Minimal surfaces from a complex analytic viewpoint
Lecture 3: The minimal metric

Franc Forstnerič

Univerza v Ljubljani

European Research Council
Executive Agency

Established by the European Commission

European School of Differential Geometry
L'Aquila, June 2026

The Schwarz–Pick lemma for conformal minimal discs

Using conformal minimal discs, possibly with branch points, one can introduce a Kobayashi–type pseudometric on domains in $\mathbb{R}^n$, $n \geq 3$, and more generally in any Riemannian manifold. For domains in $\mathbb{R}^n$, this was done in

[F. Forstnerič and D. Kalaj: Schwarz–Pick lemma for harmonic maps which are conformal at a point. Anal. PDE, 17(3):981–1003, 2024]

The motivation is given by the following Schwarz–Pick lemma.

Theorem (F. and Kalaj, 2021)

Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$ and $\mathbb{B}^n$ the unit ball of $\mathbb{R}^n$, $n \geq 2$. Assume that $f : \mathbb{D} \to \mathbb{B}^n$ is a harmonic map which is conformal at a point $z \in \mathbb{D}$. Denote by $R \in (0,1]$ the radius of the affine disc $\Sigma = (f(z) + df_z(\mathbb{R}^2)) \cap \mathbb{B}^n$. Then

$$\|df_z\| = \frac{1}{\sqrt{2}} |\nabla f(z)| \ \leq \ \frac{1}{R} \cdot \frac{1 - |f(z)|^2}{1 - |z|^2}.$$

The equality holds if and only if f is a conformal diffeomorphism of $\mathbb{D}$ onto Σ, and in this case the equality holds at every point of $\mathbb{D}$.

The case $n = 2$, $R = 1$ of the theorem generalizes the classical **Schwarz–Pick lemma** due to **H. A. Schwarz** 1869, **H. Poincaré** 1884, and **G. Pick** 1915.

Comments

Setting $\theta = \measuredangle(f(z), df_z(\mathbb{R}^2))$, we have

$$R = \sqrt{1 - |f(z)|^2 \sin^2\theta},$$

and hence the estimate is

$$\|df_z\| \;\leq\; \frac{1}{R} \cdot \frac{1-|f(z)|^2}{1-|z|^2} = \frac{1-|f(z)|^2}{\sqrt{1-|f(z)|^2\sin^2\theta}}\frac{1}{1-|z|^2} \tag{1}$$

For a fixed $|f(z)| \in [0,1)$, the maximum of the right hand side over angles $\theta \in [0, \pi/2]$ equals

$$\frac{\sqrt{1-|f(z)|^2}}{1-|z|^2},$$

and it is attained at $\theta = \pi/2$ (i.e., when $f(z)$ is orthogonal to $df_z(\mathbb{R}^2)$), unless $f(z) = 0$ when it is independent of θ.

The minimum $\frac{1-|f(z)|^2}{1-|z|^2}$ is attained at $\theta = 0$ when $f(z) \in df_z(\mathbb{R}^2)$.

The weaker estimate $\frac{1}{\sqrt{2}}\,|\nabla f(z)| \leq \frac{\sqrt{1-|f(z)|^2}}{1-|z|^2}$ holds for all harmonic maps $\mathbb{D} \to \mathbb{B}^n$ (see Theorem 2.2 in the paper by F. and Kalaj).

A differential geometric interpretation of the classical Schwarz–Pick lemma is that holomorphic maps $\mathbb{D} \to \mathbb{D}$ are distance decreasing in the Poincaré metric, and isometries coincide with holomorphic automorphisms of $\mathbb{D}$.
The analogous result holds for holomorphic maps $\mathbb{D} \to \mathbb{B}^n_{\mathbb{C}}$ with the Kobayashi metric on the complex ball $\mathbb{B}^n_{\mathbb{C}}$, where orientation preserving isometric embeddings are holomorphic embeddings onto affine complex discs in $\mathbb{B}^n_{\mathbb{C}}$.

