Every projective Oka manifold is elliptic

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Flexibility versus rigidity in complex geometry

A central question of complex geometry is to understand the space $\mathcal{O}(X, Y)$ of holomorphic maps $X \to Y$ between a pair of complex manifolds. Are there many maps, or few maps? Which properties can they have?

There are many holomorphic maps $\mathbb{C} \to \mathbb{C}$ and $\mathbb{C} \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, but there are no nonconstant algebraic maps $\mathbb{C} \to \mathbb{C}^*$ or holomorphic maps $\mathbb{C} \to \mathbb{C} \setminus \{0, 1\}$. Manifolds with the latter property are called hyperbolic. They have been studied since 1960s when Kobayashi introduced his intrinsic pseudometric on complex manifolds. Hyperbolicity is a major obstruction to solving global complex analytic problems.

On the opposite side, Oka theory studies special complex manifolds, Oka manifolds, which admit many holomorphic maps from all Stein manifolds, i.e., closed complex submanifolds of affine spaces $\mathbb{C}^N$. Oka theory gives solutions to a variety of complex analytic problems in the absence of topological obstructions.

OKA THEORY $\cong$ h-PRINCIPLE IN COMPLEX GEOMETRY

First instances of the Oka principle

Oka 1939 For complex line bundles on domains of holomorphy, the holomorphic classification agrees with the topological classification.

Grauert 1958 The same holds for principal and their associated fibre bundles (e.g. for vector bundles) on Stein manifolds and Stein spaces.

Every vector bundle on a complex manifold X is the pullback of a universal bundle on a suitable Grassmann manifold Y by a map $X \to Y$. Holomorphic maps give rise to holomorphic bundles on X, and homotopies of maps induce isomorphic bundles. Hence, Grauert's theorems follow from his result that

A complex homogeneous manifold admits many holomorphic maps from any Stein space.

What is the right way to interpret the phrase *many maps*?

Oka manifolds

A complex manifold Y is called an Oka manifold if maps $X \to Y$ from any Stein space X satisfy all forms of the Oka principle:

(a) Every continuous map $f : X \to Y$ is homotopic to a holomorphic map.

(b) If $f : X \to Y$ is holomorphic on a compact $\mathcal{O}(X)$-convex subset $K \subset X$ and on a closed complex subvariety X' of X, there is a homotopy from f to a holomorphic map $F : X \to Y$ consisting of maps with the same properties which approximate f on K and agree with f on X'.

(c) A similar statement holds for families of maps depending continuously on a parameter in a compact Hausdorff space.

For $Y = \mathbb{C}$, these properties hold by the parametric Oka–Weil–Cartan theorem.

Convex approximation property

Theorem (F. 2005-9)

A complex manifold Y is an Oka manifold iff it satisfies the

Convex Approximation Property (CAP): Every holomorphic map $f : K \to Y$ from a compact convex set $K \subset \mathbb{C}^n$ is a uniform limit of entire maps $\mathbb{C}^n \to Y$.

Furthermore, all Oka-type conditions are pairwise equivalent.

Among compact complex manifolds, hyperbolicity is closely related to the existence of hermitian metrics with negative sectional curvature.

It is therefore natural to expect that the Oka property is related to metric positivity. Mok's solution to the generalised Frankel conjecture, together with the previous theorem, implies the following.

Theorem

Every compact Kähler manifold with nonnegative holomorphic bisectional curvature is an Oka manifold.

Elliptic manifolds

Gromov 1989 A complex manifold Y is said to be elliptic if it admits a dominating holomorphic spray, i.e., a holomorphic map $s : E \to Y$ from the total space of a holomorphic vector bundle $\pi : E \to Y$ such that for all $y \in Y$, we have

$$s(0_y) = y \text{ and } s : E_y = \pi^{-1}(y) \to Y \text{ is a submersion at } 0_y \in E_y.$$

Example

1. Let G be a complex Lie group acting holomorphically transitively on a complex manifold Y. If g is the Lie algebra of G then the map

$$s : Y \times g \to Y, \quad s(y, v) = e^v y$$

is a dominating spray on Y.

