Minimal surfaces from a complex analytic viewpoint
Lecture 2: Approximation theorems

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Runge approximation theorem for minimal surfaces

In this lecture, I will outline the proof of the approximation statement in the following theorem.

Theorem (Alarcón, López, F., 2012–2017)

Let K be a compact smoothly bounded domain without holes in an open Riemann surface M (such K is a **Runge compact** *in M), and $n \geq 3$. Then:*

- *Every conformal minimal immersion $X : K \to \mathbb{R}^n$ can be approximated uniformly on K by proper conformal minimal immersions $\widetilde{X} : M \to \mathbb{R}^n$.*
- **General position theorem:** *$\widetilde{X}$ can be chosen to have only simple double points if $n = 4$, and to be an embedding if $n \geq 5$.*
- *Analogous results hold for* **nonorientable minimal surfaces** *in $\mathbb{R}^n$ and for* **holomorphic null curves** *in $\mathbb{C}^n$, $n \geq 3$.*

2019 **Alarcón, Castro-Infantes**: In addition, one can prescribe the values of $\widetilde{X}$ on any closed discrete subset of M (Weierstrass-type interpolation).

2019 **Alarcón, López**: The analogous result holds for complete minimal surfaces of finite total Gaussian curvature. In this case, M is a finitely punctured compact Riemann surface and the 1-form $\partial\widetilde{X}$ on M with values in $\mathcal{A}_*$ is algebraic (rational), with an effective pole at every puncture.

Techniques used in the proof

Fix a nonvanishing holomorphic 1-form θ on M. Recall that every conformal minimal immersion $X = (X_1, X_2, \cdots, X_n) : M \to \mathbb{R}^n$ is of the form

$$X = \Re \int \Phi : M \to \mathbb{R}^n, \qquad 2\partial X = \Phi = f\theta,$$

where $f : M \to \mathcal{A}_*$ is a holomorphic map into the punctured null quadric satisfying the period vanishing conditions

$$\Re \int_C f\theta = 0 \in \mathbb{R}^n \quad \text{for every closed curve } C \text{ in } M.$$

We shall use two main properties of punctured null quadric:

The punctured null quadric $\mathcal{A}_*$ is an Oka manifold.

The convex hull of $\mathcal{A}_*$ equals $\mathbb{C}^n$.

The first property is explained below. The second property is obvious, and it implies that for any pair of points $p, q \in \mathcal{A}_*$ and vector $v \in \mathbb{C}^n$ there is a path $\gamma : [0, 1] \to \mathcal{A}_*$ with $\gamma(0) = p$, $\gamma(1) = q$, and $\int_0^1 \gamma(t)dt = v$.

[M. Gromov: Convex Integration of Differential relations. I. (Russian) Izv. Akad. Nauk SSSR, Ser. Mat. 37, 329–343 (1973)]

The Oka principle and Oka manifolds

A complex manifold Y is an **Oka manifold** if maps $X \to Y$ from any Stein manifold X (a closed complex submanifold of $\mathbb{C}^N$) satisfy the **Oka principle**:

- (a) Every continuous map $f : X \to Y$ is homotopic to a holomorphic map.
- (b) If $f : X \to Y$ is holomorphic on a compact holomorphically convex subset $K \subset X$ and on a closed complex subvariety X_0 of X, there is a homotopy from f to a holomorphic map $F : X \to Y$ consisting of maps with the same properties which approximate f on K and agree with f on X_0.
- (c) A similar statement holds for maps $f : X \times P \to Y$ depending continuously on a parameter in a compact Hausdorff space P. If Q is a closed subspace of P and $f(\cdot, p) : X \to Y$ is holomorphic for all $p \in Q$, then the homotopy from f to F can be chosen fixed on $X \times Q$.

Oka–Weil–Cartan 1936–53: $\mathbb{C}$ is an Oka manifold.

Grauert 1958: Every complex homogeneous manifold is an Oka manifold. In particular, the punctured null quadric $\mathcal{A}_*$ is a complex homogeneous space of the complex orthogonal group $O_n(\mathbb{C})$, so it is an Oka manifold.