In the same spirit, we can interpret the above result as the distance decreasing property of conformal harmonic maps $\mathbb{D} \to \mathbb{B}^n$ with respect to the Poincaré metric on $\mathbb{D}$ and the classical **Cayley–Klein metric**[1] on $\mathbb{B}^n$, $n \geq 3$:

$$\mathrm{CK}(\mathbf{x}, \mathbf{v}) = \frac{\sqrt{1 - |\mathbf{x}|^2 \sin^2 \phi}}{1 - |\mathbf{x}|^2} \, |\mathbf{v}|, \qquad \mathbf{x} \in \mathbb{B}^n, \ \mathbf{v} \in \mathbb{R}^n,$$

where $\phi = \measuredangle(\mathbf{x}, \mathbb{R}\mathbf{v}) \in [0, \pi/2]$. Equivalently,

$$\mathrm{CK}(\mathbf{x}, \mathbf{v})^2 = \frac{(1 - |\mathbf{x}|^2)|\mathbf{v}|^2 + |\mathbf{x} \cdot \mathbf{v}|^2}{(1 - |\mathbf{x}|^2)^2} = \frac{|\mathbf{v}|^2}{1 - |\mathbf{x}|^2} + \frac{|\mathbf{x} \cdot \mathbf{v}|^2}{(1 - |\mathbf{x}|^2)^2}.$$

[1] The **Beltrami–Cayley–Klein model of hyperbolic geometry** was developed by **Arthur Cayley** (1859), **Eugenio Beltrami** (1868), and **Felix Klein** (1871–1873). The underlying space is the ball $\mathbb{B}^n$, and geodesics are line segments with endpoints on the sphere. This is a special case of a metric on convex domains introduced by **David Hilbert** in 1895. Up to constants, the Cayley–Klein metric is the restriction to $\mathbb{B}^n$ of the Kobayashi and the Bergman metric on the complex ball $\mathbb{B}^n_{\mathbb{C}} \subset \mathbb{C}^n$.

Proof of the Schwarz–Pick lemma, 1

Precomposing with a conformal automorphism of $\mathbb{D}$, it suffices to take $z = 0$ and prove the estimate

$$\|df_0\| \;\le\; \frac{1-|f(0)|^2}{\sqrt{1-|f(0)|^2\sin^2\theta}} \tag{2}$$

On the image side, we may use postcomposition of maps $\mathbb{D} \to \mathbb{B}^n$ by elements of the orthogonal group O_n.

Fix a point $\mathbf{q} \in \mathbb{B}^n$ and a 2-plane $0 \in \Lambda \subset \mathbb{R}^n$, and consider the affine disc $\Sigma = (\mathbf{q} + \Lambda) \cap \mathbb{B}^n$. Let us identify conformal parameterizations $\mathbb{D} \to \Sigma$ sending 0 to $\mathbf{q}$. Let $\mathbf{p} \in \Sigma$ be the closest point to the origin. If $n = 2$ then $\mathbf{p} = 0$ and $\Sigma = \mathbb{D}$. Suppose now that $n \ge 3$. Up to an orthogonal rotation, we may assume that

$$\mathbf{p} = (0, 0, p, 0 \ldots, 0) \quad \text{and} \quad \Sigma = \left\{(x, y, p, 0, \ldots, 0) : x^2 + y^2 < 1 - p^2\right\}.$$

Let $\mathbf{q} = (b_1, b_2, p, 0, \ldots, 0) \in \Sigma$, and let $\theta = \measuredangle(\mathbf{q}, \Sigma)$. Set

$$c = \sqrt{1-p^2} = \sqrt{1-|\mathbf{q}|^2\sin^2\theta}, \quad a = \frac{b_1 + \mathrm{i} b_2}{c} \in \mathbb{D}, \quad |a| = \frac{|\mathbf{q}|\cos\theta}{c}. \tag{3}$$