2. If V_i $(i = 1, \dots, k)$ are complete holomorphic vector fields on Y spanning TY at every point and ϕ^i_t $(t \in \mathbb{C})$ is the flow of V_i then the map

$$Y \times \mathbb{C}^k \to Y, \quad (y, t_1, \dots, t_k) \mapsto \phi^1_{t_1} \circ \cdots \circ \phi^k_{t_k}(y)$$

is a dominating holomorphic spray.

Gromov's theorem

Theorem (Gromov 1989)

Every elliptic complex manifold is an Oka manifold.

By Example 1, this generalises Grauert's Oka principle. A detailed proof was given by Jasna Prezelj and myself (2000 & 2002). I also proved that subellipticity (the existence of a finite dominating family of sprays) implies the Oka property.

The modern proof consists of two parts. The first part is the implication

$$\text{(sub)elliptic} \Longrightarrow \text{h-Runge approximation} \Longrightarrow \text{CAP}$$

This is fairly elementary by using dominating sprays and the Oka–Weil theorem.

The implication CAP $\Longrightarrow$ OKA is nontrivial, while the converse is a tautology.

Gromov 1989: Is every Oka manifold elliptic?

Andrist, Shcherbina, Wold, 2016 If $n \geq 3$ and $K \subset \mathbb{C}^n$ is a compact set with infinite limit set, then $\mathbb{C}^n \setminus K$ is not elliptic or subelliptic.

Kusakabe 2024 If $K \subset \mathbb{C}^n$, $n > 1$, is polynomially convex then $\mathbb{C}^n \setminus K$ is Oka.

These results give examples of noncompact Oka manifolds of any dimension ≥ 3 which fail to be elliptic. Kusakabe also proved that the Oka property is Zariski local, while no such result is known for ellipticity.

Theorem (Lárusson & F., 2025)

Every projective Oka manifold is elliptic.

Is there a compact nonprojective Oka manifold which fails to be elliptic?

Scheme of proof

Let $Y \subset \mathbb{CP}^n$ be a projective manifold and $\pi : E \to Y$ a holomorphic vector bundle. Along the zero section $E(0) \cong Y$ of E we have a natural splitting

$$TE|E(0) \cong E \oplus TY.$$

Let V be a holomorphic vector field on E that vanishes on $E(0)$. There is a neighbourhood $\Omega \subset E$ of $E(0)$ such that for any $e \in \Omega$, the flow $\phi_\tau(e)$ of V with $\phi_0(e) = e$ exists for all $0 \leq \tau \leq 1$. The holomorphic map

$$s = \pi \circ \phi_1 : \Omega \to Y$$

is then a local spray on Y.
Assuming that E is sufficiently negative, we shall construct s whose vertical derivative

$$Vds_y : T_{0_y} E_y \cong E_y \to T_y Y$$

is surjective for every $y \in Y$, that is, s is dominating.

Since E is a 1-convex manifold with the exceptional subvariety $E(0)$, the Oka principle (Prezelj 2010, 2016; Stopar 2013) gives a global holomorphic spray $\tilde{s} : E \to Y$ which agrees with s to the second order along $E(0)$. Hence, Y is elliptic.

Proof, 1

Let $Y \subset \mathbb{CP}^n$ be a smooth projective variety. Denote by $z = [z_0 : z_1 : \cdots : z_n]$ the homogeneous coordinates on $\mathbb{CP}^n$. Set $\Lambda_\alpha = \{z_\alpha = 0\}$ and $U_\alpha = \mathbb{CP}^n \setminus \Lambda_\alpha \cong \mathbb{C}^n$ with affine coordinates $(z_0/z_\alpha, \ldots, z_n/z_\alpha)$. Let $x = (x_1, \ldots, x_n)$ with $x_i = z_i/z_0$ be affine coordinates on U_0. There are finitely many polynomial vector fields

$$W_j(x) = \sum_{i=1}^{n} V_{i,j}(x)\partial_{x_i}, \quad j = 1, \ldots, m$$

on $U_0 \cong \mathbb{C}^n$, tangent to $Y_0 = Y \cap U_0$ and spanning $T_y Y$ at every point $y \in Y_0$.

