Minimal surfaces from a complex analytic viewpoint
Lecture 4: Complex analytic tools

Franc Forstnerič

Univerza v Ljubljani

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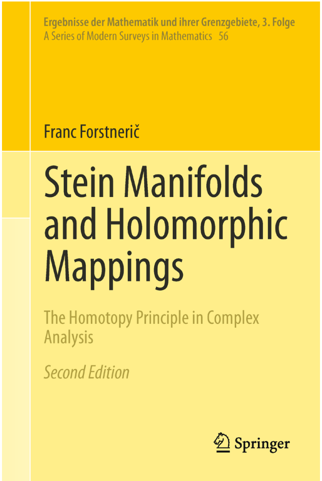
Reference book



In this lecture, I will describe some complex analytic tools used in the theory of minimal surfaces in Euclidean space. Most of them belong to **Oka theory** and are presented in my Ergebnisse monograph from 2011 & 2017.

Recent developments are summarised in:

Recent developments on Oka manifolds, Indag. Math., 34(2) (2023) 367–417 `https://doi.org/10.1016/j.indag.2023.01.005`

From Stein manifolds to Oka manifolds: the h-principle in complex analysis. To appear in the Proceedings ICM 2026. `https://arxiv.org/abs/2509.21197`.

The main tools from complex analysis

The main tools used in the theory of minimal surfaces are the following:

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2. Exposing boundary points of bordered Riemann surfaces
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In forthcoming lectures, **Antonio Alarcón** will describe applications of these results in the **Calabi–Yau problem for minimal surfaces** and in other advanced results on the subject.

Cartan pairs

A pair (A, B) of compact subsets in a complex manifold X is a **Cartan pair** if it satisfies the following two conditions:

1. The sets $C = A \cap B$ and $D = A \cup B$ are Stein compacts (i.e., they have bases of open Stein neighbourhoods), and
2. A and B are *separated* in the sense that $\overline{A \setminus B} \cap \overline{B \setminus A} = \varnothing$.

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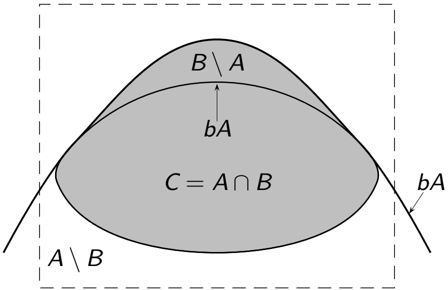
A Cartan pair (A, B) is **special** if the sets $C \subset B$ are convex in some holomorphic coordinates on a neighbourhood of B. Such a set B is called a **convex bump on** A.



Solution of the Cousin-I problem on Cartan pairs

Given a compact set K in a complex manifold X, set

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Lemma

Let $(A, B, C = A \cap B)$ be a Cartan pair in X such that $D = A \cup B$ is strongly pseudoconvex (that is, strongly convex near each point $p \in bD$ in suitable local holomorphic coordinates). There is a const > 0 such that any $c \in \mathscr{A}(C)$ can be written in the form

$$c = b - a \;\; \text{on } C, \quad a \in \mathscr{A}(A), \;\; b \in \mathscr{A}(B),$$
$$\|a\|_A \le const\,\|c\|_C, \quad \|b\|_B \le const\,\|c\|_C.$$

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$$\|a\|_A \le const\,\|c\|_C, \quad \|b\|_B \le const\,\|c\|_C.$$

Proof.

Let $\chi : X \to [0,1]$ be a smooth function such that $\chi = 0$ on $A \setminus B$ and $\chi = 1$ on $B \setminus A$. Then, $\omega = \bar\partial(c\chi) = c\bar\partial\chi$ is a continuous $(0,1)$-form on D, smooth in the interior, with $\bar\partial\omega = 0$. We can solve the $\bar\partial$-equation

$$\bar\partial h = c\bar\partial\chi \ \text{ on } D, \qquad \|h\|_D \le const\|c\bar\partial\chi\|_D \le const\,\|c\|_C.$$

The functions $a = -\chi c + h, \quad b = (1-\chi)c + h$ then satisfy the lemma. □

Corollary

Let (A, B) *be as above and* $D = A \cup B$. *Given functions* $f \in \mathscr{A}(A)$ *and* $g \in \mathscr{A}(B)$, *there is a function* $F \in \mathscr{A}(D)$ *such that*

$$\|F - f\|_A \le const\|f - g\|_C, \qquad \|F - g\|_B \le const\|f - g\|_C,$$

where $const > 0$ *only depends on* (A, B).

