The universal family of punctured Riemann surfaces is Stein

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Seminar for Complex Analysis
Ljubljana, 25 November 2025

What is a Teichmüller space?

The notion of a Teichmüller space originates in a paper of Oswald Teichmüller (1944), who defined a complex manifold structure on the set of isomorphism classes of marked closed Riemann surfaces of genus g.

Given a Riemann surface $M = (M, J_0)$, let

$$\mathfrak{J}(M) = \{J \text{ is a complex structure on } M \text{ quasiconformally equivalent to } J_0\}.$$

Complex structure $J_1, J_2 \in \mathfrak{J}(M)$ are Teichmüller equivalent iff there is a biholomorphism $\sigma : (M, J_1) \to (M, J_2)$ which induces the identity on the ideal boundary ∂M and is homotopic to the identity on $M \cup \partial M$ relative to ∂M.
The space of equivalence classes is the Teichmüller space $T(M)$.

Droping the requirement that the homotopy from σ to Id_M is fixed on ∂M gives the reduced Teichmüller space $T^{\#}(M)$.

Droping the requirement that σ is homotopic to Id_M gives the Riemann space $R(M)$.

$$\mathfrak{J}(M) \longrightarrow T(M) \longrightarrow T^{\#}(M) \longrightarrow R(M)$$

Examples

- If M is $\mathbb{CP}^1$ with at most three punctures then $T(M)$, and hence also $T^{\#}(M)$ and $R(M)$, are singletons. In fact, any two structure are qc-equivalent, and combination of up to three punctures can be placed in the standard position $\{\infty, 0, 1\}$ by a Möbius transformation of $\mathbb{CP}^1$.
- If M is a torus then $T(M)$ and $T^{\#}(M)$ are canonically identifiable with the upper halfplane $\mathbb{H}$, and $R(M) = \mathbb{H}/PSL(2,\mathbb{Z})$. The same holds for once-punctured torus.
- If M is the disc $\mathbb{D}$ then $T(\mathbb{D})$ is a ball in an infinite dimensional Banach space while $T^{\#}(\mathbb{D})$ and $R(\mathbb{D})$ are singletons.

The universal family over $T(M)$ is a space $V(M)$ with projection $\pi : V(M) \to T(M)$ whose fibre $\pi^{-1}(t)$, $t \in T(M)$, is the Riemann surface (M, J_t) where J_t is the complex structure determined by t.

The Teichmüller family $V(g,n) \to T(g,n)$

The Teichmüller space $T(M)$ is finite dimensional iff

$$M = \widehat{M} \setminus \{p_1, \ldots, p_n\}$$

is a compact Riemann surface $\widehat{M}$ of genus g with n punctures. Such M is said to be of finite conformal type and we write $T(M) = T(g,n)$. The universal family

$$\pi : \widehat{V}(g,n) \to T(g,n)$$

is a holomorphic submersion whose fibre over $t \in T(g,n)$ is the compact Riemann surface $(\widehat{M}, J_t)$ with the complex structure J_t determined by t, and with n holomorphic sections $s_1, \ldots, s_n : T(g,n) \to \widehat{V}(g,n)$ with pairwise disjoint images representing the punctures. The open subset

$$V(g,n) = \widehat{V}(g,n) \setminus \bigcup_{i=1}^{n} s_i(T(g,n)) \xrightarrow{\pi} T(g,n)$$

is the universal family of n-punctured compact Riemann surfaces of genus g.
If $2g + n \geq 3$ then $\pi : V(g,n) \to T(g,n)$ is the universal object in the category of holomorphically varying families of n-punctured genus g Riemann surfaces.

The main result

Behrs and Ehrenpreis 1964, Wolpert 1987 $T(g,n)$ is a Stein manifold.

By the Bers embedding theorem, $T(g,n)$ can be holomorphically realised as a bounded contractible domain in some $\mathbb{C}^N$. If $2g+n \geq 3$ then $N = 3g-3+n$.

Marković 2018 If $g \geq 2$ then $T(g,n)$ cannot be holomorphically realised as a convex domain in $\mathbb{C}^{3g-3+n}$. The reason is that the Teichmüller metric on $T(g,n)$ (which equals the Kobayashi metric) does not agree with the Carathéodory metric.

