational variety, for such are algebraically elliptic and hence Oka (Arzhantsev, Kaliman, Zaidenberg).

Let X be an open Riemann surface. A holomorphic immersion $h : X \to \mathbb{C}^n$ is said to be directed by A, or an A-immersion, if its complex derivative with respect to any local holomorphic coordinate on X assumes values in A.

Directed holomorphic immersions

Clearly, this holds iff the holomorphic map $f = dh/\theta : X \to \mathbb{C}^n$ assume values in A, where θ is any nowhere vanishing holomorphic 1-form on X.

Conversely, a holomorphic map $f : X \to A$ satisfying the period vanishing conditions

$$\int_C f\theta = 0 \quad \text{for all closed curves } C \subset X$$

integrates to a holomorphic A-immersion $h : X \to \mathbb{C}^n$ given by

$$h(x) = v + \int_{x_0}^{x} f\theta, \quad x \in X$$

for any $x_0 \in X$ and $v \in \mathbb{C}^n$.

Families of directed holomorphic immersions

Theorem

Assume that

- $A \subset \mathbb{C}^n_*$ *is a smooth Oka cone,*
- X, T, *and* $\{J_t\}_{t\in T}$ *are as in the main theorem, and*
- $\{\theta_t\}_{t\in T}$ *is a family of nowhere vanishing* J_t*-holomorphic* 1*-forms on* X.

Given a continuous map $f_0 : T \times X \to A$, *there is map* $h : T \times X \to \mathbb{C}^n$ *of class* $\mathcal{C}^{l,k+1}$ *such that* $h_t = h(t,\cdot\,)$ *is a* J_t*-holomorphic* A*-immersion for every* $t \in T$, *and the map* $f : T \times X \to A$ *defined by* $f(t,\cdot\,) = dh_t/\theta_t$ *is homotopic to* f_0.

For $A = \mathbb{C}^n_*$ we obtain families of J_t-holomorphic immersions $X \to \mathbb{C}^n$.

The special case $n = 1$ gives the Gunning–Narasimhan theorem for families.

Holomorphic null curves and conformal minimal immersions

A case of major interest is the punctured null quadric

$$\mathbb{A} = \big\{(z_1, \ldots, z_n) \in \mathbb{C}^n_* : z_1^2 + z_2^2 + \cdots + z_n^2 = 0\big\}, \quad n \geq 3.$$

Holomorphic $\mathbb{A}$-immersions are called holomorphic null curves in $\mathbb{C}^n$.

The real and imaginary part of a holomorphic null immersion $X \to \mathbb{C}^n$ are conformal harmonic (minimal) immersions $X \to \mathbb{R}^n$. Conversely, a conformal minimal immersion $X \to \mathbb{R}^n$ is locally the real part of a holomorphic null curve.

Corollary

Given a continuous map $f_0 : T \times X \to \mathbb{A}$, there is a map $F : T \times X \to \mathbb{R}^n$ of class $\mathcal{C}^{l,k+1}$ such that $F_t = F(t, \cdot\,) : (X, J_t) \to \mathbb{R}^n$ is a conformal minimal immersion with given flux for every $t \in T$, and the map

$$f : T \times X \to \mathbb{A}, \quad f(t, \cdot\,) = \partial_{J_t} F_t / \theta_t \ \ \textit{for all } t \in T$$

is homotopic to f_0. (Vanishing flux corresponds to holomorphic null curves in $\mathbb{C}^n$.)

THANK YOU FOR YOUR ATTENTION