Corollary

Let (A, B) be as above and $D = A \cup B$. Given functions $f \in \mathscr{A}(A)$ and $g \in \mathscr{A}(B)$, there is a function $F \in \mathscr{A}(D)$ such that

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where const > 0 only depends on (A, B).

Proof.

Let $c = (f - g)|_C$. By the lemma, $c = b - a$ where $a \in \mathscr{A}(A)$, $b \in \mathscr{A}(B)$, and

$$\|a\|_A \leq const\,\|c\|_C, \quad \|b\|_B \leq const\,\|c\|_C.$$

The function $F : D \to \mathbb{C}$ given by

$$F = \begin{cases} f + a & \text{on the set } A, \\ g + b & \text{on the set } B, \end{cases}$$

then solves the problem. $\square$

A nonlinear splitting lemma (generalised Cartan's lemma)

Lemma (Proposition 5.8.1 in my Ergebnisse book)

Let $(A, B, C = A \cap B)$ *be a Cartan pair in* X *with* $D = A \cup B$ *strongly pseudoconvex, and let* $0 \in V \Subset U \Subset \mathbb{C}^N$. *Given a map*

$$\gamma : C \times U \to C \times \mathbb{C}^N, \quad \gamma(x, z) = (x, c(x, z)),$$

of class $\mathscr{A}(C \times U)$ *and uniformly close to the identity map, there are maps* $\alpha \in \mathscr{A}(A \times V)$, $\beta \in \mathscr{A}(B \times V)$ *of the form*

$$\begin{aligned} \alpha(x, z) &= (x, a(x, z)), \quad x \in A,\ z \in V, \\ \beta(x, z) &= (x, b(x, z)), \quad x \in B,\ z \in V, \end{aligned}$$

close to the identity on their respective domains, such that

$$\gamma \circ \alpha = \beta \quad \textit{holds on } C \times V.$$

The maps α *and* β *may be chosen to depend smoothly on* γ.

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The maps α and β may be chosen to depend smoothly on γ.

This is a nonlinear Cousin-I problem with the **compositional splitting** of γ. It is proved using the splitting lemma for functions and the implicit function theorem in Banach spaces.

Extending a holomorphic map across a convex bump

Definition (F. 2005) A complex manifold Y enjoys the **convex approximation property (CAP)** if every holomorphic map $K \to Y$ from a neighbourhood of a compact convex set $K \subset \mathbb{C}^n$ is a uniform limit of holomorphic maps $\mathbb{C}^n \to Y$.

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Lemma

Assume that the complex manifold Y enjoys CAP. Let (A, B) be a special Cartan pair in X with A, B and $D = A \cup B$ strongly pseudoconvex. Then, every holomorphic map $f_0 : A \to Y$ can be approximated by holomorphic maps $F_0 : A \cup B \to Y$.

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Here are the main steps in the proof.

- The graph of f_0 over A in $X \times Y = Z$ has a Stein neighbourhood. This allows us to construct a holomorphic map $f\colon A \times U \to Y$, where $0 \in U \subset \mathbb{C}^N$ is a ball, such that $f(\cdot\,, 0) = f_0$ and

$$\frac{\partial}{\partial z}\Big|_{z=0} f(x, z) : \mathbb{C}^N \to T_{f_0(x)} Y \quad \text{is surjective for every } x \in A.$$

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- Let $C = A \cap B$. Since $C \subset B$ are convex sets in some local holomorphic coordinates on X and $C \times U$ is also convex, we can apply CAP to approximate f by a holomorphic map $g : B \times U \to Y$.

Conclusion of the proof

- If the approximation is close enough on $C \times U$, there are a smaller ball $0 \in U' \Subset U$ and a **holomorphic transition map**

$$\gamma : C \times U' \to C \times U, \quad \gamma(x,z) = (x, c(x,z)),$$

close to the identity map, such that

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- By the splitting lemma, given a smaller ball $0 \in V \Subset U'$ we have

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- The map $F_0 = F(\cdot\,, 0) : D = A \cup B \to Y$ then approximates f_0 on A.