To this collection we associate the polynomial vector field V on the total space $U_0 \times \mathbb{C}^m \cong \mathbb{C}^{n+m}$ of the trivial vector bundle $\pi : U_0 \times \mathbb{C}^m \to U_0$, defined by

$$V(x,t) = \sum_{j=1}^{m} t_j W_j(x) = \sum_{i=1}^{n} \sum_{j=1}^{m} t_j V_{i,j}(x)\partial_{x_i}$$

where $x \in U_0$ and $t = (t_1, \ldots, t_m) \in \mathbb{C}^m$. Note that V is horizontal, it vanishes on the zero section $U_0 \times \{0\}^m = \{t = 0\}$, and for every $(x,t) \in Y_0 \times \mathbb{C}^m$ we have

$$d\pi_{(x,t)} V(x,t) = \sum_{j=1}^{m} t_j W_j(x) \in T_x Y.$$

Proof, 2

Let $\mathbb{U} = \mathcal{O}_{\mathbb{CP}^n}(-1)$ denote the universal line bundle on $\mathbb{CP}^n$.

Lemma

For every $k \geq k_0 := \max_{i,j} \deg V_{i,j}$ *the vector field* V *extends to an algebraic vector field on the total space* E *of the vector bundle*

$$\pi : E = (\mathbb{CP}^n \times \mathbb{C}^m) \otimes \mathbb{U}^k = m\mathbb{U}^k \to \mathbb{CP}^n$$

which vanishes on $E(0) \cup E|_{\Lambda_0}$.

Proof. We have vector bundle trivialisations $\theta_\alpha : E|U_\alpha \overset{\cong}{\to} U_\alpha \times \mathbb{C}^m$ with transition maps $\theta_{\alpha,\beta} = \theta_\alpha \circ \theta_\beta^{-1}$ on $(U_\alpha \cap U_\beta) \times \mathbb{C}^m$ given by

$$\theta_{\alpha,\beta}([z], t) = \big([z], (z_\alpha / z_\beta)^k t\big), \quad t \in \mathbb{C}^m,\ 0 \leq \alpha, \beta \leq n.$$

In particular,

$$\theta_{\alpha,0}([z], t) = \big([z], (z_\alpha / z_0)^k t\big).$$

Proof, 3

We shall find the explicit expression for V on $E|U_\alpha$ for any $\alpha = 1, \ldots, n$. For simplicity, we make the calculation for $m = 1$, so $E = \mathbb{U}^k$ and $V = t\sum_{i=1}^n V_i(x)\partial_{x_i}$ on $E|U_0 \cong \mathbb{C}^n \times \mathbb{C}$. It suffices to consider the case $\alpha = 1$.

In the first step, we express V in the fibre coordinate t' on $E|U_1 \cong \mathbb{C}^n \times \mathbb{C}$ over $U_0 \cap U_1 = \{x = (x_1, \ldots, x_n) \in U_0 \cong \mathbb{C}^n : x_1 \neq 0\}$.

Recall that $\theta_{1,0}(x,t) = \left(x, x_1^k t\right)$, so $t' = x_1^k t$. We have

$$D\theta_{1,0}(x,t) = \begin{pmatrix} I_n & 0 \\ B & x_1^k \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix and $B = (kx_1^{k-1}t, 0, \ldots, 0)$.
It follows that the vector field $V' = D\theta_{1,0} \cdot V$ equals

$$V' = t\sum_{i=1}^n V_i(x)\partial_{x_i} + kt^2x_1^{k-1}V_1(x)\partial_{t'} = t'x_1^{-k}\sum_{i=1}^n V_i(x)\partial_{x_i} + (t')^2kx_1^{-k-1}V_1(x)\partial_{t'}$$

where we used that $t = x_1^{-k}t'$.