Proof of the Schwarz–Pick lemma, 2

We orient Σ by the tangent vectors (∂_x, ∂_y). Every orientation preserving conformal parameterization $f : \mathbb{D} \to \Sigma$ with $f(0) = \mathbf{q}$ is then of the form

$$f(z) = \left(c\,\Re \frac{e^{\mathrm{i}t} z + a}{1 + \bar{a} e^{\mathrm{i}t} z}, c\,\Im \frac{e^{\mathrm{i}t} z + a}{1 + \bar{a} e^{\mathrm{i}t} z}, p, 0, \ldots, 0 \right), \quad z \in \mathbb{D} \tag{4}$$

for some $t \in \mathbb{R}$. (If $n = 2$ then $p = 0,\ c = 1$, and the same holds if we drop all coordinates except the first two.) Using the complex coordinate $x + \mathrm{i}y$ in the plane $df_0(\mathbb{R}^2) = \mathbb{R}^2 \times \{0\}^{n-2}$, the map (4) can be written in the form

$$f(z) = \left(c \frac{e^{\mathrm{i}t} z + a}{1 + \bar{a} e^{\mathrm{i}t} z}, p, 0, \ldots, 0 \right) = (h(z), p, 0, \ldots, 0).$$

From (3) it follows that

$$\begin{aligned} |h'(0)| &= c\,(1 - |a|^2) = \frac{c^2 - c^2|a|^2}{c} = \frac{1 - |\mathbf{q}|^2 \sin^2\theta - |\mathbf{q}|^2 \cos^2\theta}{c} \\ &= \frac{1 - |\mathbf{q}|^2}{\sqrt{1 - |\mathbf{q}|^2 \sin^2\theta}} = \frac{1 - |f(0)|^2}{\sqrt{1 - |f(0)|^2 \sin^2\theta}}. \end{aligned}$$

Since $\|df_0\| = |h'(0)|$, this gives the equality in (2) at $z = 0$.

The main lemma

The theorem now follows from the following lemma.

Lemma

Let $f : \mathbb{D} \to \mathbb{B}^n$, $n \geq 2$, be the extremal disc (4). If $g : \mathbb{D} \to \mathbb{B}^n$ is a harmonic disc such that $g(0) = f(0)$, g is conformal at 0, and $dg_0(\mathbb{R}^2) = df_0(\mathbb{R}^2)$, then $\|dg_0\| \leq \|df_0\|$, with equality if and only if $g(z) = f(e^{\mathrm{i}s}z)$ or $g(z) = f(e^{\mathrm{i}s}\bar{z})$ for some $s \in \mathbb{R}$ and all $z \in \mathbb{D}$.

The proof uses ideas from Laszlo Lempert's seminal paper on complex geodesics of the Kobayashi metric in convex domains of $\mathbb{C}^n$:

[L. Lempert, La métrique de Kobayashi et la représentation des domaines sur la boule. Bull. Soc. Math. Fr. 109, 427–474 (1981)

Let p, c, a be as in (3) related to the extremal map f (4), where $\mathbf{q} = f(0)$. Precomposing f by a rotation, we may assume that $t = 0$ in (4). For simplicity we assume that $n = 3$; the proof for $n > 3$ is exactly the same. If $n = 2$, we delete the remaining components and take $c = 1$.

Proof of the main lemma

Consider the holomorphic disc $F : \mathbb{D} \to \Omega = \mathbb{B}^3 \times i\mathbb{R}^3$ given by

$$F(z) = \left(c\,\frac{z+a}{1+\bar{a}z}, -c\,i\frac{z+a}{1+\bar{a}z}, p \right), \quad z \in \mathbb{D}.$$

Then, $f = \Re F$. Suppose that $g : \mathbb{D} \to \mathbb{B}^3$ is as in the lemma. Up to replacing $g(z)$ by $g(e^{is}z)$ or $g(e^{is}\bar{z})$ for a suitable $s \in \mathbb{R}$, we may assume that

$$dg_0 = r \cdot df_0 \quad \text{for some } r > 0. \tag{5}$$

We must prove that $r \leq 1$, with $r = 1$ if and only if $g = f$.

Let $G : \mathbb{D} \to \Omega$ be the unique holomorphic map with $\Re G = g$ and $G(0) = F(0)$. By the Cauchy–Riemann equations, (5) implies

$$G'(0) = r \cdot F'(0).$$

It follows that the map $(F(z) - G(z))/z$ is holomorphic on $\mathbb{D}$ and

$$\lim_{z \to 0} \frac{F(z) - G(z)}{z} = F'(0) - G'(0) = (1-r)F'(0). \tag{6}$$

The bounded harmonic map $g : \mathbb{D} \to \mathbb{B}^3$ has a nontangential boundary value at almost every point of the circle $\mathbb{T} = b\mathbb{D}$. Since the Hilbert transform is an isometry on the Hilbert space $L^2(\mathbb{T})$, the same is true for its holomorphic extension G.

