

Oka manifolds and their role in complex geometry

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Flexibility versus rigidity in complex geometry

A central question of complex analysis and geometry is to understand the space $\mathcal{O}(X, Y)$ of holomorphic maps $X \rightarrow Y$ between a pair of complex manifolds. Are there many maps, or few maps? Which properties can they have?

There are many holomorphic maps $\mathbb{C} \rightarrow \mathbb{C}$ and $\mathbb{C} \rightarrow \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, but there are no nonconstant algebraic maps $\mathbb{C} \rightarrow \mathbb{C}^*$ or holomorphic maps $\mathbb{C} \rightarrow \mathbb{C} \setminus \{0, 1\}$. Manifolds with the latter property are called **hyperbolic**. Hyperbolicity is an obstruction to solving global complex analytic problems.

On the opposite side, **Oka theory** studies complex manifolds Y which admit many holomorphic maps $X \rightarrow Y$ from all **Stein manifolds**, i.e., closed complex submanifolds of euclidean spaces $\mathbb{C}^N$. It provides solutions to many complex analytic problems in the absence of topological obstructions, so it is also called the **h-principle in complex geometry**. It has numerous applications:

- classification of principal and associated fibre bundles on Stein manifolds
- the h-principles for holomorphic immersions and submersions
- the optimal embedding theorem for Stein manifolds in Euclidean spaces
- factorisation of holomorphic maps from Stein manifolds to matrix groups
- the theory of minimal surfaces in Euclidean spaces, ...

The first main protagonist: Stein manifolds



Karl Stein, 1951

A complex manifold X is holomorphically complete (Stein) if

- ① holomorphic functions on X separate points, and
- ② for every discrete sequence $p_j \in X$ there is a holomorphic function $f \in \mathcal{O}(X)$ with $\lim_{j \rightarrow \infty} |f(p_j)| = +\infty$.

Equivalently, for every compact subset $K \subset X$, its holomorphic hull $\hat{K}_{\mathcal{O}(X)}$ is also compact.

Examples and characterizations of Stein manifolds

- Open Riemann surfaces are Stein (Behnke and Stein, 1949).
- Closed complex submanifolds of Euclidean spaces $\mathbb{C}^N$ are Stein. Conversely, every Stein n -manifold is a closed complex submanifold of $\mathbb{C}^{2n+1}$ (Remmert, Bishop, Narasimhan, 1956–61).
- The category of Stein manifolds is closed under taking holomorphic covering, Cartesian products, and holomorphic vector bundles.
- A complex manifold X is Stein iff it admits a strongly plurisubharmonic exhaustion function $\rho : X \rightarrow \mathbb{R}$, $dd^c\rho = i\partial\bar{\partial}\rho > 0$ (Grauert, 1958).
- Every Stein subvariety of a complex space admits an open Stein neighbourhood (Siu, 1976).

A **Stein space** is a complex space with singularities having similar function theoretic properties as Stein manifolds.

The Oka–Grauert principle

Kiyoshi Oka 1939, Hans Grauert 1958

For principal and their associate fibre bundles (e.g. vector bundles) on Stein spaces, the holomorphic classification agrees with the topological classification.



The gist of the proof is to show that there are many holomorphic maps $X \rightarrow Y$ from any Stein manifold X to every complex homogeneous manifold Y , and many holomorphic sections of any holomorphic fibre bundle $Z \rightarrow X$ with a homogeneous fibre over a Stein base X .

The second main protagonist: Oka manifolds

A complex manifold Y is called an **Oka manifold** (F. 2009) if maps $X \rightarrow Y$ from any Stein space X satisfy all forms of the **Oka principle**:

- Ⓐ Every continuous map $f: X \rightarrow Y$ is homotopic to a holomorphic map.
- Ⓑ If $f: X \rightarrow Y$ is holomorphic on (a neighbourhood of) a compact holomorphically convex subset $K = \hat{K} \subset X$ and on a closed complex subvariety X_0 of X , then there is a homotopy from f to a holomorphic map $F: X \rightarrow Y$ consisting of maps with the same properties which approximate f on K and agree with f on X_0 .
- Ⓒ A similar statement holds for maps $f: X \times P \rightarrow Y$ depending continuously on a parameter in a compact Hausdorff space P .