[Chapter V in F. Forstnerič: Stein manifolds and holomorphic mappings. The homotopy principle in complex analysis. 2nd edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge 56. Cham: Springer (2017)]

Mergelyan approximation on admissible sets in Riemann surfaces

Definition (Admissible sets)

Let M be a smooth surface. An **admissible set** in M is a compact set of the form $S = K \cup E$, where K is a (possibly empty) finite union of pairwise disjoint compact domains with piecewise $\mathscr{C}^1$ boundaries in M and $E = S \setminus \mathring{K}$ is a union of finitely many pairwise disjoint smooth Jordan arcs and closed Jordan curves meeting K only at their endpoints (if at all) and such that their intersections with the boundary bK of K are transverse.

Theorem

Let S be an admissible set in a Riemann surface M and $r \in \mathbb{Z}_+ \in \{0, 1, 2, \ldots\}$. Every function $f \in \mathscr{A}^r(S) := \mathscr{C}^{(}S) \cap \mathscr{O}(\mathring{S})$ can be approximated in $\mathscr{C}^r(S)$ by meromorphic functions on M, and by holomorphic functions on M if S has no holes. Furthermore, the approximating functions can be chosen to agree with f on any given finite set $A \subset S$, and to agree with f to any given finite order in the points in $A \cap \mathring{S}$.

[Theorem 16 in J. E. Fornæss, F. Forstnerič, and E. F. Wold: Holomorphic approximation: the legacy of Weierstrass, Runge, Oka-Weil, and Mergelyan. In: D. Breaz, M. Th. Rassias (Eds.), Advancements in Complex Analysis, pp. 133–192. Springer, Cham, 2020.]

Lemma (Lemma 1.12.10 in the book)

A connected admissible set S has finitely generated first homology group $H_1(S,\mathbb{Z}) \cong \mathbb{Z}^l$. Furthermore, there is a homology basis $\mathcal{C} = \{C_1, \ldots, C_l\}$ consisting of closed piecewise smooth Jordan curves in S such that $C = \bigcup_{i=1}^{l} C_i$ is connected and Runge in any regular neighbourhood of S, and every curve $C_i \in \mathcal{C}$ contains a nontrivial arc I_i disjoint from $\bigcup_{j\neq i} C_j$.

When $S = K$ ($E = \varnothing$), the curves $C_1, \ldots, C_l$ may be chosen such that any two meet only at a common point $p_0 \in \mathring{K}$ and $C = \bigcup_{i=1}^{l} C_j$ is a strong deformation retract of K.

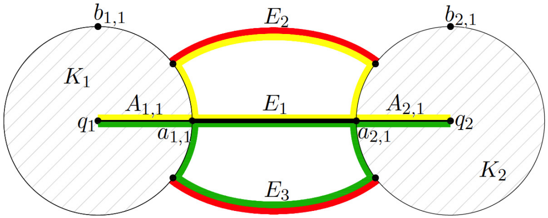


Figure: Two islands connected by three bridges

Let M be a connected open Riemann surface. Fix a nowhere vanishing holomorphic 1-form θ on M. Let $\mathcal{C} = \{C_1, \ldots, C_l\}$ be a collection of smooth oriented embedded arcs and closed Jordan curves in M. Set $C = \bigcup_{i=1}^{l} C_i$.

To any such collection and integer $n \in \mathbb{N}$ we associate the **period map**

$$\begin{aligned} \mathcal{P} &= (\mathcal{P}_1, \ldots, \mathcal{P}_l) : \mathscr{C}(C, \mathbb{C}^n) \longrightarrow (\mathbb{C}^n)^l = \mathbb{C}^{ln}, \\ \mathcal{P}_i(f) &= \int_{C_i} f\theta \in \mathbb{C}^n, \quad f \in \mathscr{C}(C, \mathbb{C}^n), \quad i = 1, \ldots, l. \end{aligned} \tag{1}$$

The following lemma plays a crucial role in the proof of the Runge approximation theorem for conformal minimal surfaces.

Lemma (Lemma 3.2.1 in the book)

Let $S = K \cup E$ be an admissible set in an open Riemann surface M, $\mathcal{C} = \{C_1, \ldots, C_l\}$ be smooth oriented Jordan curves and arcs in S such that $C = \bigcup_{i=1}^{l} C_i$ is Runge in S, and let $f \in \mathscr{A}^r(S, \mathcal{A}_*)$ for some $r \in \mathbb{Z}_+$.

Assume that every curve $C_i \in \mathcal{C}$ contains a nontrivial arc I_i disjoint from $\bigcup_{j \neq i} C_j$ such that $f(I_i)$ is not contained in any complex ray of $\mathcal{A}$.