Denote by $\langle \cdot, \cdot \rangle$ the complex bilinear form

$$\langle z, w \rangle = \sum_{i=1}^{n} z_i w_i, \quad z, w \in \mathbb{C}^n.$$

On vectors in $\mathbb{R}^n$ this is the Euclidean inner product.

For each $z = e^{\mathrm{i}t} \in b\mathbb{D}$ the vector $f(z) \in b\mathbb{B}^3$ is the unit normal vector to the sphere $b\mathbb{B}^3$ at the point $f(z)$. Since $\mathbb{B}^3$ is strongly convex and f is real-valued, we have that

$$\Re \left\langle F(z) - G(z), f(z) \right\rangle = \left\langle f(z) - g(z), f(z) \right\rangle \geq 0 \quad a.e.\ z \in b\mathbb{D}, \qquad (7)$$

and the value is positive for almost every $z \in b\mathbb{D}$ if and only if $g \neq f$.

Proof of the main lemma, 3

Consider the map $\tilde{f}$ defined on the circle $b\mathbb{D}$ by

$$\tilde{f}(z) = z|1 + \bar{a}z|^2 f(z), \qquad |z| = 1. \tag{8}$$

A calculation, taking into account $z\bar{z} = 1$, shows that

$$\tilde{f}(z) = \begin{pmatrix} \frac{c}{2}\left(1 + a^2 + 4(\Re a)z + (1 + \bar{a}^2)z^2\right) \\ \frac{c}{2}\left(\mathrm{i}(1 - a^2) + 4(\Im a)z + \mathrm{i}(\bar{a}^2 - 1)z^2\right) \\ p\,(z + a)(1 + \bar{a}z) \end{pmatrix}.$$

We extend $\tilde{f}$ to all $z \in \mathbb{C}$ by the quadratic holomorphic polynomial map on the right hand side above. Since $|1 + \bar{a}z|^2 > 0$ for $z \in \overline{\mathbb{D}}$, (7) implies

$$\begin{aligned} h(z) &:= \Re\left\langle F(z) - G(z), |1 + \bar{a}z|^2 f(z)\right\rangle \\ &= \left\langle f(z) - g(z), |1 + \bar{a}z|^2 f(z)\right\rangle \geq 0 \quad a.e.\ z \in b\mathbb{D}, \end{aligned}$$

and $h > 0$ almost everywhere on $b\mathbb{D}$ iff $g \neq f$. From (8) we see that

$$h(z) = \Re\left\langle \frac{F(z) - G(z)}{z}, \tilde{f}(z)\right\rangle \quad a.e.\ z \in b\mathbb{D} \tag{9}$$

Proof of the main lemma, 4

Since the maps $(F(z) - G(z))/z$ and $\tilde{f}(z)$ are holomorphic on $\mathbb{D}$, the formula (9) provides an extension of h from $b\mathbb{D}$ to a nonnegative harmonic function on $\mathbb{D}$ which is positive on $\mathbb{D}$, unless $f = g$ when $h = 0$.

Inserting the value $F'(0) - G'(0) = (1-r)F'(0)$ (6) into (9) gives

$$h(0) = \Re \left\langle F'(0) - G'(0), \tilde{f}(0) \right\rangle = (1-r)\, \Re \left\langle F'(0), \tilde{f}(0) \right\rangle \geq 0,$$

with equality iff $f = g$. Applying this with the linear conformal map

$$g(z) = f(0) + r\, df_0(z), \quad z \in \mathbb{D}$$

for some $0 < r < 1$, we get

$$\Re \left\langle F'(0), \tilde{f}(0) \right\rangle > 0.$$

It follows that $r \leq 1$ for every g, with equality iff $g = f$.

This completes the proof of the lemma, and hence of the theorem.