Oka–Weil–Cartan 1936–53: $\mathbb{C}$ is an Oka manifold.

Oka 1939: $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is an Oka manifold.

Grauert 1958: Every complex homogeneous manifold is an Oka manifold.

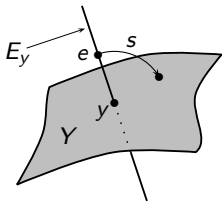
Kobayashi and Ochiai 1977: A compact complex manifold of general type is not dominable by the affine space, so it is not an Oka manifold.

Oka manifolds are rare in the universe of complex manifolds

Elliptic manifolds

Gromov 1989 A complex manifold Y is called **elliptic** if it admits a **dominating holomorphic spray** (E, π, s) : a holomorphic vector bundle $\pi : E \rightarrow Y$ and a holomorphic map $s : E \rightarrow Y$ such that

$$s(0_y) = y \text{ and } ds_{0_y}(E_y) = T_y Y \text{ for all } y \in Y.$$



An ostensibly weaker condition, **subellipticity**, asks for a finite family of sprays $s_j : E_j \rightarrow Y$ with

$$\sum_j (ds_j)_{0_y}(E_{j,y}) = T_y Y \text{ for all } y \in Y.$$

On algebraic manifolds, one defines **algebraic (sub-) ellipticity**.

Ellipticity of Y implies the relative Oka principle

Lemma

Given a holomorphic map $f_0 : X \rightarrow Y$ from a Stein manifold X , a compact $\mathcal{O}(X)$ -convex subset $K \subset X$, and a homotopy of holomorphic maps $f_t : K \rightarrow Y$, $t \in [0, 1]$, we can approximate f_1 uniformly on K by holomorphic maps $X \rightarrow Y$. The analogous result holds for continuous families of maps $X \rightarrow Y$.

$$\begin{array}{ccc} f_0^* E & \xhookrightarrow{\iota} & E \\ \downarrow & & \downarrow \pi \\ X & \xrightarrow{f_0} & Y \end{array} \quad \begin{array}{c} \searrow s \\ \nearrow \end{array}$$

The holomorphic map

$$F = s \circ \iota : f_0^* E \rightarrow Y$$

agrees with f_0 on the zero section X of $f_0^* E$,

and

$$dF_x : f_0^* E_x \rightarrow T_{f_0(x)} Y$$

is surjective for every $x \in X$.

By the implicit function theorem, there is $t_0 \in [0, 1]$ and a homotopy of holomorphic sections $g_t : K \rightarrow f_0^* E$, $t \in [0, t_0]$, such that

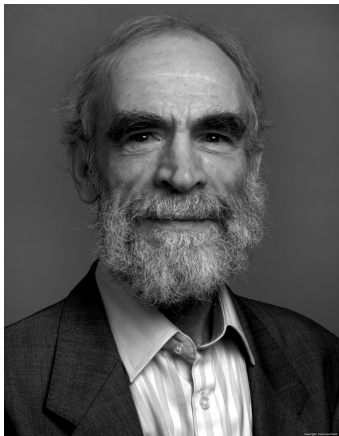
g_0 is the zero section, and $F \circ g_t = f_t$ for all $t \in [0, t_0]$.

Approximating g_t by global holomorphic sections $\tilde{g}_t : X \rightarrow f_0^* E$ gives a family

$$\tilde{f}_t := F \circ \tilde{g}_t : X \rightarrow Y, \quad t \in [0, t_0],$$

satisfying the lemma. The proof is completed in finitely many such steps.

Gromov's Oka principle



Gromov 1989: Every elliptic manifold is an Oka manifold. The Oka principle also holds for sections of elliptic submersions $Z \rightarrow X$ over a Stein base.

Prezelj & F., 2000–2: detailed proof.