Then there exist an open ball $0 \in U \subset (\mathbb{C}^n)^l$ and a map $\Phi_f : S \times U \to \mathcal{A}_*$ of class $\mathscr{A}^r(S \times U, \mathcal{A}_*)$ such that $\Phi_f(\cdot\,, 0) = f$ and

$$\frac{\partial}{\partial t}\Big|_{t=0} \mathcal{P}(\Phi_f(\cdot\,, t)) : (\mathbb{C}^n)^l \longrightarrow (\mathbb{C}^n)^l \quad \text{is an isomorphism.} \tag{2}$$

Furthermore, given a finite set $P \subset S$, we may choose Φ_f such that each map $\Phi_f(\cdot\,, t) : S \to \mathcal{A}_*,\ t \in U$, agrees with f at every point of P, and it agrees with f to any given finite order at every point of $P \cap \mathring{S}$.

Every $f_0 \in \mathscr{A}^r(S, \mathcal{A}_*)$ satisfying the above assumptions has a neighbourhood $\Omega \subset \mathscr{A}^r(S, \mathcal{A}_*)$ and a holomorphic map $\Omega \ni f \mapsto \Phi_f$ having these properties.

Proof, 1

We first show how to find a spray Φ_f satisfying the **period domination property** (2). Interpolation is a simple addition which will be explained at the end. We assume that $r = 0$ since the proof for $r > 0$ is the same.

For every $i = 1, \ldots, l$ we can find points $p_{i,k} \in I_i$ and holomorphic vector fields $V_{i,k}$ on $\mathbb{C}^n$ $(k = 1, \ldots, n)$, tangent to the null quadric $\mathcal{A}$ and satisfying

$$\mathrm{span}\{V_{i,k}(f(p_{i,k})) : k = 1, \ldots, n\} = \mathbb{C}^n.$$

Let $\phi_t^{i,k}$ denote the flow of $V_{i,k}$. Write $t = (t_1, \ldots, t_l) \in (\mathbb{C}^n)^l$ where $t_i = (t_{i,1}, \ldots, t_{i,n}) \in \mathbb{C}^n$. Pick an open ball $0 \in U_0 \subset (\mathbb{C}^n)^l$ such that the map

$$S \times U_0 \ni (p, t) \longmapsto \phi_{t_{1,1}}^{1,1} \circ \cdots \circ \phi_{t_{l,n}}^{l,n}(f(p)) \in \mathcal{A}_*$$

is well defined for all $t \in U_0$ and $p \in S$. (We take the composition of flows $\phi_{t_{i,k}}^{i,k}$ for all $i = 1, \ldots, l$ and $k = 1, \ldots, n$.) For every such pair (i, k) we choose a smooth function $h_{i,k} : C = \bigcup_{i=1}^l C_i \to \mathbb{C}$ supported on a short arc around the point $p_{i,k} \in I_i \subset C_i$. Then, the map

$$\Phi(p, t) = \phi_{h_{1,1}(p)t_{1,1}}^{1,1} \circ \cdots \circ \phi_{h_{l,n}(p)t_{l,n}}^{l,n}(f(p)) \in \mathcal{A}_* \tag{3}$$

is defined for all $p \in C$ and all t in a ball $0 \in U_1 \subset \mathbb{C}^{ln}$, and $\Phi(\cdot\,, 0) = f$.

Proof, 2

Clearly, $\Phi(p, \cdot\,)$ is holomorphic on U_1 for each $p \in C$ and

$$\left.\frac{\partial \Phi(p,t)}{\partial t_{j,k}}\right|_{t=0} = h_{j,k}(p)\, V_{j,k}(f(p)).$$

Let $\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_l)$ denote the period map (1) associated to the family of curves $\mathcal{C}$ in the lemma. Note that

$$\left.\frac{\partial \mathcal{P}_i(\Phi(\cdot\,,t))}{\partial t_{j,k}}\right|_{t=0} = \int_{C_i} h_{j,k} \cdot (V_{j,k} \circ f) \cdot \theta \in \mathbb{C}^n \tag{4}$$

for all $i, j = 1, \ldots, l$ and $k = 1, \ldots, n$.

A suitable choice of the function $h_{i,k}$ supported on a short arc around $p_{i,k}$ in $I_i \subset C_i$ ensures that this vector is as close as desired to $V_{i,k}(f(p_{i,k})) \in \mathbb{C}^n$ if $i = j$, and it equals zero if $i \neq j$.