Discussion

The main point in the above proof is that a complexification of a conformal proper affine disc in $\mathbb{B}^n$ is a **stationary disc** in the tube $\mathcal{T}_{\mathbb{B}^n} = \mathbb{B}^n \times i\mathbb{R}^n$. In Lempert's terminology in

[Laszlo Lempert: La métrique de Kobayashi et la représentation des domaines sur la boule. Bull. Soc. Math. France, 109(4):427–474, 1981]

a proper holomorphic disc $F : \mathbb{D} \to \Omega$ in a smoothly bounded convex domain $\Omega \subset \mathbb{C}^n$, extending continuously to $\overline{\mathbb{D}}$, is a stationary disc if, denoting by $\nu : b\mathbb{D} \to \mathbb{C}^n$ the unit normal vector field to $b\Omega$ along the boundary circle $F(b\mathbb{D}) \subset b\Omega$, there is a positive continuous function $q > 0$ on $b\mathbb{D}$ such that the function

$$z\, q(z)\overline{\nu(z)}, \quad |z| = 1,$$

extends from the circle $b\mathbb{D} = \{|z| = 1\}$ to a holomorphic function $\tilde{f}(z)$ on $\mathbb{D}$.

Lempert showed that every stationary disc F in a bounded strongly convex domain is the unique Kobayashi extremal disc through the point $F(a)$ in the tangent direction $F'(a)$ for every $a \in \mathbb{D}$. Lempert's theory also works on tubes over bounded strongly convex domains; however, our proof is self-contained. In our case, a suitable holomorphic function $\tilde{f}$ is given by (8):

$$\tilde{f}(z) = z|1 + \bar{a}z|^2 f(z), \qquad |z| = 1.$$

Dimension $n = 2$

Let D be a connected and simply connected domain in $\mathbb{R}^2$. Given $p \in D$, set

$$M_D(p) = \sup\big\{\|df_0\| : \ f : \mathbb{D} \to D \text{ harmonic}, \ f(0) = p, \ df_0 \text{ conformal}\big\}.$$

For $D = \mathbb{D}$, our main result shows that for any $p \in \mathbb{D}$, a map $f : \mathbb{D} \to \mathbb{D}$ in the definition of $M_{\mathbb{D}}(p)$ satisfies $\|df_0\| = M_{\mathbb{D}}(p)$ if and only if f is a conformal diffeomorphism. This fails for more general domains $D \subset \mathbb{R}^2$.

Example

Let $U : \mathbb{D} \to [-1, 1]$ be the harmonic function

$$U(z) = \Im \frac{2}{\pi} \log \frac{1+z}{1-z} = \frac{2}{\pi} \arctan \frac{2y}{1 - x^2 - y^2}.$$

Its boundary values equal $+1$ on the upper semicircle, -1 on the lower semicircle, $\nabla U(0) = \frac{4}{\pi}(0, 1)$, and $|\nabla U(0)| = \frac{4}{\pi}$. The harmonic map

$$f(x, y) = (U(y, x), U(x, y))$$

takes $\mathbb{D}$ onto the square $P = \{(x, y) \in \mathbb{R}^2 : |x| < 1, \ |y| < 1\}$, f is conformal at $(0, 0)$, and $\|df_0\| = 4/\pi \approx 1.27$. On the other hand, a biholomorphism $\Phi : \mathbb{D} \to P$ with $\Phi(0) = 0$ satisfies $|\Phi'(0)| \approx 1.08$.

Classification of convex domains in $\mathbb{R}^2$ with the extremal property

Theorem (F. and Kalaj, May 2026)

Let $a, \lambda \in \mathbb{C}$ *be such that the quadratic polynomial* $q(z) := 1 + az + \lambda z^2$ *has no zeros on* $\overline{\mathbb{D}}$. *By the Schur–Cohn criterion, this holds iff*

$$|\lambda| < 1 \quad \text{and} \quad |a - \bar{a}\lambda| < 1 - |\lambda|^2.$$

Given $c > 0$, *let* $F : \mathbb{D} \to \mathbb{C}$ *be the holomorphic map determined by*

$$F(0) = 0 \quad \text{and} \quad F'(z) = \frac{c}{1 + az + \lambda z^2}. \tag{10}$$

Then, F *is a biholomorphic map onto the bounded strongly convex domain* $D = F(\mathbb{D}) \subset \mathbb{C}$, *and*

$$M_D(0) = |F'(0)| = c.$$

The harmonic extremal maps attaining $M_D(0)$ *are*

$$z \longmapsto F(e^{i\theta} z) \quad \text{and} \quad z \longmapsto F(e^{i\theta} \bar{z}), \qquad \theta \in \mathbb{R}.$$

Conversely, if $\Phi : \mathbb{D} \to D$ *is a biholomorphic map onto a bounded convex domain with* $\Phi(0) = 0$ *and* $|\Phi'(0)| = M_D(0)$, *then, up to a rotation,* Φ *equals one of the maps in* (10).