F. 2003: The same holds for subelliptic manifolds and submersions.

Problems: (1) Can Oka manifolds be characterised by the Runge approximation property from simple domains in Euclidean spaces?

(2) Is every Oka manifold elliptic?

Examples of elliptic manifolds

- **Every $\mathbb{C}$ -homogeneous manifold Y is elliptic.** If G is a complex Lie group acting holomorphically transitively on Y and $\mathfrak{g}$ is the Lie algebra of G , then the map $Y \times \mathfrak{g} \xrightarrow{s} Y$, $(y, v) \mapsto e^v \cdot y$, is a dominating spray.
- **A flexible manifold Y is elliptic.** Such Y admits $\mathbb{C}$ -complete holomorphic vector fields $V_1, \dots, V_k$ spanning the tangent space $T_y Y$ at every point. Let ϕ_t^j denote the flow of V_j for time $t \in \mathbb{C}$. The map $s: Y \times \mathbb{C}^k \rightarrow Y$,

$$s(y, t_1, \dots, t_k) = \phi_{t_1}^1 \circ \dots \circ \phi_{t_k}^k(y),$$

is then a dominating spray on Y .

- **An algebraically flexible manifold is algebraically elliptic.**
The tangent bundle of such a manifold is pointwise spanned by complete algebraic vector fields with algebraic flows.
- **Arzhantsev, Kaliman and Zaidenberg 2024, Banecki 2025**
Every rational projective manifold is algebraically elliptic.
- **Kaliman and Zaidenberg 2024** Every algebraically subelliptic manifold is also algebraically elliptic.

CAP $\iff$ OKA

Definition (F. 2005) A complex manifold Y enjoys the **convex approximation property (CAP)** if every holomorphic map $K \rightarrow Y$ from a neighbourhood of a compact convex set $K \subset \mathbb{C}^n$ is a uniform limit of holomorphic maps $\mathbb{C}^n \rightarrow Y$.

The implication **Elliptic** $\implies$ **CAP** is immediate from the relative Oka principle.

Theorem (F. 2005–2010, 2017)

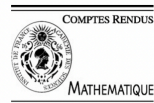
- $CAP \iff OKA$. (*This answers Gromov's first question.*)
- *The individual Oka properties are pairwise equivalent.*
- *If $Y \rightarrow Z$ is a holomorphic fibre bundle with an Oka fibre, then Y is Oka iff Z is Oka. This generalises to **Oka maps**.*
- *Every Oka manifold Y is the image of a strongly dominating holomorphic map $\mathbb{C}^{\dim Y} \rightarrow Y$.*



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Complex Analysis

Oka manifolds

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Presented by Mikhaël Gromov

To Mikhaël Gromov on the occasion of his receiving the Abel Prize

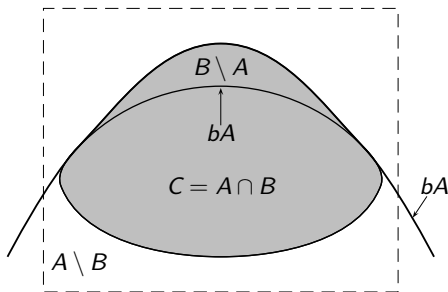
MSC 2020: 32Q56 Oka principle and Oka manifolds

Sketch of proof $\text{CAP} \implies \text{OKA}$

Every Stein manifold X admits a smooth strongly plurisubharmonic Morse exhaustion function $\rho : X \rightarrow [0, \infty)$, $dd^c \rho > 0$. Every sublevel set

$$X_a = \{x \in X : \rho(x) \leq a\}, \quad a \in \mathbb{R},$$

is **strongly pseudoconvex**: at any point $x_0 \in \partial X_a$ where $d\rho_{x_0} \neq 0$, X_a is locally biholomorphic to a strongly convex domain. If ρ has no critical values on $[a, b]$, then X_b is obtained from X_a by successively attaching **convex bumps**.