The matrix of the differential of $\frac{\partial}{\partial t}\big|_{t=0} \mathcal{P}(\Phi(\cdot\,,t))$ has a block structure, with a block of size $n \times n$ for each pair of indices $i, j \in \{1, \ldots, l\}$; its total size is $ln \times ln$. The off-diagonal blocks for $i \neq j$ vanish by what has been said above, while the diagonal blocks corresponding to $i = j$ are invertible. Hence, the differential is invertible.

Proof, 3

Since the set $C = \bigcup_{i=1}^{l} C_i$ is Runge in S, the Bishop–Mergelyan theorem shows that we can approximate each function $h_{i,k}$ uniformly on C by a holomorphic function $\tilde{h}_{i,k} \in \mathscr{O}(S)$ on a neighbourhood of S. Inserting these new functions into Φ (3) furnishes a spray $\Phi_f : S \times U \to \mathcal{A}_*$ satisfying the conclusion of the lemma on a smaller ball $0 \in U \subset U_1$.

It remains to explain how to interpolate on a finite set $P \subset S$. Fix $s \in \mathbb{N}$. Let $g \in \mathcal{O}(S)$ be a holomorphic function vanishing to order s in each point of P and having no other zeros. Choose points $p_{i,k} \in I_i \setminus P$ and vector fields $V_{i,k}$ as above. As before, we can find continuous functions $h'_{i,k} : C \to \mathbb{C}$ supported on short arcs in I_i around the points $p_{i,k}$ such that, setting $h_{i,k} = gh'_{i,k}$ for all $i = 1, \ldots, l$ and $k = 1, \ldots, n$, conditions (4) hold.

We then approximate each $h'_{i,k}$ by $\tilde{h}'_{i,k} \in \mathscr{O}(S)$ and set $\tilde{h}_{i,k} = g\tilde{h}'_{i,k} \in \mathscr{O}(S)$. Inserting into (3) yields a spray Φ_f with the desired properties such that $\Phi_f(\cdot\,, t) : S \to \mathcal{A}_*,\ t \in U$, agrees with f at each point $p \in P$, to order s if $p \in P \cap \mathring{S}$.

Proof of the Runge approximation theorem for minimal surfaces

We exhaust M by smoothly bounded compact Runge sets

$$K = K_1 \subset K_2 \subset \cdots\cdots \bigcup_{i=1}^{\infty} K_i = M$$

such that for every $i \in \mathbb{N}$ we have one of the following possibilities.

(a) The noncritical case: there is no change of topology from K_i to K_{i+1}. This means that $K_{i+1} \setminus K_i$ is a union of finitely many pairwise disjoint annuli.

(a) The critical case: K_{i+1} admits a deformation retraction onto an admissible set $S_i = K_i \cup E_i$ obtained by attaching smooth arcs to K_i.

We must explain how to approximate a conformal minimal immersion $X_i : K_i \to \mathbb{R}^n$ as closely as desired by $X_{i+1} : K_{i+1} \to \mathbb{R}^n$; the limit $X = \lim_{i\to\infty} X_i : M \to \mathbb{R}^n$ is then a conformal minimal immersion.

Fix $i \in \mathbb{N}$ and write $K = K_i,\ L = K_{i+1}$. Let $X : K \to \mathbb{R}^n$ be a conformal minimal immersion. Let $C_1, \ldots, C_\ell \subset K$ be closed curves forming a homology basis of K such that $C = \bigcup_{j=1}^{\ell} C_j$ is Runge in M. We may assume that X does not map any curve C_i to a ray of $\mathcal{A}$.

The noncritical case

Let $\mathbb{B}^n$ denote the unit ball of $\mathbb{C}^n$. By the lemma, there is a holomorphic map

$$F : K \times \mathbb{B}^{n\ell} \to \mathcal{A}_*, \quad F(\cdot\,, 0) = f = 2\partial X/\theta$$

such that the period map

$$\mathbb{B}^{n\ell} \ni t \;\longmapsto\; \left(\int_{C_j} F(\cdot\,, t)\theta \right)_{j=1}^{\ell} \in \mathbb{C}^{n\ell}$$

is biholomorphic onto its image.

Since $\mathcal{A}_*$ is an Oka manifold, we can approximate F by a holomorphic map $\widetilde{F} : M \times \mathbb{B}^{n\ell} \to \mathcal{A}_*$.