The minimal metric on domains in $\mathbb{R}^n$

In the paper

[F. Forstnerič and D. Kalaj: Schwarz–Pick lemma for harmonic maps which are conformal at a point. Anal. PDE, 17(3):981–1003, 2024]

we introduced an intrinsic Finsler pseudometric g_Ω on any domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, in terms of conformal minimal discs $\mathbb{D} \to \Omega$. It is modelled on the definition of the Kobayashi pseudometric on complex manifolds, which uses holomorphic discs.

The pseudometric g_Ω, and the associated pseudodistance $\rho_\Omega : \Omega \times \Omega \to \mathbb{R}_+$, are the largest ones having the property that any conformal harmonic map $M \to \Omega$ from a hyperbolic conformal surface with the Poincaré metric is metric and distance decreasing.

With some more work, the same definition of g_Ω applies in any Riemannian manifold of dimension at least three (see the lectures by **Hervé Gaussier**). This provides the basis for hyperbolicity theory of domains in Euclidean spaces and, more generally, of Riemannian manifolds, in terms of minimal surfaces.

The minimal metric on domains in $\mathbb{R}^n$

Given a domain $\Omega \subset \mathbb{R}^n$, let $\mathrm{CH}(\mathbb{D}, \Omega)$ denote the space of conformal harmonic maps $\mathbb{D} \to \Omega$, possibly with branch points.

The **minimal (pseudo) metric** $g_\Omega : \Omega \times \mathbb{R}^n \to \mathbb{R}_+$ is defined for any $\mathbf{x} \in \Omega$ and $\mathbf{u} \in T_{\mathbf{x}}\Omega = \mathbb{R}^n$ by

$$g_\Omega(\mathbf{x}, \mathbf{u}) = \inf\left\{1/r > 0 : \exists f \in \mathrm{CH}(\mathbb{D}, \Omega),\ f(0) = \mathbf{x},\ f_x(0) = r\mathbf{u}\right\}.$$

It follows that every conformal harmonic map $f : \mathbb{D} \to \Omega$ satisfies

$$g_\Omega\big(f(z), df_z(\xi)\big) \ \leq\ \mathcal{P}(z, \xi) = \frac{|\xi|}{1 - |z|^2}, \qquad z \in \mathbb{D},\ \xi \in \mathbb{R}^2,$$

and g_Ω is the biggest pseudometric on Ω with this property.

By integrating g_Ω, we get the **minimal pseudodistance** $\rho_\Omega : \Omega \times \Omega \to \mathbb{R}_+$:

$$\rho_\Omega(\mathbf{x}, \mathbf{y}) = \inf_\gamma \int_0^1 g_\Omega(\gamma(t), \dot\gamma(t))\, dt, \qquad \mathbf{x}, \mathbf{y} \in \Omega,$$

where the infimum is over paths $\gamma : [0,1] \to \Omega$ with $\gamma(0) = \mathbf{x},\ \gamma(1) = \mathbf{y}$.

Metric decreasing properties

A conformal surface M is said to be **hyperbolic** if its universal covering space is the disc $\mathbb{D}$. Such a surface carries the **Poincaré metric**, $\mathcal{P}_M$, the unique Riemannian metric on M such that the universal covering map $h : \mathbb{D} \to M$ is a local isometry.