CAP $\implies$ OKA: the noncritical case

- We can extend a holomorphic map $f: A \rightarrow Y$ from a strongly pseudoconvex domain $A \subset X$ to a **dominating holomorphic spray** $F: A \times U \rightarrow Y$, where $0 \in U \subset \mathbb{C}^N$ is a ball, such that $F(\cdot, 0) = f$ and

$$\left. \frac{\partial}{\partial z} \right|_{z=0} F(x, z) : \mathbb{C}^N \rightarrow T_{f(x)} Y \text{ is surjective for every } x \in A.$$

- Let B be a convex bump on A . Since $C = A \cap B$ is convex, we can apply CAP to approximate F on $C \times U$ by holomorphic maps $G: B \times U \rightarrow Y$.
- There exists $0 < r_0 < 1$ such that if G is close enough to F on $C \times U$, there is a holomorphic map

$$\gamma: C \times r_0 U \rightarrow C \times U, \quad \gamma(x, z) = (x, c(x, z)),$$

close to the identity map, such that

$$F = G \circ \gamma \quad \text{holds on } C \times r_0 U.$$

- **The splitting lemma:** Given $r \in (0, r_0)$, we have

$$\gamma \circ \alpha = \beta \quad \text{on } C \times rU$$

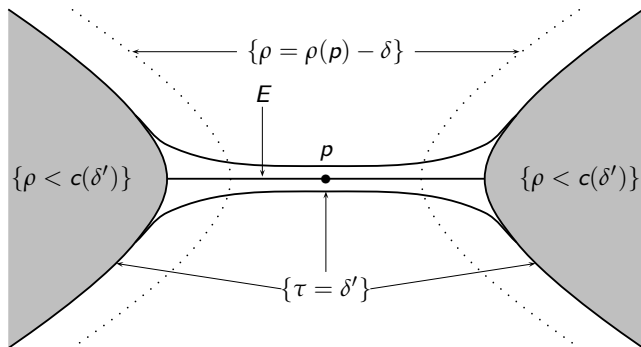
where α and β are of the same form as γ over A and B , respectively, and close to the identity. Then,

$$F \circ \alpha = G \circ \beta \quad \text{holds on } C \times rU.$$

CAP $\implies$ OKA: the critical case

This gives a holomorphic map $A \cup B \rightarrow Y$ approximating f on A . In finitely many steps of this kind, we can approximate the holomorphic map $f: X_a \rightarrow Y$ by a holomorphic map $\tilde{f}: X_b \rightarrow Y$.

To cross the Morse critical points of ρ , we use a Mergelyan type approximation theorem for maps on Stein handlebodies, which effect the change of topology of the sublevel set.



Examples of Oka manifolds

- A Riemann surface is Oka iff it is not hyperbolic.
- **Grauert 1958** Complex Lie groups and homogeneous manifolds are Oka.
- **Gromov 1989** Elliptic complex manifolds are Oka.
- Hirzebruch surfaces ($\mathbb{P}^1$ bundles over $\mathbb{P}^1$) are Oka.
- Hopf manifolds (quotients of $\mathbb{C}^n \setminus \{0\}$ by cyclic groups) are Oka.
- Complex tori of dimension > 1 with finitely many points removed, or blown up in finitely many points, are Oka.
- **Lárusson 2013** Smooth toric varieties are Oka.
- **Mok 1988** Every compact Kähler manifold with nonnegative holomorphic bisectional curvature is an Oka manifold.
- **Campana & Peternell 1991** Every compact projective manifold with $\dim \leq 3$ and with nef tangent bundle is an Oka manifold.
- **Kaliman and Zaidenberg, 2025** Every smooth cubic projective hypersurface of dimension at least two is algebraically elliptic, hence Oka.
- **Banacki 2025** Rational projective manifolds are Oka.

Kusakabe's characterisation of Oka manifolds

Since 2020, major contributions have been made by
Yuta Kusakabe (2026 Math. Soc. Japan Spring Prize).