By the implicit function theorem, there exists $\tilde{t} \in \mathbb{B}^{n\ell}$ close to 0 such that the map $\tilde{f} = F(\cdot\,, \tilde{t}) : M \to \mathcal{A}_*$ has vanishing real periods on the curves $C_1, \ldots, C_\ell$.

Hence, fixing a point $p_0 \in K$, the map $\widetilde{X} : L \to \mathbb{R}^n$ given by

$$\widetilde{X}(p) = X(p_0) + \Re \int_{p_0}^{p} \tilde{f}\theta, \qquad p \in L$$

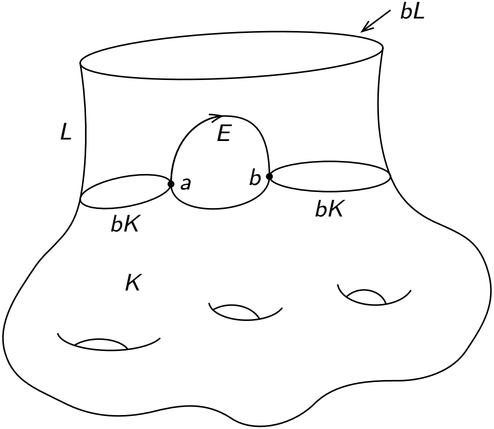
is a conformal minimal immersion which approximates $X : K \to \mathbb{R}^n$ on K.

Assume now that $E \subset L \setminus \mathring{K}$ is an arc attached with its endpoints to K such that $K \cup E$ is a deformation retract of L. This situation arises when passing a critical point of index 1 of an exhaustion function on M.
We illustrate **attaching a pair of pants**, which increases the genus by one.

1° The domain K

2° Add the arc E – the seam of the pants

3° Add the pair of pants



The critical case

Let $a, b \in bK$ be the endpoints of the arc E. Fix $p_0 \in K$ and write

$$X(p) = X(p_0) + \Re \int_{p_0}^{p} f\theta, \qquad p \in K.$$

Extend f smoothly across E to a map $f : K \cup E \to \mathcal{A}_*$ such that

$$\Re \int_E f\theta = X(b) - X(a) \in \mathbb{R}^n.$$

This is possible by Gromov's convex integration lemma since the convex hull of $\mathcal{A}_*$ equals $\mathbb{C}^n$.

Now, proceed as in the noncritical case, constructing a period dominating spray and applying Mergelyan approximation on $K \cup E$.

The proof of the approximation theorem follows by induction on an exhaustion of M such that every step is of one of the two types described above. Interpolation on a discrete set of points is obtained by the same scheme.

However, general position theorems, and the existence of proper conformal minimal immersions or embeddings, require substantial additional work.

A parametric h-principle for minimal surfaces

By using parametric extensions of our techniques, one obtains various parametric h-principles for minimal surfaces, and for holomorphic immersions $M \to \mathbb{C}^n$ directed by any complex cone $A \subset \mathbb{C}^n$ such that $A_* = A \setminus \{0\}$ is a connected Oka manifold. Here is an example.

Theorem

Assume that M is an open Riemann surface, θ is a nonvanishing holomorphic 1-form on M, $Q \subset P$ are Euclidean compacts (Q may be empty), and $f : M \times P \to \mathcal{A}_*$ is a continuous map such that for all $p \in Q$, $f(\cdot\,, p) : M \to \mathcal{A}_*$ is a nonflat holomorphic map with vanishing real periods.

Then, there is a continuous map $X : M \times P \to \mathbb{R}^n$ such that

- (a) $X(\cdot\,, p) : M \to \mathbb{R}^n$ *is a conformal minimal immersion for every* $p \in P$,
- (b) $\partial X(\cdot\,, p)/\theta = f(\cdot\,, p) : M \to \mathcal{A}_*$ *for all* $p \in Q$,
- (c) $\partial X/\theta$ *is homotopic to* f *by a homotopy which is fixed on* $M \times Q$.

Theorem 5.3 in [F. Forstnerič and F. Lárusson: The parametric h-principle for minimal surfaces in $\mathbb{R}^n$ and null curves in $\mathbb{C}^n$. Comm. Anal. Geom. **27:1** (2019) 1–45.]