Proof, 4

We now express V' in the affine coordinates $x' = (x'_1, x'_2, \dots, x'_n)$ on U_1. Note that

$$\begin{aligned} x'_1 &= \frac{z_0}{z_1} = \frac{1}{x_1}, \\ x'_i &= \frac{z_i}{z_1} = \frac{x_i}{x_1}, \quad i = 2, \dots, n. \end{aligned}$$

Write $x' = \psi(x)$ and $(x', t') = \tilde{\psi}(x, t') = (\psi(x), t')$. We have

$$D\psi(x) = \begin{pmatrix} -\frac{1}{x_1^2} & 0 & 0 & \cdots & 0 \\ -\frac{x_2}{x_1^2} & \frac{1}{x_1} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\frac{x_n}{x_1^2} & 0 & 0 & \cdots & \frac{1}{x_1} \end{pmatrix}.$$

Recall that

$$V' = t' x_1^{-k} \sum_{i=1}^{n} V_i(x) \partial_{x_i} + (t')^2 k x_1^{-k-1} V_1(x) \partial_{t'}.$$

Proof, 5

Hence, the vector field $\widetilde{V} = D\tilde{\psi} \cdot V'$ equals

$$\begin{aligned}\widetilde{V}(x,t') &= -\frac{t'}{x_1^{k+2}}V_1(x)\partial_{x_1'} + \sum_{i=2}^{n}\left[-\frac{x_i}{x_1^{k+2}}V_1(x) + \frac{1}{x_1^{k+1}}V_i(x)\right]t'\partial_{x_i'} \\ &\quad + \frac{k(t')^2}{x_1^{k+1}}V_1(x)\partial_{t'}.\end{aligned}$$

Note that

$$x = \psi^{-1}(x') = (1/x_1', x_2'/x_1', \cdots, x_n'/x_1').$$

Inserting in the above expression gives

$$\begin{aligned}\widetilde{V}(x',t') &= -t'(x_1')^{k+2}V_1(\psi^{-1}(x'))\partial_{x_1'} \\ &\quad + \sum_{i=2}^{n} t'(x_1')^{k+1}\left[-x_i'V_1(\psi^{-1}(x')) + V_i(\psi^{-1}(x'))\right]\partial_{x_i'} \\ &\quad + k(t')^2(x_1')^{k+1}V_1(\psi^{-1}(x'))\partial_{t'}.\end{aligned}$$

Proof, 6

Note that the affine hyperplane $\{z_0 = 0,\ z_1 \neq 0\}$ corresponds to $\{x_1' = 0\}$.

Since ψ^{-1} is a fractional linear map with a simple pole along $x_1' = 0$, the functions $V_i(\psi^{-1}(x'))$ have a pole of degree at most k_0 along $x_1' = 0$ and no other singularities.

It follows that for $k \geq k_0$ the vector field $\widetilde{V}(x', t')$ is polynomial in (x', t') and it vanishes on $\{x_1' = 0\} \cup \{t' = 0\}$.

The calculation is similar for arbitrary $m \in \mathbb{N}$. This proves the lemma.

Proof, 7

Since the vector field V vanishes on the zero section $E(0)$ of E, there is a neighbourhood $\Omega \subset E$ of $E(0)$ such that the flow $\phi_\tau(e)$ of V, starting at time $\tau = 0$ in a point $e \in \Omega$, exists for all $\tau \in [0, 1]$. The map

$$s := \pi \circ \phi_1 : \Omega \to \mathbb{CP}^n$$

is then a local holomorphic spray on $\mathbb{CP}^n$. On $E(0) \cong \mathbb{CP}^n$ we have $TE|E(0) = E \oplus T\mathbb{CP}^n$. Identifying a vector $e \in E_x = \pi^{-1}(x)$ with $e \in T_{0_x} E_x$, we let

$$(Vds)_x(e) = (ds)_{0_x}(e) \in T_x\mathbb{CP}^n$$

denote the vertical derivative of s at 0_x applied to the vector e.
We claim that for every $e = (x, t) \in \Omega$, with $x \in U_0$, we have

$$(Vds)_x(t_1, \ldots, t_m) = d\pi_{(x,t)} V(x, t) = \sum_{j=1}^{m} t_j W_j(x). \qquad (1)$$

Proof, 8

To see this, note that in the vector bundle chart on $E|U_0$ the vector field V is horizontal and its coefficients are linear in the fibre variable t. Hence,

$$\pi \circ \phi_\tau(x, \delta t) = \pi \circ \phi_{\delta\tau}(x, t)$$

holds for every $(x, t) \in E|U_0 \cap \Omega$ and $0 \leq \delta, \tau \leq 1$. At $\tau = 1$ we obtain

$$s(x, \delta t) = \pi \circ \phi_1(x, \delta t) = \pi \circ \phi_\delta(x, t), \quad 0 \leq \delta \leq 1.$$

Differentiating with respect to δ at $\delta = 0$ and noting that $\frac{d}{d\delta}\Big|_{\delta=0} \phi_\delta(x, t) = V(x, t)$ and $d\pi_{(x,t)} V(x, t) = \sum_{j=1}^m t_j\, W_j(x)$ gives (1).