Let $\pi : V(g,n) \to T(g,n)$ denote the holomorphic Teichmüller submersion.

Theorem (F. 2025)

If $n \geq 1$ then $V(g,n)$ is a Stein manifold.

`https://arxiv.org/abs/2511.07963`, November 2025

Corollaries

Example

If $g = 0$ and $n \in \{1, 2, 3\}$ then $T(0, n)$ is a singleton and $V(0, n)$ equals $\mathbb{C}$, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, and $\mathbb{C} \setminus \{0, 1\}$, respectively. These are evidently Stein (every open Riemann surface is Stein).

$V(1, 1) \to T(1, 1) \cong \mathbb{H}$ is the universal family of once-punctured tori. It is no longer evident that the complex surface $V(1, 1)$ is Stein.

Corollary (1)

If $n \geq 1$ then $V(g, n)$ admits a proper holomorphic embedding in some $\mathbb{C}^N$. If $2g + n \geq 3$ we can take $N = \left[\frac{3 \dim V(g,n)}{2}\right] + 1 = \left[\frac{9g-9+3n}{2}\right] + 1$.

Problem

Does $V(g, n)$ for $n \geq 1$ admit a proper holomorphic embedding in some $\mathbb{C}^N$ which is algebraic on every fibre?

Corollaries, 2

Since $T(g,n)$ is contractible, the submersion $\pi : V(g,n) \to T(g,n)$ is smoothly trivial and the inclusion of any fibre of π in $V(g,n)$ is a homotopy equivalence. It follows that for any manifold Y, the restrictions of a continuous map $V(g,n) \to Y$ to the fibre of $V(g,n)$ lie in the same homotopy class.

Corollary (2)

Let Y be an Oka manifold and $n \geq 1$.

- *There is a holomorphic map $V(g,n) \to Y$ in every homotopy class.*
- *A holomorphic map $M_t \to Y$, $t \in T(g,n)$, from any fibre $M_t = \pi^{-1}(T)$ of $V(g,n)$ extends to a holomorphic map $V(g,n) \to Y$.*
- *More generally, given a closed complex subvariety T' of $T(g,n)$, every continuous map $F_0 : V(g,n) \to Y$ which is holomorphic on $\pi^{-1}(T')$ is homotopic to a holomorphic map $F : V(g,n) \to Y$ by a homotopy which is fixed on $\pi^{-1}(T')$.*

Corollaries, 3

Note that $V(0,1) = \mathbb{C}$. If $n \geq 1$ and $(g,n) \neq (0,1)$ then $V(g,n)$ is not simply connected, and its homotopy type is a finite bouquet of circles. It follows that homotopically nontrivial maps $V(g,n) \to Y$ exist whenever the manifold Y is not simply connected. This gives the following corollary. The last statement follows by taking the Oka manifold $Y = \mathbb{C}^*$.

Corollary (3)

If $n \geq 1$ and Y is an Oka manifold which is not simply connected, there is a holomorphic map $V(g,n) \to Y$ which is nonconstant on every fibre.
In particular, $V(g,n)$ admits a nowhere vanishing holomorphic function which is nonconstant on every fibre.

One can also construct fibrewise nonconstant holomorphic maps by using the Oka principle with approximation.

Corollaries, 4

The fact that $V(g, n)$ for $n \geq 1$ is homotopy equivalent to a bouquet of circles implies that every complex vector bundle on $V(g, n)$ is topologically trivial. Since $V(g, n)$ is Stein, the Oka–Grauert principle implies the following.

Corollary (4)

Every holomorphic vector bundle on $V(g, n)$ for $n \geq 1$ is holomorphically trivial.

Corollary (5)

Assume that $n \geq 1$.

- (a) *There exists a nowhere vanishing holomorphic vector field ξ on $V(g, n)$ which is tangent to the fibres of the projection $\pi : V(g, n) \to T(g, n)$, that is, $d\pi(\xi) = 0$.*
- (b) *With ξ as in (a), there exists a holomorphic 1-form θ on $V(g, n)$ satisfying $\langle \theta, \xi \rangle = 1$. In particular, θ is nowhere vanishing on the tangent bundle to any fibre of π.*

Proof of Corollary 5

Part (a) follows from the fact that the holomorphic line bundle $\ker d\pi \to V(g, n)$ is trivial.