Topological structure of the space of conformal minimal immersions

$\mathrm{NC}_*(M,\mathbb{C}^n)$ the space of nonflat holomorphic null curves $M \to \mathbb{C}^n$
$\mathrm{CMI}_*(M,\mathbb{R}^n)$ the space of nonflat conformal minimal immersions $M \to \mathbb{R}^n$

Theorem (F. Forstnerič and F. Lárusson, CAG, 2019)

All maps in the following diagram are weak homotopy equivalences, and genuine homotopy equivalences if $H_1(M,\mathbb{Z})$ is finitely generated.

$$\begin{tikzcd}
\Re\mathrm{NC}_*(M,\mathbb{C}^n) \arrow[r, hook] \arrow[dr, "\partial"'] & \mathrm{CMI}_*(M,\mathbb{R}^n) \arrow[d, "\partial"] & \\
 & \mathcal{O}(M,\mathcal{A}_*) \arrow[r, hook] & \mathscr{C}(M,\mathcal{A}_*)
\end{tikzcd}$$

Furthermore, the map $\mathrm{CMI}(M,\mathbb{R}^n) \to \mathscr{C}(M,\mathcal{A}_*)$ induces a bijection of path components of the two spaces (Corollary 5.7.7 in the book).
Let $H_1(M,\mathbb{Z}) = \mathbb{Z}^\ell$. Then,

$$\pi_0(\mathrm{CMI}(M,\mathbb{R}^n)) = \begin{cases} \mathbb{Z}_2^\ell, & n = 3; \\ 0, & n > 3. \end{cases}$$

Complete minimal surfaces of finite total curvature

Let $M = \overline{M} \setminus \{p_1, \dots, p_m\}$ be a compact Riemann surface with finitely many punctures. Consider the diagram

$$\begin{tikzcd} \Re\mathrm{NC}_*(M, \mathbb{C}^n) \arrow[r, hook] \arrow[dr, "\partial"'] & \mathrm{CMI}_*(M, \mathbb{R}^n) \arrow[d, "\partial"] \\ & \mathscr{A}^1(M, \mathcal{A}_*) \end{tikzcd}$$

- ∂ is the $(1, 0)$-differential,
- $\mathrm{CMI}_*(M, \mathbb{R}^n)$ is the space of nonflat complete conformal minimal immersions $M \to \mathbb{R}^n$ with finite total Gaussian curvature,
- $\mathrm{NC}_*(M, \mathbb{C}^n)$ is the space of nonflat complete algebraic null immersons,
- $\mathscr{A}^1(M, \mathcal{A})$ is the space of algebraic 1-forms on M with values in $\mathcal{A}_*$.

Theorem (Alarcón, Lárusson, F. 2024)

The maps in the above diagram are weak homotopy equivalences.

Problem: What can be said about the inclusion $\mathscr{A}^1(M, \mathcal{A}_*) \hookrightarrow \mathcal{O}^1(M, \mathcal{A}_*)$?

Families of proper minimal surfaces

Assume that

- M is a connected, open, orientable smooth surface,
- B is a compact subset of a Euclidean space $\mathbb{R}^m$ which admits a retraction from an open neighbourhood $U \subset \mathbb{R}^m$ (every finite CW complex is such),
- $\mathscr{J} = \{J_b\}_{b \in B}$ is a continuous family of complex structures on M of local Hölder class $\mathscr{C}^\alpha$ for some $0 < \alpha < 1$.

Theorem (Alarcón & F., May 2026)

Assume that M, B, and $\mathscr{J}$ are as above. Then there exists a continuous map $u = (u_1, u_2, u_3) : B \times M \to \mathbb{R}^3$ such that the map

$$u_b = u(b, \cdot\,) = (u_{b,1}, u_{b,2}, u_{b,3}) : M \to \mathbb{R}^3$$

is a proper J_b-conformal minimal immersion for every $b \in B$.

The proof relies on a recent Oka principle for maps from families of open Riemann surfaces to Oka manifolds:

[F.F., Runge and Mergelyan theorems for families of open Riemann surfaces, 2024, https://arxiv.org/abs/2412.04608]

Isotopies to proper CMIs

We proved that every open Riemann surface M admits a proper conformal minimal immersion $M \to \mathbb{R}^3$. The following is an interesting improvement.

Theorem (Tjaša Vrhovnik, 2025)

Let M be an open Riemann surface and $X : M \to \mathbb{R}^n$, $n \geq 3$, be a conformal minimal immersion. Then, there exists a homotopy of conformal minimal immersions $X_t : M \to \mathbb{R}^n$, $t \in [0, 1]$, such that $X_0 = X$ and X_1 is proper.