Kusakabe 2021 A complex manifold Y is said to enjoy **convex relative ellipticity, CRE**, if for every holomorphic map $f: U \rightarrow Y$ from a bounded convex set $U \subset \mathbb{C}^m$ there is a holomorphic spray $F: U \times \mathbb{C}^N \rightarrow Y$ with $F(\cdot, 0) = f$ satisfying

$$\left. \frac{\partial}{\partial \bar{\zeta}} \right|_{\zeta=0} F(z, \zeta) : \mathbb{C}^N \rightarrow T_{f(z)} Y \text{ is surjective for every } z \in U.$$

Kusakabe 2021: CRE $\implies$ CAP. It follows that

$$\text{CRE} \iff \text{CAP} \iff \text{OKA}$$

Corollary (Kusakabe's localization theorem, 2021)

If a complex manifold Y is a union of Zariski open Oka domains, then Y is Oka.

Complements of polynomially convex sets are Oka

Theorem (Kusakabe 2020; Ann. Math. 2024)

If K is a compact polynomially convex set in $\mathbb{C}^n$ ($n > 1$) then $\mathbb{C}^n \setminus K$ satisfies CRE, and hence is an Oka manifold.

The analogous result holds if $\mathbb{C}^n$ is replaced by any Stein manifold X having many complete holomorphic vector fields (Varolin's density property).

In fact, CRE of $\mathbb{C}^n \setminus K$ holds in a stronger form, which gives a holomorphically varying family of Fatou–Bieberbach domains with given centres in $\mathbb{C}^n \setminus K$.

Theorem (Wold & F. 2020)

Every holomorphic map $f: U \rightarrow \mathbb{C}^n \setminus K$ from a convex domain $U \subset \mathbb{C}^m$ extends to a holomorphic map $F: U \times \mathbb{C}^n \rightarrow \mathbb{C}^n \setminus K$ such that for all $\zeta \in U$,

- $F(\zeta, \cdot) : \mathbb{C}^n \rightarrow \mathbb{C}^n \setminus K$ is injective (a Fatou–Bieberbach map), and*
- $F(\zeta, 0) = f(\zeta)$ for all $\zeta \in U$.*

What is the shape of Oka domains?

Conjecture: Every Oka domain in a complex manifold is pseudoconcave.

Theorem (Wold & F., 2023)

If $E \subset \mathbb{C}^n$, $n > 1$, is a closed convex set which does not contain any affine real line, then the concave domain $\mathbb{C}^n \setminus E$ is an Oka manifold.

Example

If $\phi : \mathbb{C}^{n-1} \times \mathbb{R} \rightarrow \mathbb{R}_+$ is a strictly convex function then the concave domain

$$\{(z, x + iy) \in \mathbb{C}^n : y < \phi(z, x)\} \text{ is Oka}$$

while $\{y > \phi(z, x)\}$ is hyperbolic. The halfspace $\{y > 0\}$ has neither property.

Theorem (Kusakabe & F. 2024)

Let $X \subset \mathbb{CP}^n$ be a flag manifold and $L \rightarrow X$ an ample holomorphic line bundle. For any semipositive hermitian metric h on L , the tube $\{h < 1\}$ is Oka.

Every projective Oka manifold is elliptic

Gromov 1989 Is every Oka manifold elliptic?

A negative answer for open manifolds follows by combining the following result.

Andrist, Shcherbina & Wold 2016 If K is a compact set in $\mathbb{C}^n$ ($n \geq 3$) with infinitely many limit points then $\mathbb{C}^n \setminus K$ is not elliptic or subelliptic.

Kusakabe 2024 If K is a compact polynomially convex set in $\mathbb{C}^n$, $n > 1$, then $\mathbb{C}^n \setminus K$ is an Oka domain.

However, we have a positive answer for projective manifolds.

Theorem (Lárusson & F. 2026)

Every projective Oka manifold $Y \subset \mathbb{P}^n$ is elliptic. In fact, there exists a dominating holomorphic spray $s: N\mathbb{U}^{\otimes k}|_Y \rightarrow Y$, where $\mathbb{U} \cong \mathcal{O}_{\mathbb{P}^n}(-1) \rightarrow \mathbb{P}^n$ is the universal line bundle and $N, k > 0$ are sufficiently large integers.