To see (b), consider the short exact sequence of vector bundles over $V(g, n)$:

$$0 \longrightarrow \ker d\pi \hookrightarrow TV(g, n) \xrightarrow{\alpha} H := TV(g, n)/\ker \pi \longrightarrow 0.$$

Here, $TV(g, n)$ denotes the tangent bundle of $V(g, n)$.

Since $V(g, n)$ is Stein, Cartan's Theorem B implies that the sequence splits, i.e. there is a holomorphic vector bundle injection

$$\sigma : H \hookrightarrow TV(g, n), \quad \alpha \circ \sigma = \mathrm{Id}_H.$$

Hence,

$$TV(g, n) = \ker d\pi \oplus \sigma(H) = \mathbb{C}\xi \oplus \sigma(H),$$

where ξ is as in (a). The unique holomorphic 1-form θ on $V(g, n)$ with $\langle \theta, \xi \rangle = 1$ and $\xi = 0$ on $\sigma(H)$ clearly satisfies condition (b).

A problem

Problem

Let $n \geq 1$. Does there exist a holomorphic function $f : V(g, n) \to \mathbb{C}$ whose restriction to any fibre of $\pi : V(g, n) \to T(g, n)$ is an immersion?

There exists a holomorphic function $f : V(g, n) \to \mathbb{C}$ without critical points. The problem is to ensure that $\ker df_z$ is transverse to $\ker d\pi_z$ at every point $z \in V(g, n)$. In my paper *Runge and Mergelyan theorems on families of open Riemann surfaces*, I constructed a smooth function on $V(g, n)$ whose restriction to any fibre of $\pi : V(g, n) \to T(g, n)$ is a holomorphic immersion.

This is related to the question whether a holomorphic 1-form θ in Corollary 5 (b) can be made exact on every fibre of π by multiplying it with a suitably chosen nowhere vanishing holomorphic function on $V(g, n)$. However, this is not the only problem. Since the submersion $\pi : V(g, n) \to T(g, n)$ does not admit a holomorphic section when $g \geq 3$, there is no natural way of choosing the initial point for computing the fibrewise integrals of θ.

Towards the main theorem

Recall that

$$\pi : \widehat{V}(g,n) \to T(g,n)$$

is a holomorphic submersion whose fibres are compact Riemann surfaces, with n holomorphic sections $s_1, \dots, s_n : T(g,n) \to \widehat{V}(g,n)$ with pairwise disjoint images such that

$$V(g,n) = \widehat{V}(g,n) \setminus \bigcup_{i=1}^{n} s_i(T(g,n)).$$

Since $T(g,n)$ is Stein, the fact that $V(g,n)$ is Stein is an immediate consequence of the following result.

Theorem

Assume that X is a Stein manifold, Z is a connected complex manifold with $\dim Z = \dim X + 1$, $\pi : Z \to X$ is a surjective proper holomorphic submersion, and $s_1, \dots, s_n : X \to Z$ are holomorphic sections with pairwise disjoint images for some $n \geq 1$. Then, the domain $\Omega = Z \setminus \bigcup_{i=1}^{n} s_i(X)$ is Stein.

Proof, 1

the conditions imply that every fibre $Z_x = \pi^{-1}(x)$, $x \in X$, is a compact Riemann surface, and the fibres are diffeomorphic but not necessarily biholomorphic to each other. Hence, $\{Z_x\}_{x \in X}$ is a holomorphic family of compact Riemann surfaces and $\Omega_x = Z_x \setminus \bigcup_{i=1}^n s_i(x)$ $(x \in X)$ is a holomorphic family of n-punctured Riemann surfaces.

Each $H_i = s_i(X)$ is a closed complex hypersurface in Z whose ideal sheaf is a principal ideal sheaf, that is, it is locally near each point of H_i generated by a single holomorphic function.