Set $E|Y = \pi^{-1}(Y)$. Since $d\pi_{(x,t)} V(x, t) \in T_x Y$ for $x \in Y$, the spray $s = \pi \circ \phi_1$ maps the domain $\Omega \cap E|Y$ to Y. Since the vector fields $W_1, \ldots, W_m$ generate the tangent space $T_x Y$ every point $x \in Y_0 = Y \cap U_0$, the restricted spray $s : \Omega \cap E|Y \to Y$ is dominating on Y_0. On the other hand, since V vanishes on $E|\Lambda_0$, ϕ_1 is the identity on this set and the spray $s = \pi$ is trivial over Λ_0.

Proof, 9

In order to find a local dominating spray on Y, we proceed as follows. For $\alpha \in \{0, 1, \ldots, n\}$ set $Y_\alpha = Y \cap U_\alpha$. Choose $m \in \mathbb{N}$ big enough that for every α the tangent bundle TY_α is generated by m polynomial vector fields

$$W_j^\alpha(x) = \sum_{i=1}^{n} V_{i,j}^\alpha(x)\partial_{x_i}$$

in the affine coordinates $x = (x_1, \ldots, x_n) = (z_0/z_\alpha, \ldots, z_n/z_\alpha)$ on U_α. Let

$$k_0 := \max_{\alpha,i,j} \deg V_{i,j}^\alpha.$$

For every $k \geq k_0$ the above argument gives an algebraic vector field V^α on the vector bundle $E^\alpha = m\mathbb{U}^k$ which is of the form

$$V^\alpha(x, t^\alpha) = \sum_{i=1}^{n} \sum_{j=1}^{m} t_j^\alpha V_{i,j}^\alpha(x)\partial_{x_i}$$

in the chart $E^\alpha|U_\alpha \cong U_\alpha \times \mathbb{C}^m$. In other charts $E^\alpha|U_\beta$ for $\beta \neq \alpha$, V^α is of the same form but also has a vertical component of size $|t|^2$.

Proof, 10

Set

$$\pi : E = E^0 \oplus E^1 \oplus \cdots \oplus E^n = (n+1)m\mathbb{U}^{k_0} \to \mathbb{CP}^n.$$

The algebraic vector field V^α on E^α extends to an algebraic vector field on E by first extending it trivially (horizontally) to each of the summands $E^\beta|U_\alpha$ of $E|U_\alpha$ for $\beta \neq \alpha$ (these are trivial bundles), and then observing that the resulting vector field on $E|U_\alpha$ extends to an algebraic vector field on E. With these extensions in place, we consider the vector field $V = \sum_{\alpha=0}^n V^\alpha$ on E. The construction implies that

$$d\pi_e V(e) \in T_y Y \quad \text{for every } y \in Y \text{ and } e \in E_y = \pi^{-1}(y).$$

Since each V^α vanishes on the zero section of E_0 of E, so does V. Hence, there is a neighbourhood $\Omega \subset E$ of E_0 such that the flow $\phi_\tau(e)$ of V exists for any initial point $e \in \Omega$ and every $\tau \in [0,1]$. Consider the holomorphic spray

$$s = \pi \circ \phi_1 : \Omega \to \mathbb{CP}^n.$$

We claim that the restricted spray $s : \Omega \cap \pi^{-1}(Y) \to Y$ is dominating. To see this, consider V on a chart $E|U_\alpha$. Let $\alpha = 0$ for simplicity of notation.

