Minimal surfaces from a complex analytic viewpoint
Lecture 1: The basic notions and examples.

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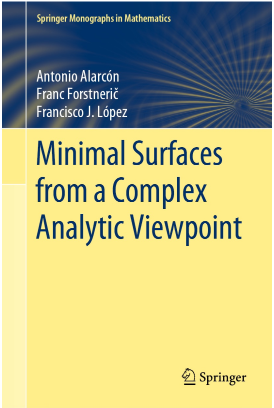
In these lectures...



I will describe some recent developments in the theory of minimal surfaces in Euclidean spaces $\mathbb{R}^n$ which have been obtained by complex analytic methods. My collaborators on these projects:

- Antonio Alarcón and Francisco J. López, Granada
- Barbara Drinovec Drnovšek, Ljubljana
- David Kalaj, Podgorica
- Finnur Lárusson, Adelaide

In the lectures, this book will be refereed as **the book**.
The lectures will be continued in week 2 by Antonio Alarcón.

The modern history of minimal surfaces started with Leonhard Euler

1744 **Euler** A surface in $\mathbb{R}^3$ is called **minimal** if it locally minimizes the area among all nearby surfaces with the same boundary.

The only minimal surfaces of rotation are planes and catenoids.

$x^2 + y^2 = \cosh^2 z$

$(t, z) \mapsto (\cos t \cdot \cosh z, \sin t \cdot \cosh z, z)$

The catenoid is a paradigmatic example in the theory. Besides **Enneper's surface**, it is the only complete, nonflat, orientable minimal surface in $\mathbb{R}^3$ with the smallest absolute total Gaussian curvature 4π.

Lagrange's equation of minimal graphs

1760 **Lagrange** Let $\Omega \subset \mathbb{R}^2$ be a smoothly bounded domain.

A smooth graph $\{(x, y, f(x, y)) : (x, y) \in \overline{\Omega}\}$ is a **critical point of the area functional** with prescribed boundary values iff

$$(1 + f_y^2) f_{xx} - 2 f_x f_y f_{xy} + (1 + f_x^2) f_{yy} = 0.$$

Equivalently,

$$\mathrm{div} \left(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} \right) = 0.$$

This is the **equation of minimal graphs**.

1776 **Meusnier** A smooth surface $M \subset \mathbb{R}^3$ locally satisfies the equation of minimal graphs iff its mean curvature function vanishes identically.

A smoothly immersed surface $M \to \mathbb{R}^3$ is called a **minimal surface** if its **mean curvature function** $\mathrm{H} : M \to \mathbb{R}$ **is identically zero**: $\mathrm{H} = 0$.

Recall that

$$\mathrm{H} = \frac{\kappa_1 + \kappa_2}{2}$$

where κ_1, κ_2 are the **principal curvatures**. Their product

$$\mathrm{K} = \kappa_1 \kappa_2 : M \to \mathbb{R}$$

is the **Gaussian curvature** of M. Note that

$$\mathrm{H} = 0 \implies \mathrm{K} \leq 0.$$

The helicoid (Archimedes' screw)

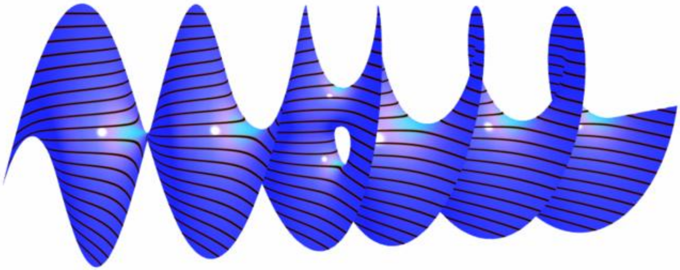
1776 **Meusnier** The helicoid is a minimal surface.

The helicoid is obtained by rotating a line and displacing it along the axis of rotation.

$(u, v) \mapsto$
$(\cos u \cdot \sinh v, \sin u \cdot \sinh v, u)$

1842 **Catalan** The helicoid and the plane are the only ruled minimal surfaces in $\mathbb{R}^3$.

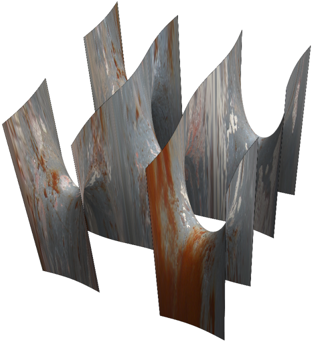
Many mathematicians (Hoffman, Karcher, Wei, White) described helicoidal minimal surfaces of higher genus.