Grauert and Remmert: If Z is a Stein space and H is a closed complex hypersurface in Z (of pure codimension one) whose ideal sheaf is a principal ideal sheaf, then $Z \setminus H$ is also Stein. If Z is nonsingular then the ideal sheaf of any closed complex hypersurface in Z is a principal ideal sheaf.

Hence, it suffices to prove the theorem for $n = 1$, that is, to show that the complement of the image of a single section $s : X \to Z$ is Stein.

Proof, 2

By a theorem of Siu 1976, the Stein hypersurface $H = s(X)$ has a basis of open Stein neighbourhoods U in Z. Since $U \setminus H$ is a Stein manifold by the aforementioned theorem of Grauert and Remmert, it admits a strongly plurisubharmonic exhaustion function $\phi : U \setminus H \to \mathbb{R}_+$.

To prove the theorem, we shall construct a strongly plurisubharmonic exhaustion function $Z \setminus H \to \mathbb{R}_+$; a theorem of Grauert 1958 will then imply that $Z \setminus H$ is Stein.

Fix a point $x_0 \in X$ and set $z_0 = s(x_0) \in H \subset Z$. Since $\pi : Z \to X$ is a holomorphic submersion with compact one-dimensional fibres, it is a smooth fibre bundle whose fibre M is a compact smooth surface. In particular, there is a neighbourhood $X_0 \subset X$ of x_0 such that the restricted bundle $Z|X_0 = \pi^{-1}(X_0) \to X_0$ can be smoothly identified with the trivial bundle $X_0 \times M \to X_0$. In this identification, $z_0 = (x_0, p_0)$ with $p_0 \in M$.

Proof, 3

Since ϕ tends to $+\infty$ along H, there are small smoothly bounded open discs $D \Subset D' \Subset M$ with $p_0 \in D$ such that

$$\inf_{p \in bD} \phi(x_0, p) > \max_{p \in bD'} \phi(x_0, p).$$

The set $O = M \setminus D$ is a compact bordered Riemann surface with smooth boundary $bO = bD$, endowed with the complex structure inherited by the identification $M \cong Z_{x_0} = \pi^{-1}(x_0)$. Note that $bD' \subset \mathrm{Int}(O)$. There is a smooth strongly subharmonic function $u_0 : O \to \mathbb{R}_+$ such that

$$\phi(x_0, \cdot\,) > u_0 \text{ on } bO = bD, \qquad u_0 > \phi(x_0, \cdot\,) \text{ on } bD'.$$

Shrinking X_0 around x_0, the following conditions hold for every $x \in X_0$:

(a) $s(x) \in D$,

(b) the function $u(x, \cdot\,) = u_0$ is strongly subharmonic on O in the complex structure on $Z_x \cong M$,

(c) $\phi(x, \cdot\,) > u(x, \cdot\,)$ on $bO = bD$ and $u(x, \cdot\,) > \phi(x, \cdot\,)$ on bD'.

Proof, 4

We define a function $\rho_0 : (X_0 \times M) \setminus H \to \mathbb{R}_+$ by taking for every $x \in X_0$:

$$\rho_0(x,p) = \begin{cases} \phi(x,p), & p \in D \setminus \{s(x)\}; \\ \max\{\phi(x,p), u(x,p)\}, & p \in D' \setminus D; \\ u(x,p), & p \in M \setminus D'. \end{cases}$$

Note that ρ_0 is well defined, piecewise smooth, strongly subharmonic on each fibre $Z_x \setminus \{s(x)\}$ $(x \in X_0)$, and it agrees with ϕ on $(X_0 \times D) \setminus H$. By using the regularised maximum in the definition of ρ_0, we may assume that ρ_0 is smooth.

This gives an open locally finite cover $\{X_j\}_{j=1}^{\infty}$ of X with smooth fibre bundle trivialisations $Z|X_j \cong X_j \times M$, discs $D_j \subset M$ such that $s(x) \in D_j$ for all $x \in X_j$, and smooth functions $\rho_j : (Z|X_j) \setminus H = \pi^{-1}(X_j) \setminus H \to \mathbb{R}_+$ such that

- ρ_j is strongly subharmonic on each fibre $Z_x \setminus \{s(x)\}$ $(x \in X_j)$, and
- $\rho_j = \phi$ on $(X_j \times D_j) \setminus H$, so it is strongly plurisubharmonic there.

