The universal family of punctured Riemann surfaces is Stein

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What is a Teichmüller space?

Fix a Riemann surface $M = (M, J_0)$ and let $\kappa = T^*M$. Any other complex structure J on M in the same orientation class is determined by a section $\mu = \mu_J$ of the line bundle $\kappa^{-1} \otimes \overline{\kappa} \to M$, with $|\mu| < 1$. In any local holomorphic coordinate z on (M, J_0),

$$\mu = h(z) d\bar{z}/dz \quad \text{with } |h| < 1.$$

A diffeomorphism $w = f(z)$ is a J-holomorphic chart on M iff $h = f_{\bar{z}}/f_z$ is the Beltrami coefficient of f.

The structure J is said to be quasiconformally equivalent to J_0 if $\|\mu_J\|_\infty < 1$. Let

$$\mathcal{J}(M) = \{\text{complex structures on } M \text{ quasiconformally equivalent to } J_0\}.$$

Complex structure $J_1, J_2 \in \mathcal{J}(M)$ are Teichmüller equivalent iff there is a biholomorphism $\sigma : (M, J_1) \to (M, J_2)$ which induces the identity on the ideal boundary ∂M and is homotopic to the identity on $M \cup \partial M$ relative to ∂M.
The space of equivalence classes is the Teichmüller space $T(M) = T(M, J_0)$.

Droping the requirement that the homotopy from σ to Id_M is fixed on ∂M gives the reduced Teichmüller space $T^\#(M)$. (If M is compact then $T(M) = T^\#(M)$.)

Droping the requirement that σ is homotopic to Id_M gives the Riemann space $R(M)$.

$$\mathcal{J}(M) \longrightarrow T(M) \longrightarrow T^\#(M) \longrightarrow R(M)$$

Examples

- If M is $\mathbb{CP}^1$ with at most three punctures then $T(M)$, and hence also $T^\#(M)$ and $R(M)$, are singletons. This follows from classical results of Ahlfors and Bers 1960 on quasiconformal diffeomorphisms of $\mathbb{C} = \mathbb{CP}^1 \setminus \{\infty\}$, along with the fact any combination of up to three punctures can be mapped to $\{\infty, 0, 1\}$ by a Möbius transformation.
- If M is a torus then $T(M)$ and $T^\#(M)$ are canonically identifiable with the upper halfplane $\mathbb{H}$, and $R(M) = \mathbb{H}/PSL(2, \mathbb{Z})$. The same holds for once-punctured torus.
- If M is the disc $\mathbb{D}$ then $T(\mathbb{D})$ is a ball in an infinite dimensional Banach space while $T^\#(\mathbb{D})$ and $R(\mathbb{D})$ are singletons.

The universal family over $T(M)$ is a space $V(M)$ with projection

$$\pi : V(M) \to T(M)$$

whose fibre $\pi^{-1}(t)$, $t \in T(M)$, is the Riemann surface (M, J_t) where J_t is the complex structure determined by t.

The Teichmüller family $V(g, n) \to T(g, n)$

The Teichmüller space $T(M)$ is a finite dimensional complex manifold iff

$$M = \widehat{M} \setminus \{p_1, \ldots, p_n\}$$

is a compact Riemann surface $\widehat{M}$ of genus g with n punctures.

By the Bers embedding theorem, $T(M) = T(g, n)$ is a bounded contractible Stein domain in some $\mathbb{C}^N$. If $2g + n \geq 3$ then $N = 3g - 3 + n$.

Marković 2018 If $g \geq 2$ then $T(g, n)$ is not a convex domain in $\mathbb{C}^{3g-3+n}$. Indeed, the Kobayashi metric on $T(g, n)$ differs from the Carathéodory metric. By Lempert 1981, these metric agree on any bounded convex domain.

Ahlfors and Bers 1960 The universal family

$$\pi : \widehat{V}(g, n) \to T(g, n),$$

whose fibre over $t \in T(g, n)$ is the compact Riemann surface $(\widehat{M}, J_t)$, is a holomorphic submersion.

Furthermore, there are n canonical holomorphic sections

$$s_1, \dots, s_n : T(g,n) \to \widehat{V}(g,n)$$

with pairwise disjoint images, representing the punctures. The open subset

$$V(g,n) = \widehat{V}(g,n) \setminus \bigcup_{i=1}^{n} s_i(T(g,n)) \tag{1}$$

is the universal family of n-punctured compact Riemann surfaces.

If $2g+n \geq 3$ then the Teichmüller projection $\pi : V(g,n) \to T(g,n)$ is the universal object in the category of topologically marked holomorphically varying families of n-punctured genus g Riemann surfaces, in the sense that any other such family is obtained by pulling back the universal family by a holomorphic map to $T(g,n)$.