Source: Wikipedia

Scherk's surface

1835 **Scherk** The first two new minimal surfaces since Meusnier (1776).



The first Scherk's surface is doubly periodic.

Its main branch is a graph over the square $P = (-\pi/2, \pi/2)^2$ given by

$$x_3 = \log \frac{\cos x_2}{\cos x_1}$$

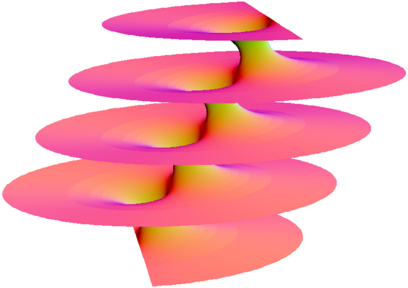
Finn and Osserman (1964)
Kalaj and Melentijević (2025)

Sherk's surface S has the biggest absolute Gaussian curvature at $0 \in \mathbb{R}^3$ over all minimal graphs over P.

1917 **Bernstein** A minimal graph over $\mathbb{R}^2$ in $\mathbb{R}^3$ is a plane.

Riemann's minimal examples

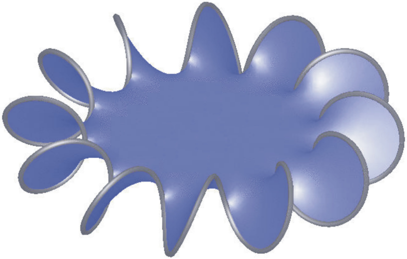
1867 **Riemann** discovered a family $R_\lambda,\ \lambda > 0$, of periodic planar domains, properly embedded as minimal surfaces in $\mathbb{R}^3$ such that every horizontal plane intersects each R_λ in a circle or a line. His surfaces converge to a vertical catenoid as $\lambda \to 0$, and to a vertical helicoid as $\lambda \to \infty$.



2015 **Meeks, Pérez, Ros** Planes, catenoids, helicoids, and Riemann's examples are the only planar domains which can be properly embedded as minimal surfaces in $\mathbb{R}^3$.

The Plateau Problem

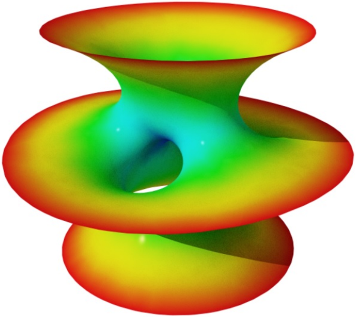
1873 **Plateau** Soap films are minimal surfaces. A conformally parameterized minimal disc with a given Jordan boundary curve minimizes the internal tension within the surface.



1932 **Douglas, Radó** Every Jordan curve $\Gamma \subset \mathbb{R}^3$ spans a minimal surface.

1976 **Meeks, S.-T. Yau** If Γ lies in the boundary of a convex domain, then the Douglas–Morrey solution of the Plateau problem (the disc surface of smallest area with boundary Γ) is embedded.

1982 **Costa** discovered a complete embedded minimal surface in $\mathbb{R}^3$ of genus one, a middle planar end and two catenoidal ends, and total Gaussian curvature -12π. Its conformal type is that of a thrice-punctured torus. Hoffman and Meeks proved in 1985 that it is embedded.



1991 **Costa** Complete embedded minimal surfaces in $\mathbb{R}^3$ having genus one and three ends form the 1-parameter family of Costa–Hoffman–Meeks surfaces.

A digression: Riemann surfaces in Riemannian manifolds

Assume that M is a surface embedded or immersed into a Riemannian manifold (N, ds^2); for example, into $\mathbb{R}^n$ with the Euclidean metric.

Let g be the induced metric on M (the **first fundamental form**). Then, g determines on M a **conformal structure** (a way of measuring angles), and the given immersion is conformal from (M, g) to (N, ds^2).