Tjaša Vrhovnik, Ph.D., University of Granada, 23 June 2026

Everybody is the Gauss map

Let $X = (X_1, \ldots, X_n) : M \to \mathbb{R}^n$ be a conformal minimal immersion.

Since the 1-form $\partial X = (\partial X_1, \ldots, \partial X_n)$ is holomorphic and nowhere vanishing, it determines the holomorphic **Gauss map** of X:

$$G_X : M \to \mathbb{CP}^{n-1}, \quad G_X(p) = [\partial X_1(p) \colon \cdots : \partial X_n(p)], \quad p \in M,$$

Since $\sum_{j=1}^n (\partial X_j)^2 = 0$, G_X assumes values in the hyperquadric

$$Q^{n-2} = \big\{[z_1 : z_2 : \cdots : z_n] \in \mathbb{CP}^{n-1} : z_1^2 + \cdots + z_n^2 = 0\big\}.$$

Theorem (Alarcón, Lopez., F.; J. Geom. Anal. 2017)

Let M be an open Riemann surface and $n \geq 3$.

For every holomorphic map $\mathscr{G} : M \to Q^{n-2} \subset \mathbb{CP}^{n-1}$ there exists a conformal minimal immersion $X : M \to \mathbb{R}^n$ with $G_X = \mathscr{G}$.

In particular, if $n = 3$ then every holomorphic map $M \to Q^1 \cong \mathbb{CP}^1$ is the Gauss map of a conformal minimal immersion $M \to \mathbb{R}^3$.

If $\mathscr{G}(M)$ is not contained in a proper projective subspace of $\mathbb{CP}^{n-1}$, then X can be chosen to have arbitrary flux, and to be an embedding if $n \geq 5$ and an immersion with simple double points if $n = 4$.

The first step in the proof is to find a holomorphic map $f : M \to \mathcal{A}_*$ such that $G_f = [f_1 : \cdots : f_n] = \mathscr{G}$. This is an application of the Oka principle.

The second and the main step is to find a holomorphic function $h : M \to \mathbb{C}^*$ such that $\int_C hf\theta = 0$ for all closed curves $C \subset M$. This a highly technical but the basic idea of killing the periods is similar to the one in the proof of the Runge approximation theorem for conformal minimal immersions.

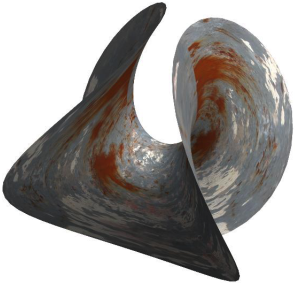
The conformal minimal immersions in this theorem cannot be complete or proper in general. There is a vast literature on the question how big a set can be omitted by the Gauss map of a complete minimal surface. We mention two of them.

1990 **Fujimoto** The Gauss map of a complete nonflat minimal surface in $\mathbb{R}^3$ can omit at most 4 values in $\mathbb{CP}^1$, and the bound 4 is sharp.

1991 **Min Ru** If $X : M \to \mathbb{R}^n$ is a complete nonflat minimal surface then its Gauss map $G_X : M \to \mathbb{CP}^{n-1}$ can omit at most $n(n+1)/2$ hyperplanes in general position.

Different minimal surfaces may share the same Gauss map

1886 **Alfred Enneper** Besides the catenoid, Enneper's surface is one of two most basic properly embedded minimal surfaces in $\mathbb{R}^3$. It is parameterized by $\mathbb{C}$.



$\mathfrak{g}(z) = z$ the Gauss map

$\mathrm{TC} = -4\pi$ the total curvature

$\partial X_3 = z dz$ for Enneper

$\partial X_3 = \frac{dz}{2z}$ for catenoid

Enneper $\quad X(\zeta) = \Re \int_0^\zeta \left(\frac{1}{2}\left(\frac{1}{z} - z\right), \frac{i}{2}\left(\frac{1}{z} + z\right), 1 \right) 2z dz, \qquad \zeta \in \mathbb{C};$

Catenoid $\quad X(\zeta) = (1,0,0) - \Re \int_1^\zeta \left(\frac{1}{2}\left(\frac{1}{z} - z\right), \frac{i}{2}\left(\frac{1}{z} + z\right), 1 \right) \frac{dz}{z}, \quad \zeta \in \mathbb{C}^*.$