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Theorem (F. 2005–2010, 2017)

- *A complex manifold satisfying CAP is an* **Oka manifold**, *that is, holomorphic maps* $X \to Y$ *from any Stein manifold* X *satisfy all forms of the h-principle (also called Oka principle). The converse is a tautology.*

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MSC 2020: New subfield **32Q56 Oka principle and Oka manifolds**.

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A related splitting lemma was used in the construction of noncritical holomorphic functions on Stein manifolds, for exposing boundary points, etc.

Lemma (F., Acta Math 2003; Theorem 9.7.1 in the Ergebnisse book)

Let $(A, B, C = A \cap B)$ be a Cartan pair in X. Given a neighbourhood $U \subset X$ of C and a holomorphic map $\gamma : U \to X$ sufficiently close to the identity map, there are holomorphic maps $\alpha : A' \to X$ and $\beta : B' \to X$ on neighbourhoods of A and B, respectively, each close to the identity map, such that

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This gives the existence of holomorphic functions $f : X \to \mathbb{C}$ without critical points on any Stein manifold, and the h-principle for holomorphic submersions $X \to \mathbb{C}^q$ when $1 \leq q < \dim X$.

Theorem (Theorem 6.7.1 in the book on minimal surfaces)

Assume that M is bordered Riemann surfaces, i.e. a compact, connected, smoothly bounded domain in a Riemann surface R.

Choose finitely many pairwise distinct points $a_1, \ldots, a_k \in M \setminus bM$ and $p_1, \ldots, p_m \in bM$, and let $E_1, \ldots, E_m$ be smooth embedded pairwise disjoint arcs in R such that $E_i \cap M = \{p_i\}$ for $i = 1, \ldots, m$. Let $q_i \in E_i$ be the other endpoint of E_i, and let b_i be a point in the relative interior of E_i.

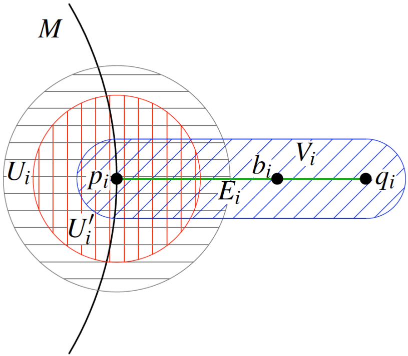
Suppose that for every $i = 1, \ldots, m$ we are given neighbourhoods $U_i' \Subset U_i$ of the point p_i and a neighbourhood $V_i \subset R$ of the arc E_i.

Given $s \in \mathbb{N}$ there exists a smooth conformal diffeomorphism $\phi : M \to \phi(M) \subset R$ satisfying the following conditions.

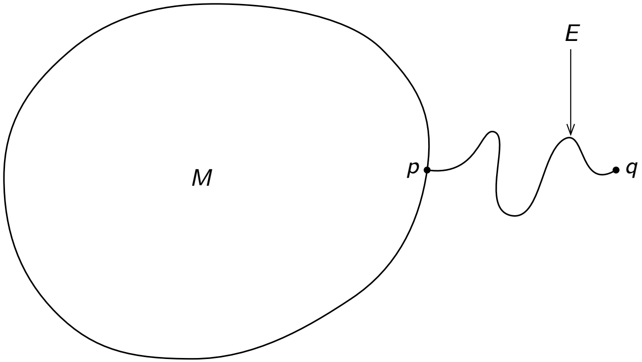
- a. *ϕ is close to the identity in the smooth topology on $M \setminus \bigcup_{i=1}^m U_i'$.*
- b. *ϕ is tangent to the identity map to order s at each point $a_1, \ldots, a_k$.*
- c. *For every $i = 1, \ldots, m$ we have that*

$$\phi(M \cap U_i') \subset U_i \cup V_i, \quad \phi(p_i) = q_i, \quad b_i \in \phi(\mathring{M}).$$