It follows that every conformal harmonic map f from $(M, \mathcal{P}_M)$ to (Ω, g_Ω) is metric and distance decreasing:

$$g_\Omega\big(f(z), df_z(\xi)\big) \ \le \ \mathcal{P}_M(z, \xi), \qquad z \in M,\ \xi \in T_z M.$$

Any rigid map $R : \mathbb{R}^n \to \mathbb{R}^m$ $(n \le m)$ with $R(\Omega) \subset \Omega'$ is obviously metric decreasing:

$$g_{\Omega'}(R(\mathbf{x}), R(\mathbf{v})) \le g_\Omega(\mathbf{x}, \mathbf{v}), \qquad \mathbf{x} \in \Omega,\ \mathbf{v} \in \mathbb{R}^n.$$

A domain $\Omega \subset \mathbb{R}^n$ is **hyperbolic** if the pseudodistance ρ_Ω is a distance function on Ω, and is **complete hyperbolic** if (Ω, ρ_Ω) is a complete metric space. A hyperbolic domain does not admit any conformal harmonic maps $\mathbb{C} \to \Omega$.

Minimal (complete) hyperbolic domains

On the ball $\mathbb{B}^n$, $g_{\mathbb{B}^n}$ is the Cayley–Klein metric. This is an immediate consequence of our Schwarz–Pick lemma for conformal harmonic discs. This seems to be the only domain for which an explicit formula is known.

In [B. Drinovec Drnovšek and F. Forstnerič: Hyperbolic domains in real Euclidean spaces. Pure Appl. Math. Q., 19:6 (2023) 2689–2735], several results on (complete) minimal hyperbolicity were established. For example:

Theorem

For a (not necessarily bounded) convex domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, the following conditions are equivalent:

- (a) *Ω is minimal hyperbolic.*
- (b) *Ω is complete minimal hyperbolic.*
- (c) *Ω does not contain any affine plane.*

There is a similar result for Kobayashi hyperbolicity of convex domains $\Omega \subset \mathbb{C}^n$, $n \geq 2$, with (c) replaced by

- (c') Ω does not contain any complex line.

The proof uses the following lemma of independent interest.

Lemma

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ *be coordinates on* $\mathbb{R}^n$, $n \geq 3$, *and let*

$$\mathbb{H} = \left\{ \mathbf{x} = (x_1, \mathbf{x}') \in \mathbb{R}^n : x_1 > 0 \right\}.$$

If $\mathbf{x}(t) = (x_1(t), \mathbf{x}'(t)) \in \mathbb{H}$ $(t \in [0,1))$ *is a smooth path such that* $x_1(t)$ *clusters at* 0 *or* $+\infty$, *(i.e., there is a sequence* $t_j \in [0,1)$ *with* $\lim_{j\to\infty} t_j = 1$ *and* $\lim_{j\to\infty} x_1(t_j) \in \{0, +\infty\}$*), then*

$$\int_0^1 g_{\mathbb{H}}(\mathbf{x}(t), \dot{\mathbf{x}}(t))\, dt = +\infty.$$

Proof that the minimal distance to a hyperplane is infinite

Denote by $z = x + \mathrm{i}y$ the complex coordinate on $\mathbb{C}$. Fix a point $\mathbf{x} = (x_1, \mathbf{x}') \in \mathbb{H}$ and a vector $\mathbf{v} = (v_1, \mathbf{v}') \in \mathbb{R}^n$, and consider a harmonic map $f = (f_1, f_2, \ldots, f_n) : \mathbb{D} \to \mathbb{H}$ such that $f(0) = \mathbf{x}$ and $f_x(0) = r\mathbf{v}$ for some $r = r_f > 0$. Hence, $f_1 : \mathbb{D} \to (0, +\infty)$ is a positive harmonic function with $f_1(0) = x_1$ and $(f_1)_x(0) = rv_1$. It follows that

$$r|v_1| = |(f_1)_x(0)| \le |\nabla f_1(0)| \le 2|f_1(0)| = 2x_1,$$

where the second inequality holds by the Schwarz lemma for positive harmonic functions. Therefore,

$$\frac{1}{r} \ge \frac{|v_1|}{2x_1} \implies g_{\mathbb{H}}(\mathbf{x}, \mathbf{v}) = \inf_f \frac{1}{r_f} \ge \frac{|v_1|}{2x_1}.$$