In any smooth local coordinates (x, y) on M we have

$$g = Edx \otimes dx + F(dx \otimes dy + dy \otimes dx) + Gdy \otimes dy = Edx^2 + 2Fdxdy + Gdy^2$$

where E, F, G are real functions satisfying $EG - F^2 > 0$.
Taking $z = x + \imath y$ as a complex coordinate on U, we have

$$g = \lambda |dz + \mu d\bar{z}|^2$$

for a positive function $\lambda > 0$ and the complex function

$$\mu = \frac{1 - c + \imath b}{1 + c + \imath b} : U \to \mathbb{D}$$

with values in the unit disc, where

$$\delta = EG - F^2 > 0, \quad b = F/\sqrt{\delta}, \quad c = G/\sqrt{\delta} > 0.$$

Isothermal coordinates and Riemann surfaces structure

The Euclidean metric on $\mathbb{R}^2 \cong \mathbb{C}$ with the coordinate $z = x + iy$ is

$$g_{\mathrm{st}} = ds^2 = dx^2 + dy^2 = |dz|^2.$$

A diffeomorphism $f : U \to f(U) \subset \mathbb{C}$ is conformal from the g-structure on M to the standard structure g_{st} on $\mathbb{C}$ iff

$$g = h \cdot f^*(g_{\mathrm{st}}) = h|df|^2 \quad \text{for a positive function } h > 0.$$

Such a chart f is said to be **isothermal** for g.

Assume that f is orientation preserving, $|f_z| > |f_{\bar{z}}|$. Then

$$|df|^2 = |f_z dz + f_{\bar{z}} d\bar{z}|^2 = |f_z|^2 \cdot \left| dz + \frac{f_{\bar{z}}}{f_z} d\bar{z} \right|^2.$$

A comparison with $g = \lambda |dz + \mu d\bar{z}|^2$ shows that f is isothermal iff

$$f_{\bar{z}} = \mu f_z.$$

This is the **Beltrami equation** with the Beltrami coefficient μ given above. The transition map between isothermal charts is a conformal diffeomorphism between plane domains, hence holomorphic or antiholomorphic. If M is orientable, g determines on M the structure of a **Riemann surface**.

Analytic description of conformal minimal surfaces

Assume that $D \subset \mathbb{R}^2_{(u,v)}$ is a bounded domain with smooth boundary and $X : \overline{D} \to \mathbb{R}^n$ is a smooth immersion. Precomposing X with a diffeomorphism from another such domain in $\mathbb{R}^2$, we may assume that X is **conformal**:

$$|X_u| = |X_v|, \qquad X_u \cdot X_v = 0.$$

Consider the **area functional**

$$\mathrm{Area}(X) = \int_D |X_u \times X_v| \, dudv = \int_D \sqrt{|X_u|^2 |X_v|^2 - |X_u \cdot X_v|^2} \, dudv$$

and the **Dirichlet energy functional**

$$\mathscr{D}(X) = \frac{1}{2} \int_D |\nabla X|^2 \, dudv = \frac{1}{2} \int_D \left(|X_u|^2 + |X_v|^2 \right) \, dudv.$$

From the elementary inequalities

$$|x|^2|y|^2 - |x \cdot y|^2 \le |x|^2|y|^2 \le \frac{1}{4} \left(|x|^2 + |y|^2 \right)^2, \quad x, y \in \mathbb{R}^n,$$

which are equalities iff x, y is a conformal frame, we infer that

$$\mathrm{Area}(X) \le \mathscr{D}(X), \quad \text{with equality iff } X \text{ is conformal.}$$

A conformal immersion is minimal iff it is harmonic

Hence, these two functionals have the same critical points on the set of conformal immersions.

If $G : \overline{D} \to \mathbb{R}^n$ is a smooth map vanishing on bD, then

$$\frac{d}{dt}\Big|_{t=0} \mathscr{D}(X + tG) = \int_D (X_u \cdot G_u + X_v \cdot G_v)\, dudv = -\int_D \Delta X \cdot G\, dudv.$$

We integrated by parts and used $G|_{bD} = 0$. This vanishes for all such G iff $\Delta X = 0$, thereby proving the first part of the following theorem.

Theorem

A smooth conformal immersion $X : \overline{D} \to \mathbb{R}^n$, $n \geq 3$, parameterizes a minimal surface iff X is harmonic $(\Delta X = 0)$. This holds iff the mean curvature vector field of X vanishes identically.