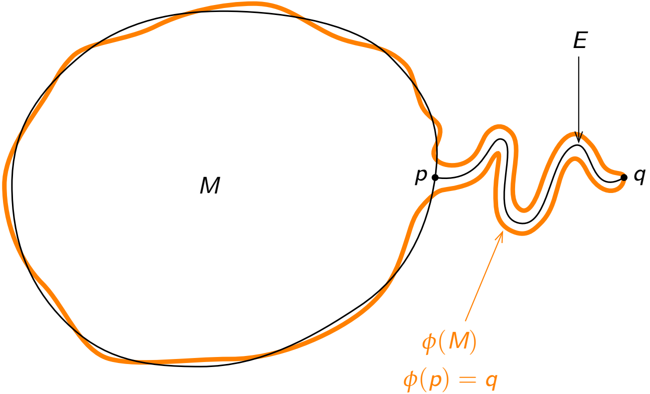
Exposing a boundary point



An application to minimal surfaces



Assume that M is a compact bordered domain in a Riemann surface R and $X: M \to \mathbb{R}^n$ is a conformal minimal immersion. We attach to M a smooth arc E in the complement of $\overline{M}$ with one endpoint $p \in bM$. We extend $X: M \to \mathbb{R}^n$ smoothly to E such that the derivative of $X|_E$ lies in the null quadric, and the arc $X(E) \subset \mathbb{R}^n$ is long but stays close to $X(M)$. By Mergelyan approximation, X can be made a conformal minimal immersion in a neighbourhood of $M \cup E$.



Pick a conformal diffeomorphism $\phi : \overline{M} \to \phi(\overline{M}) \subset R$ which maps p to the other endpoint q of the arc E, it adds a thin tube around E, and it is close to the identity on $\overline{M}$ outside a small neighbourhood of p.
The conformal minimal immersion $X \circ \phi : \overline{M} \to \mathbb{R}^n$ then has a much larger intrinsic distance to $p \in bM$.

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The Riemann–Hilbert method

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Let $\mathbb{T} = b\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$.

Lemma

Let $f \in \mathscr{A}(\mathbb{D}, \mathbb{C}^n)$, and let $g : \mathbb{T} \times \overline{\mathbb{D}} \to \mathbb{C}^n$ be a continuous map such that for each $\zeta \in \mathbb{T}$ we have $g(\zeta, \cdot\,) \in \mathscr{A}(\mathbb{D}, \mathbb{C}^n)$ and $g(\zeta, 0) = f(\zeta)$.

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Given numbers $\epsilon > 0$, $0 < r < 1$ and a closed arc $I \subset \mathbb{T}$, there are a number $r_1 \in [r, 1)$ and a holomorphic disc $h \in \mathscr{A}(\mathbb{D}, \mathbb{C}^n)$ with $h(0) = f(0)$ satisfying

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The Riemann–Hilbert method

The Riemann-Hilbert problem refers to a collection of boundary value problems for holomorphic functions. Here is a basic case.

Let $\mathbb{T} = b\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$.

Lemma

Let $f \in \mathscr{A}(\mathbb{D}, \mathbb{C}^n)$, and let $g : \mathbb{T} \times \overline{\mathbb{D}} \to \mathbb{C}^n$ be a continuous map such that for each $\zeta \in \mathbb{T}$ we have $g(\zeta, \cdot\,) \in \mathscr{A}(\mathbb{D}, \mathbb{C}^n)$ and $g(\zeta, 0) = f(\zeta)$.

Given numbers $\epsilon > 0$, $0 < r < 1$ and a closed arc $I \subset \mathbb{T}$, there are a number $r_1 \in [r, 1)$ and a holomorphic disc $h \in \mathscr{A}(\mathbb{D}, \mathbb{C}^n)$ with $h(0) = f(0)$ satisfying

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4. *if $J \subset \mathbb{T}$ is an open arc containing I and $g(\zeta, \cdot\,) = f(\zeta)$ is the constant disc for all $\zeta \in \mathbb{T} \setminus J$, then we can choose h such that $|h - f| < \epsilon$ on $\mathbb{T} \setminus J$.*

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The model case is the disc $\overline{\mathbb{D}} \ni z \to (z, 0) \in \mathbb{C}^2$ deformed to the discs $z \mapsto (z, c\,z^k)$ for $|c| = 1$ and large $k \in \mathbb{N}$.