Given a smooth path $\mathbf{x}(t) = (x_1(t), \mathbf{x}'(t)) \in \mathbb{H}$ $(t \in [0, 1))$, the above shows

$$g_{\mathbb{H}}(\mathbf{x}(t), \dot{\mathbf{x}}(t)) \ge \frac{|\dot{x}_1(t)|}{2x_1(t)}, \qquad t \in [0, 1).$$

If $x_1(t) \in (0, \infty)$ clusters at 0 or $+\infty$ as $t \to 1$, then

$$\int_0^1 g_{\mathbb{H}}(\mathbf{x}(t), \dot{\mathbf{x}}(t))dt \ge \frac{1}{2}\int_0^1 \frac{|\dot{x}_1(t)|}{x_1(t)}dt = \frac{1}{2}\int_0^1 |d\log x_1(t)| = +\infty.$$

This proves the lemma.

Strongly minimally convex domains are complete hyperbolic

A domain $\Omega \subset \mathbb{R}^n,\ n \geq 3$, with $\mathscr{C}^2$ boundary is said to be **strongly minimally convex** if at every point $\mathbf{p} \in b\Omega$ the interior principal curvatures $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_{n-1}$ of $b\Omega$ satisfy

$$\nu_1 + \nu_2 > 0.$$

When $n = 3$, such a domain is called **strongly mean convex**. This class of domains plays the analogous role in the theory of minimal surfaces as strongly pseudoconvex domains play for holomorphic curves.

Theorem (Drinovec Drnovšek & Forstnerič, 2023)

Let Ω be a (not necessarily bounded) domain in $\mathbb{R}^n,\ n \geq 3$. If the boundary $b\Omega$ is strongly minimally convex at a point $\mathbf{p} \in b\Omega$, then $\mathbf{p}$ is at infinite minimal distance from Ω. In particular, every bounded strongly minimally convex domain is complete hyperbolic.

If at some point $\mathbf{p} \in b\Omega$ we have $\nu_1 + \nu_2 < 0$, there is an embedded conformal minimal disc in $\Omega \cup \{\mathbf{p}\}$ centred at $\mathbf{p}$, so the result fails.

Problem

What can be said about the completeness of the minimal metric on bounded weakly minimally convex domains in $\mathbb{R}^n$?

In particular, assume that $\Sigma \subset \mathbb{R}^3$ is an embedded minimal surface. Is the minimal distance from $p \in \mathbb{R}^3 \setminus \Sigma$ to $q \in \Sigma$ infinite?

We have seen above that the answer is affirmative if Σ is a plane. This seems to be the only case when the answer is known.

The Levi flat hypersurfaces in $\mathbb{C}^n$ may be considered as complex-analytic analogues of minimal hypersurfaces. Such a hypersurface is foliated by complex hypersurfaces. The Kobayashi distance to a complex hypersurface, and hence to a Levi flat hypersurface, is infinite.

Gromov hyperbolicity of the minimal metric

A metric space is said to be **Gromov hyperbolic** if there exists a constant $\delta > 0$ such that for any geodesic triangle, every point on any of its sides is within a distance δ from at least one of the other two sides. This is a much studied topic in hyperbolic geometry.

A domain $\Omega \subset \mathbb{C}^n$ with $\mathscr{C}^2$ boundary is said to be **strongly pseudoconvex** if at every point $p \in b\Omega$ there is a local holomorphic coordinate system (U, ϕ), with $p \in U$ and $\phi : U \to \mathbb{C}^n$ injective holomorphic, such that $\phi(U \cap \Omega)$ is strongly convex along $\phi(U \cap b\Omega)$. An intrinsic differential-theoretic characterisation is that the Levi form $dd^c\rho$ of a $\mathscr{C}^2$ defining function for Ω is strongly positive at every point $p \in b\Omega$ and in every tangent direction.

By **Graham (1975)**, every bounded strongly pseudoconvex domain in $\mathbb{C}^n$ is complete Kobayashi hyperbolic. In 2000, **Balogh and Bonk** proved that the Kobayashi metric on any such domain is Gromov hyperbolic. By **Bertrand and Gaussier (2015)**, the same holds in almost complex manifolds.

Theorem (Matteo Fiacchi 2025)

A bounded strongly minimally convex domain is complete Gromov hyperbolic.