Proof, 11

In the affine coordinates $x = (z_1/z_0, \dots, z_n/z_0)$ on U_0 and fibre coordinates $t = (t^0, t^1, \dots, t^n)$ on $E|U_0$, where $t^\alpha = (t_1^\alpha, \dots, t_m^\alpha)$ are fibre coordinates on the direct summand $E^\alpha|U_0$ of $E|U_0$, we have

$$V(x,t) = \sum_{\alpha=0}^{n} \sum_{i=1}^{n} \sum_{j=1}^{m} t_j^\alpha \, V_{i,j}^\alpha(x) \partial_{x_i} + \widetilde{V}(x,t) = \Theta(x,t) + \widetilde{V}(x,t)$$

where $\widetilde{V}$ is vertical and $|\widetilde{V}(x,t)| = O(|t|^2)$.
Since $\Theta(x,t)$ is linear in t, its flow ψ_τ satisfies

$$\pi \circ \psi_1(x, \delta t) = \pi \circ \psi_\delta(x,t),$$

and hence the vertical derivative of the spray $\tilde{s} = \pi \circ \psi_1 : \Omega \to \mathbb{CP}^n$ equals

$$Vd\tilde{s}_x(t) = \sum_{\alpha=0}^{n} \sum_{j=1}^{m} t_j^\alpha \, W_j^\alpha(x).$$

Since the vectors $W_j^0(x)$ for $j = 1, \dots, m$ span $T_x Y$ for every $x \in Y_0$, $\tilde{s}$ is dominating over Y_0. Since $|\widetilde{V}(x,t)| = O(|t|^2)$, the flow ϕ_τ of V satisfies

$$\phi_\tau(x,t) = \psi_\tau(x,t) + O(|t|^2) \;\text{ as } |t| \to 0 \text{ and } \tau \in [0,1].$$

Proof, 12

Hence, the spray $s = \pi \circ \phi_1 : \Omega \to \mathbb{CP}^n$ satisfies

$$Vds_x(t) = Vd\tilde{s}_x(t) \quad \text{for } x \in U_0,$$

so it is dominating on $Y_0 = Y \cap U_0$. The same argument holds on every chart $E|U_\alpha$, which proves that the spray $s : \Omega \cap \pi^{-1}(Y) \to Y$ is dominating.

Set $X = E|Y = \pi^{-1}(Y) \to Y$ and replace s by its restriction $s : X \cap \Omega \to Y$. Since $X \to Y$ is a Griffiths negative vector bundle, X is a 1-convex manifold with the exceptional subset $X(0) \cong Y$.

Assuming that Y is an Oka manifold, the Oka principle of Prezelj 2010, 2016 and Stopar 2013 gives a holomorphic map $\tilde{s} : X \to Y$ which agrees with the spray s to the second order along the zero section $X(0) \cong Y$. Hence, $\tilde{s}$ is dominating, so Y is elliptic. This completes the proof.

Remark. Our proof gives holomorphic (nonalgebraic) sprays. There exist projective Oka manifolds which are not algebraically elliptic; for example, abelian varieties.

Local dominating sprays on positive vector bundles

Proposition

Assume that Y is a compact complex manifold and the vector bundle $\pi : E \to Y$ is generated by global holomorphic sections (this holds if E is sufficiently positive). If there is a local dominating holomorphic spray $s : U \to Y$ defined on a neighbourhood $U \subset E$ of the zero section $E(0)$, then Y is a complex homogeneous manifold.

Proof. The vertical derivative $Vds|E(0) : E \to TY$ is a vector bundle epimorphism. Given a holomorphic section $\xi : Y \to E$, the map

$$Y \ni y \mapsto V_\xi(y) := Vds_y(\xi(y)) \in T_y Y$$

is a holomorphic vector field on Y.

Applying this argument to sections $\xi_1, \ldots, \xi_m : Y \to E$ generating E gives holomorphic vector fields $V_1, \ldots, V_m$ on Y spanning the tangent bundle TY.

Since Y is compact, their flows are complex 1-parameter subgroups of the holomorphic automorphism group $\mathrm{Aut}(Y)$, a finite-dimensional complex Lie group. The spanning property implies that $\mathrm{Aut}(Y)$ acts transitively on Y.

THANK YOU FOR YOUR ATTENTION