Problem

Is there a compact nonprojective Oka manifold which fails to be elliptic?

Scheme of proof

Let $Y \subset \mathbb{CP}^n$ be a projective manifold and $\pi : E \rightarrow Y$ a holomorphic vector bundle. Along the zero section $E(0) \cong Y$ of E , we have a natural splitting

$$TE|_{E(0)} \cong E \oplus TY.$$

Let V be a holomorphic vector field on E that vanishes on $E(0)$. There is a neighbourhood $\Omega \subset E$ of $E(0)$ such that for any $e \in \Omega$, the flow $\phi_\tau(e)$ of V with $\phi_0(e) = e$ exists for all $0 \leq \tau \leq 1$. The holomorphic map

$$s = \pi \circ \phi_1 : \Omega \rightarrow Y$$

is then a local spray on Y . Taking $E = N \cdot \mathbb{U}^k|_Y$, where $\mathbb{U} \rightarrow \mathbb{CP}^n$ is the universal line bundle and $N, k \gg 0$, we construct s whose vertical derivative

$$ds_y : T_{0_y}E_y \cong E_y \rightarrow T_yY$$

is surjective for every $y \in Y$. This means that s is dominating.

Since E is a 1-convex manifold with the exceptional subvariety $E(0)$, the Oka principle (**Prezelj 2010, 2016; Stopar 2013**) gives a global holomorphic spray $E \rightarrow Y$ which agrees with s to the second order along $E(0)$. Hence, Y is elliptic.

A tsunami of August–September 2026

The following results were obtained in the last two months.

***** Caution: strong use of AI *****

Song-Yan Xie, Shengyuan Zhao: In every local Kuranishi family of K3 surfaces (these are Calabi–Yau manifolds), the points with Oka fibers form a dense G_δ subset of the base. A similar result holds for Kummer surfaces.

Song-Yan Xie: The complement of any cubic hypersurface in $\mathbb{P}^n$, $n > 1$, is elliptic, and hence an Oka manifold. (By Kaliman and Zaidenberg 2025, smooth cubic hypersurfaces in $\mathbb{P}^n$ are algebraically elliptic, hence Oka.)

Yun-Heng Du, Bin Guo, Song-Yan Xie:

Smooth quartic hypersurfaces in $\mathbb{P}^n$, $n \geq 4$, are Oka manifolds.

Yun-Heng Du, Bin Guo, Song-Yan Xie:

- Every rationally connected smooth projective complex manifold is an Oka-1 manifold. (This confirms a conjecture by Alarcón and F., 2025.)
- Every Fano manifold is an Oka-1 manifold.

An Oka complex structure on the 6-sphere

Claude and Levent Alpöge, August 2026

The $(3, 4, \infty)$ modular family of 2-tori $Y \rightarrow \mathbb{P}^1$, ramified over three special points in $\mathbb{P}^1$, is a complex structure on S^6 . <https://alpo.ge/s6.pdf>

Philip Engel, 1 October 2026 Complex structures on S^6 .
<https://arxiv.org/pdf/2609.38442>

If correct, this solves the problem posed by Heinz Hopf in 1948.

Yun-Heng Du, Bin Guo, Yuta Kusakabe, Song-Yan Xie; Zhangchi Chen:

The 3-fold of Alpöge is an Oka manifold.

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Franz Forstnerič

Stein Manifolds and Holomorphic Mappings

The Homotopy Principle in Complex
Analysis

Second Edition

 Springer

The main results on this subject up to 2017 are presented in my *Ergebnisse* monograph.

Most developments during 2017–25 are summarised in my surveys

Recent developments on Oka manifolds.
Indag. Math., 34(2) (2023) 367–417.

From Stein manifolds to Oka manifolds: the h-principle in complex analysis. *Proc. ICM* 2026.

Thank you for your attention.