The second part is an elementary calculation; see Theorem 2.3 in

F. Forstnerič: Minimal surfaces in Euclidean spaces by way of complex analysis. European Congress of Mathematics, 9–43. EMS Press, Berlin, [2023], ©2023.
https://users.fmf.uni-lj.si/forstneric/papers/2021-8ECM.pdf

Connection to complex analysis

Let $z = u + \mathrm{i}v$ be a complex coordinate on $\mathbb{C}$ and $dz = du + \mathrm{i}dv$. Then,

$$dX = \partial X + \bar\partial X = X_z dz + X_{\bar z} d\bar z = \frac{1}{2}\left(X_u - \mathrm{i}X_v\right) dz + \frac{1}{2}\left(X_u + \mathrm{i}X_v\right) d\bar z.$$

From

$$\Delta X = X_{uu} + X_{vv} = 4(X_z)_{\bar z}$$

we see that

X is harmonic ($\Delta X = 0$) iff X_z is holomorphic.

Furthermore:

$$\textbf{X is conformal} \iff X_u \cdot X_v = 0,\ |X_u|^2 = |X_v|^2 \iff \sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{n}} (\partial \mathbf{X_k})^{\mathbf{2}} = \mathbf{0}.$$

The analogous conclusions hold if D is any **open conformal surface**.

Every immersed surface in $\mathbb{R}^n$ admits a conformal parameterization by a Riemann surface. For nonorientable surfaces, we pass to their orientable double cover.

Hence, a smooth immersion $X = (X_1, X_2, \cdots, X_n) : M \to \mathbb{R}^n$ from an open Riemann surface parameterizes a conformal minimal surface if and only if

$$\partial X = (\partial X_1, \ldots, \partial X_n) \text{ is a holomorphic 1-form and } \sum_{k=1}^{n} (\partial X_k)^2 = 0.$$

Conversely: a nowhere vanishing holomorphic 1-form $\Phi = (\phi_1, \ldots, \phi_n)$ on M satisfying the nullity condition $\sum_{k=1}^n \phi_k^2 = 0$ and the period vanishing conditions

$$\Re \int_C \Phi = 0 \in \mathbb{R}^n \quad \text{for every closed curve } C \text{ in } M$$

determines a conformal minimal immersion

$$X = \Re \int \Phi : M \to \mathbb{R}^n, \qquad 2\partial X = \Phi.$$

Since Φ is holomorphic, it suffices to test the period conditions on a basis of the homology group $H_1(M, \mathbb{Z}) \cong \mathbb{Z}^l$, $l \in \mathbb{Z}_+ \cup \{\infty\}$.

The null quadric

Let us introduce the **null quadric**:

$$\mathcal{A} = \left\{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{j=1}^{n} z_j^2 = 0 \right\}.$$

Fix a nowhere vanishing holomorphic 1-form θ on M. (Such exists by a theorem of Gunning and Narasimhan (1967).) With Φ as above, we have

$$\Phi = f\theta$$

where

$$f : M \to \mathcal{A}_* = \mathcal{A} \setminus \{0\}$$

is a holomorphic map having vanishing real periods:

$$\Re \int_C \Phi = \Re \int_C f\theta = 0 \in \mathbb{R}^n \quad \text{for every } [C] \in H_1(M, \mathbb{Z}).$$

The **flux homomorphism** $\mathrm{Flux}_\Phi : H_1(M, \mathbb{Z}) \to \mathbb{R}^n$ is given by

$$\mathrm{Flux}_\Phi(C) = \Im \int_C \Phi \in \mathbb{R}^n, \quad [C] \in H_1(M, \mathbb{Z}).$$

Holomorphic null curves

A holomorphic immersion

$$Z = (Z_1, \ldots, Z_n) : M \to \mathbb{C}^n, \quad n \geq 3$$

is said to be a **holomorphic null curve** if

$$\sum_{k=1}^{n} (\partial Z_k)^2 = 0.$$

Every such curve is of the form $Z = \int f\theta$, where $f : M \to \mathcal{A}_*$ is a holomorphic map such that

$$\int_C f\theta = 0 \quad \text{for all } [C] \in H_1(M, \mathbb{Z}).$$

Equivalently, $\mathrm{Flux}_f = 0$. Hence, a conformal minimal immersion $X : M \to \mathbb{R}^n$ is the real part of a holomorphic null curve $Z : M \to \mathbb{C}^n$ iff

$$\mathrm{Flux}_X := \mathrm{Flux}_{\partial X} = 0.$$

The real and the imaginary part of a null curve are conformal minimal surfaces.

The Gauss map of a minimal surface

Let $X = (X_1, \ldots, X_n) : M \to \mathbb{R}^n$ be a conformal minimal immersion.