Proof, 5

Let $\{\chi_j\}_j$ be a smooth partition of unity on X with $\operatorname{supp}(\chi_j) \subset X_j$ for each j. Set

$$\rho = \sum_{j=1}^{\infty} \chi_j \rho_j : Z \setminus H \to \mathbb{R}_+.$$

The restriction of ρ to each fibre $Z_x \setminus \{s(x)\}$ $(x \in X)$ is strongly subharmonic, and there is an open neighbourhood $U_0 \subset U \subset Z$ of H such that $\rho = \phi$ holds on $U_0 \setminus H$. In particular, ρ is strongly plurisubharmonic on $U_0 \setminus H$.

Note that for every compact set $K \subset X$, the set $\pi^{-1}(K) \setminus U_0 \subset Z \setminus H$ is compact. Hence, by choosing a strongly plurisubharmonic exhaustion function $\tau : X \to \mathbb{R}_+$ whose Levi form $dd^c\tau$ grows fast enough, we can ensure that

$$\rho + \tau \circ \pi : Z \setminus H \to \mathbb{R}_+ \ \text{ is a strongly plurisubharmonic exhaustion function.}$$

Indeed, denoting by J the almost complex structure on Z and

$$(d^c\rho)(z, \xi) = -d\rho(z, J\xi), \quad z \in Z, \ \xi \in T_z Z,$$

A function ρ is strongly plurisubharmonic at $z \in Z$ iff

$$(dd^c\rho)(z, \xi \wedge J\xi) > 0 \ \text{ for every } 0 \neq \xi \in T_z Z.$$

Proof, 6

Since ρ is strongly subharmonic on every fibre $Z_x \setminus \{s(x)\}$, $x \in X$, we have

$$(dd^c\rho)(z, \xi \wedge J\xi) > 0 \;\text{ if } z \in Z \setminus H \text{ and } 0 \neq \xi \in \ker d\pi_z.$$

Hence, the eigenvectors of the Levi form $(dd^c\rho)(z, \cdot\,)$ associated to non-positive eigenvalues lie in a closed cone $C_z \subset T_zZ$ which intersects $\ker d\pi_z$ only in the origin.

It follows that if $\tau : X \to \mathbb{R}$ is such that $dd^c\tau > 0$ is sufficiently big on T_xX, $x = \pi(z)$, then

$$dd^c\rho + dd^c(\tau \circ \pi) > 0 \;\text{ on }\; T_zZ.$$

Furthermore, the estimates are uniform on the compact set $\pi^{-1}(K) \setminus U_0$, where $U_0 \subset Z$ is a neighbourhood of H as above such that $dd^c\rho > 0$ on $U_0 \setminus H$.

Proof, 7

To see that $\tau : X \to \mathbb{R}_+$ can be chosen such that $dd^c\tau$ grows as fast as desired, note that if $h : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^2$ function then for each point $x \in X$ and tangent vector $\xi \in T_xX$ we have that

$$\begin{aligned} dd^c(h \circ \tau)(x, \xi \wedge J\xi) &= h'(\tau(x))\,(dd^c\tau)(x, \xi \wedge J\xi) \\ &\quad + h''(\tau(x))\left(|d\tau(x,\xi)|^2 + |d\tau(x, J\xi)|^2\right). \end{aligned}$$

Hence, if τ is a strongly plurisubharmonic exhaustion function on X and the function $h : \mathbb{R} \to \mathbb{R}$ is chosen such that $h'' \geq 0$ and h' grows sufficiently fast, then $dd^c(h \circ u)$ also grows as fast as desired. This completes the proof.

A minor modification of the proof gives the following more general result.

Theorem

Assume that X is a Stein manifold, Z is a connected complex manifold with $\dim Z = \dim X + 1$, $\pi : Z \to X$ is a surjective proper holomorphic submersion, and H is a closed complex subvariety of Z of pure codimension one which does not contain any fibre of π. Then, the domain $\Omega = Z \setminus H$ is Stein.

THAT'S ALL FOLKS!

THANKS FOR YOUR ATTENTION