Proof

Write

$$g(\zeta, z) = f(\zeta) + \lambda(\zeta, z), \qquad \zeta \in \mathbb{T},\ z \in \overline{\mathbb{D}},$$

where $\lambda(\zeta, z)$ is continuous on $(\zeta, z) \in \mathbb{T} \times \overline{\mathbb{D}}$, and for every fixed $\zeta \in \mathbb{T}$ the function $\overline{\mathbb{D}} \ni z \mapsto \lambda(\zeta, z)$ is holomorphic on $\mathbb{D}$ and satisfies $\lambda(\zeta, 0) = 0$.

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We approximate λ uniformly on $\mathbb{T}\times\overline{\mathbb{D}}$ by Laurent polynomials

$$\widetilde{\lambda}(\zeta,z)=\frac{1}{\zeta^m}\sum_{j=1}^{N}A_j(\zeta)z^j=\frac{z}{\zeta^m}\sum_{j=1}^{N}A_j(\zeta)z^{j-1}$$

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with polynomial coefficients $A_j(\zeta)$.

Hence, we can choose a map $\widetilde{\lambda}$ as above and a number $r_1 \in [r, 1)$ such that

$$\left|\widetilde{\lambda}(\rho\, e^{it}, z) - \lambda(e^{it}, z)\right| < \frac{\epsilon}{2}, \qquad t \in \mathbb{R},\ r_1 \le \rho \le 1,\ |z| \le 1 \tag{1}$$

and

$$\left|f(\rho e^{it}) - f(e^{it})\right| < \frac{\epsilon}{2}, \qquad t \in \mathbb{R},\ r_1 \le \rho \le 1. \tag{2}$$

Proof, 2

Choose an integer $k > m$ and set

$$h_k(\zeta) = f(\zeta) + \widetilde{\lambda}(\zeta, \zeta^k) = f(\zeta) + \zeta^{k-m} \sum_{j=1}^{N} A_j(\zeta) \left(\zeta^k\right)^{j-1}, \quad |\zeta| \leq 1.$$

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$$h_k\big(e^{\mathrm{i}t}\big) = f\big(e^{\mathrm{i}t}\big) + \widetilde{\lambda}\big(e^{\mathrm{i}t}, e^{k\mathrm{i}t}\big) \approx g\big(e^{\mathrm{i}t}, e^{\mathrm{i}kt}\big), \tag{3}$$

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By using approximation and gluing, this technique extends to an arbitrary bordered Riemann surface M in place of the disc.

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The Riemann–Hilbert method

The Riemann-Hilbert problem refers to a class of boundary value problems for holomorphic functions. We adapted it to null curves and minimal surfaces.

Let M be a bordered Riemann surface and $X\colon M \to \mathbb{R}^n$ be a conformal minimal immersion. Choose an arc $I \subset bM$. Let $G\colon I \times \overline{\mathbb{D}} \to \mathbb{R}^n$ be a smooth map such that for every $p \in I$,

$$G(p,\cdot\,) : \overline{\mathbb{D}} \to \mathbb{R}^n \text{ is a conformal minimal disc and } G(p,0) = X(p).$$

If $n > 3$, assume in addition that these discs lie in parallel planes.

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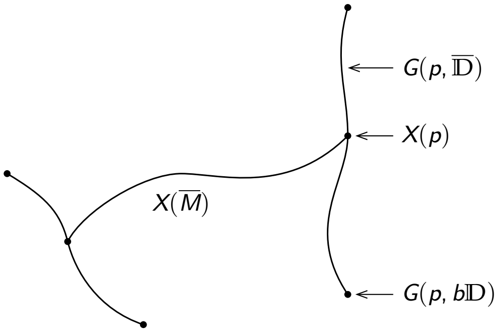
Then, there exists a conformal minimal immersion $Y\colon M \to \mathbb{R}^n$ such that:

- $Y(p)$ lies close to $G(p, b\mathbb{D})$ for $p \in I$,
- Y is uniformly close to X on $M \setminus U$, where $U \subset M$ is a given neighbourhood of the arc $I \subset bM$.
- $Y(U)$ lies close to $G(I \times \overline{\mathbb{D}})$.
- Y agrees with X in finitely many points of $M \setminus I$.