Since the 1-form $\partial X = (\partial X_1, \ldots, \partial X_n)$ is holomorphic and nowhere vanishing, it determines the holomorphic map

$$G_X : M \to \mathbb{CP}^{n-1}, \quad G_X(p) = [\partial X_1(p) \colon \cdots : \partial X_n(p)], \quad p \in M,$$

called the **(generalised) Gauss map of X**. It is of great importance in the theory of minimal surfaces, especially when $n = 3$.

Since $\sum_{j=1}^n (\partial X_j)^2 = 0$, G_X assumes values in the complex hyperquadric

$$Q^{n-2} = \big\{ [z_1 : \ldots : z_n] \in \mathbb{CP}^{n-1} : z_1^2 + \cdots + z_n^2 = 0 \big\}.$$

The classical Enneper–Weierstrass formula for $n = 3$

If $n = 3$ then $Q^1 \cong \mathbb{CP}^1 \overset{\iota}{\hookrightarrow} \mathbb{CP}^2$ is a quadratically embedded rational curve in $\mathbb{CP}^2$, and we have $G_X = \iota \circ \mathfrak{g}_X$ where $\mathfrak{g}_X : M \to \mathbb{CP}^1$ is a holomorphic map (a meromorphic function on M), called the **complex Gauss map** of X.

Let $X = (X_1, X_2, X_3) : M \to \mathbb{R}^3$ be a conformal minimal immersion, and let

$$(N_1, N_2, N_3) : M \to S^2 \subset \mathbb{R}^3$$

denote its **classical Gauss map**. Then, the **complex Gauss map** of X is given by the stereographic projection

$$\mathfrak{g} = \frac{N_1 + \mathfrak{i} N_2}{1 - N_3} = \frac{\partial X_3}{\partial X_1 - \mathfrak{i}\,\partial X_2} : M \to \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}.$$

In terms of the complex Gauss map $\mathfrak{g}$, we have the following **classical Enneper–Weierstrass formula** of the minimal surface $X : M \to \mathbb{R}^3$:

$$X = 2\Re \int \left(\frac{1}{2}\left(\frac{1}{\mathfrak{g}} - \mathfrak{g}\right), \frac{\mathfrak{i}}{2}\left(\frac{1}{\mathfrak{g}} + \mathfrak{g}\right), 1 \right) \partial X_3.$$

Many important quantities of a minimal surface $X : M \to \mathbb{R}^3$ can be computed from its Gauss map, in particular:

$$g = X^*(ds^2) = \frac{(1+|\mathfrak{g}|^2)^2}{4|\mathfrak{g}|^2}|\partial X_3|^2 \qquad \text{(the induced Riemannian metric)}$$

$$Kg = -\frac{4|d\mathfrak{g}|^2}{(1+|\mathfrak{g}|^2)^2} = -\mathfrak{g}^*(\sigma^2_{\mathbb{CP}^1}) \quad \text{(the Gaussian curvature)}$$

($\sigma^2_{\mathbb{CP}^1}$ is the spherical metric), and the **total Gaussian curvature**

$$\mathrm{TC}(X) = \int_M K dA = -\mathrm{Area}_{\mathbb{CP}^1}(\mathfrak{g}(M)).$$

Complete minimal surfaces of finite total curvature

An immersed surface $X : M \to \mathbb{R}^n$ is said to be **complete** if the induced metric $g = X^* ds^2$ is a complete Riemannian metric on M.

1957 **Huber** If (M, g) is a complete oriented Riemannian surface with $\int_M K^- dA > -\infty$, then M is conformally equivalent to the complement of finitely many points in a compact Riemann surface $\overline{M}$.

1967 **Chern-Osserman**; 1983 **Jorge-Meeks**
If $X : M \to \mathbb{R}^3$ is a complete minimal surface of finite total curvature, then

- its Gauss map extends to a holomorphic map $\mathfrak{g} : \overline{M} \to \mathbb{CP}^1$,
- its total curvature is -4π times the degree of the Gauss map:

$$\mathrm{TC}(X) = -4\pi \deg(\mathfrak{g}),$$

- ∂X has an effective pole of degree ≥ 2 at every point end in $\overline{M} \setminus M$,
- $X : M \to \mathbb{R}^3$ is a proper map which approaches a limit tangent plane sublinearly at every end.

Example: the catenoid, helicoid, and helicatenoid

By rotating the catenal curve

$$\mathbb{R} \ni v \mapsto (\cosh v, 0, v) \in \mathbb{R}^3$$

around the x_3-axis, we obtain the vertical catenoid in $\mathbb{R}^3$ with the axis $x_1 = x_2 = 0$ and the implicit equation

$$x_1^2 + x_2^2 = \cosh^2 x_3.$$

This is the real part of the holomorphic null curve $z : \mathbb{C} \to \mathbb{C}^3$ given by

$$Z(\zeta) = (\cos\zeta, \sin\zeta, -\mathrm{i}\zeta) \in \mathbb{C}^3, \qquad \zeta = u + \mathrm{i}v \in \mathbb{C},$$

called **helicatenoid**. The 1-parameter family of minimal surfaces in $\mathbb{R}^3$ associated to the helicatenoid is

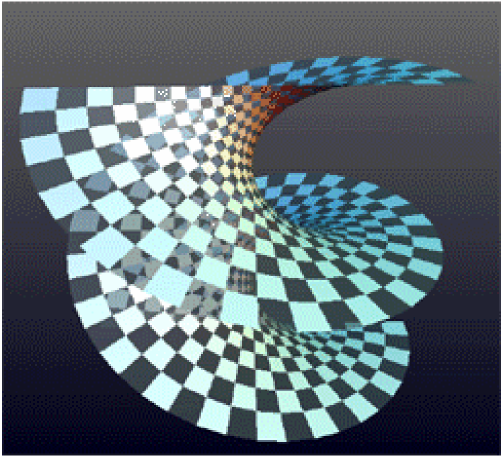
$$X^t(\zeta) = \Re\left(\mathrm{e}^{\mathrm{i}\,t} z(\zeta)\right) = \cos t \begin{pmatrix} \cos u \cdot \cosh v \\ \sin u \cdot \cosh v \\ v \end{pmatrix} + \sin t \begin{pmatrix} \sin u \cdot \sinh v \\ -\cos u \cdot \sinh v \\ u \end{pmatrix}$$

with $t \in \mathbb{R}$. At $t = k\pi \ (k \in \mathbb{Z})$ we have the catenoid, while at $t = \pm\pi/2 + k\pi \ (k \in \mathbb{Z})$ we have a right or left helicoid.

The catenoid and the helicoid are conjugate minimal surfaces.

Helicatenoid (Source: Wikipedia)

The family of minimal surfaces $X_t(\zeta) = \Re\left(e^{it} Z(\zeta)\right), \quad t \in \mathbb{R}$:



$$\begin{aligned} Z(\zeta) &= (1,0,0) + \int_0^\zeta (-\sin\xi, \cos\xi, -\mathrm{i})\, d\xi \\ &= (1,0,0) + \int_0^\zeta \left(\frac{1}{2}\left(\frac{1}{\mathrm{e}^{\mathrm{i}\xi}} - \mathrm{e}^{\mathrm{i}\xi} \right), \frac{\mathrm{i}}{2}\left(\frac{1}{\mathrm{e}^{\mathrm{i}\xi}} + \mathrm{e}^{\mathrm{i}\xi} \right), 1 \right) (-\mathrm{i}) d\xi \end{aligned}$$

with $\zeta = u + \mathrm{i}v \in \mathbb{C}$. It follows that the Gauss map of the helicatenoid, and hence of all its associated minimal surfaces, is $\mathfrak{g}(\zeta) = \mathrm{e}^{\mathrm{i}\zeta}$. The above parameterization of the catenoid is 2π-periodic. Introducing the variable

$$w = \mathrm{e}^{\mathrm{i}\zeta} = \mathrm{e}^{-v+\mathrm{i}u} \in \mathbb{C}^*,$$

we obtain a single sheeted parameterization with the Weierstrass representation

$$X(w) = (1,0,0) - \Re \int_1^w \left(\frac{1}{2}\left(\frac{1}{\eta} - \eta \right), \frac{\mathrm{i}}{2}\left(\frac{1}{\eta} + \eta \right), 1 \right) \frac{d\eta}{\eta}.$$

The complex Gauss map of this catenoid is $\mathfrak{g}(w) = w,\ w \in \mathbb{C}^*$, with $\deg \mathfrak{g} = 1$, so the catenoid has total Gaussian curvature $\mathrm{TC}(X) = -4\pi$. The catenoid is the only surface in the above family which factors through $\mathbb{C}^*$, has algebraic Weierstrass data, and is of finite total curvature.