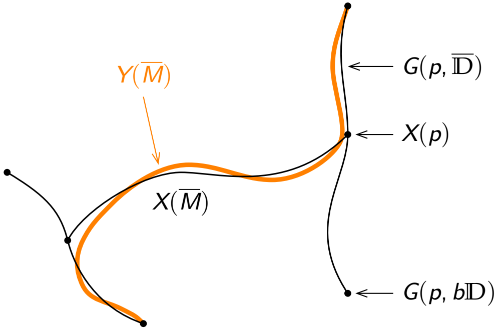
The Riemann–Hilbert deformation, I



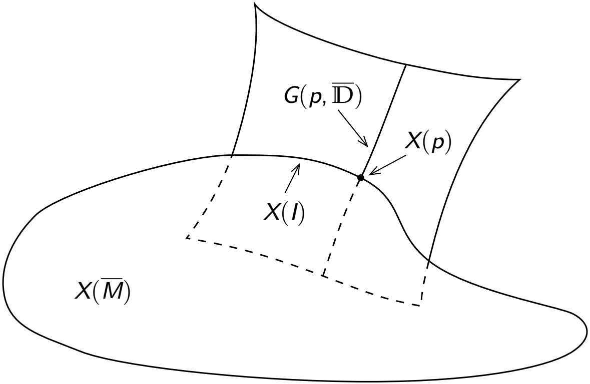
On each of the short arcs $I \subset bM$ between two consecutive exposed points, we push the image $X(\overline{M})$ in the direction roughly orthogonal to the position vector in order to enlarge the intrinsic boundary distance by a given amount.

This is done by attaching to $X(\overline{M})$ a family of conformal minimal discs $G(p, \cdot\,) : \overline{\mathbb{D}} \to \mathbb{R}^n$, $p \in I$, and taking an approximate solution of the Riemann–Hilbert problem as shown on the next slide.

The Riemann–Hilbert deformation, I

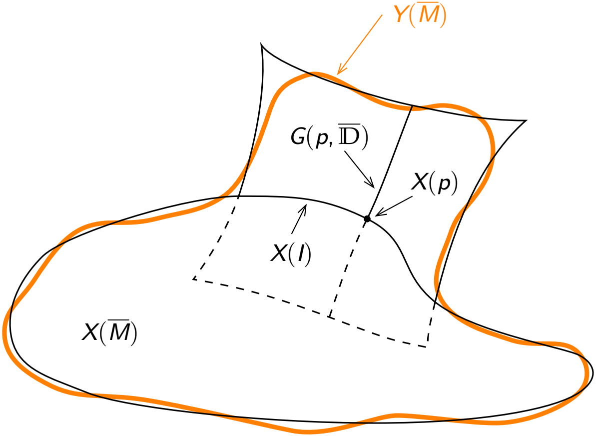


The new disc Y is close to X outside a small neighbourhood of the arc $I \subset bM$, and for $p \in I$ the point $Y(p)$ is close to the circle $G(p, b\mathbb{D})$. The modification is tempered at the endpoints of I to keep what was done in the first step.



On this 3-dimensional picture of the deformation, we see a family of conformal minimal discs $G(p, \cdot\,) : \overline{\mathbb{D}} \to \mathbb{R}^n$ centred at points $G(p, 0) = X(p)$, $p \in I$.

The Riemann–Hilbert deformation, II



The modification Y has now been made. Its image follows closely that of X until we come near the arc I, where it turns in the direction of the discs $G(p, \cdot\,)$ and maps the point $p \in I$ close to the circle $G(p, b\mathbb{D})$.

The effect of the Riemann–Hilbert deformation

The effect of this deformation is that for any path $\gamma : [0,1] \to M$ from $\gamma(0) = p_0 \in \mathring{M}$ to a point $\gamma(1) = p \in I$, we have

$$\mathrm{length}(Y \circ \gamma) \approx \mathrm{length}(X \circ \gamma) + \mathrm{diam}\, G(p, \overline{\mathbb{D}}).$$

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Repeating this deformation on other arcs for a sequence $\delta_j > 0$ with

$$\sum_j \delta_j = +\infty \quad \text{and} \quad \sum_j \delta_j^2 < \epsilon,$$

we find a sequence of conformal minimal immersions $X_j : M \to \mathbb{R}^n$ such that that the boundaries $X_j(bM)$ spiral around $\Gamma(bM)$ and the intrinsic diameter of $(M, X_j^* ds^2)$ grows to infinity, while $|X_j - X|$ remains uniformly small.

In a universe of minimal surfaces

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