The first main result

Theorem (F. 2026)

For every $g \geq 0$ and $n \geq 1$ the universal family $V(g,n)$ is a Stein manifold.

Since $T(g,n)$ is a Stein manifold, this is a special case of the following result.

Theorem (F. 2026)

Assume that X is a Stein manifold, Z is a connected complex manifold with $\dim Z = \dim X + 1$, $\pi : Z \to X$ is a surjective proper holomorphic submersion, and $s_1, \ldots, s_n : X \to Z$ are holomorphic sections with pairwise disjoint images for some $n \geq 1$. Then, the domain $\Omega = Z \setminus \bigcup_{i=1}^n s_i(X)$ is Stein.

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Corollary 1

Example

- If $g = 0$ and $n \in \{1, 2, 3\}$ then $T(0, n)$ is a singleton and $V(0, n)$ equals $\mathbb{C}$, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, and $\mathbb{C} \setminus \{0, 1\}$, respectively.
- $V(1, 1) \to T(1, 1) \cong \mathbb{H}$ is the universal family of once-punctured tori. It is no longer evident that the complex surface $V(1, 1)$ is Stein.

Corollary (1)

If $2g + n \geq 3$ then $V(g, n)$ admits a proper holomorphic embedding in $\mathbb{C}^N$ with $N = \left[\frac{3 \dim V(g,n)}{2}\right] + 1 = \left[\frac{9g-9+3n}{2}\right] + 1$.

This follows from a theorem of Eliashberg and Gromov 1992 and Schürmann 1997.

Oka manifolds

A complex manifold Y is called an Oka manifold (F. 2009) if maps $X \to Y$ from any Stein space X satisfy all forms of the Oka principle:

(a) Every continuous map $f : X \to Y$ is homotopic to a holomorphic map.

(b) If $f : X \to Y$ is holomorphic on (a neighbourhood of) a compact holomorphically convex subset $K = \widehat{K} \subset X$ and on a closed complex subvariety X_0 of X, then there is a homotopy from f to a holomorphic map $F : X \to Y$ consisting of maps with the same properties which approximate f on K and agree with f on X_0.

(c) A similar statement holds for maps $f : X \times P \to Y$ depending continuously on a parameter in a compact Hausdorff space P.

Oka–Weil–Cartan 1936–53: $\mathbb{C}^n$ is an Oka manifold.

Oka 1939: $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is an Oka manifold.

Grauert 1958: Every complex homogeneous manifold is an Oka manifold.

Gromov 1989: Every elliptic complex manifold is Oka.

F. 2006: A complex manifold Y is Oka iff every holomorphic map from a neighbourhood of a compact convex set $K \subset \mathbb{C}^n$ to Y is a uniform limit of holomorphic maps $\mathbb{C}^n \to Y$.

Corollary 2

Since $T(g,n)$ is contractible, the submersion $\pi : V(g,n) \to T(g,n)$ is smoothly trivial and the inclusion of any fibre of π in $V(g,n)$ is a homotopy equivalence. It follows that for any manifold Y, the restrictions of a continuous map $V(g,n) \to Y$ to the fibre of $V(g,n)$ lie in the same homotopy class.

Corollary (2)

Let Y be an Oka manifold and $n \geq 1$.

- *There is a holomorphic map $V(g,n) \to Y$ in every homotopy class.*
- *A holomorphic map $M_t \to Y$, $t \in T(g,n)$, from any fibre $M_t = \pi^{-1}(T)$ of $V(g,n)$ extends to a holomorphic map $V(g,n) \to Y$.*
- *More generally, given a closed complex subvariety T' of $T(g,n)$, every continuous map $F_0 : V(g,n) \to Y$ which is holomorphic on $\pi^{-1}(T')$ is homotopic to a holomorphic map $F : V(g,n) \to Y$ by a homotopy which is fixed on $\pi^{-1}(T')$.*

Corollary 3

Note that $V(0,1) = \mathbb{C}$. If $n \geq 1$ and $(g,n) \neq (0,1)$ then $V(g,n)$ is not simply connected, and its homotopy type is that of a finite bouquet of circles.
It follows that homotopically nontrivial continuous maps $V(g,n) \to Y$ exist whenever the manifold Y is not simply connected.

Corollary (3)

If $n \geq 1$ and Y is an Oka manifold which is not simply connected, there is a holomorphic map $V(g,n) \to Y$ which is homotopically nontrivial on every fibre.

This holds in particular for $Y = \mathbb{C}^*$.

Corollary 4

The fact that $V(g, n)$ for $n \geq 1$ is homotopy equivalent to a bouquet of circles implies that every complex vector bundle on $V(g, n)$ is topologically trivial. Since $V(g, n)$ is Stein, the Oka–Grauert principle implies the following.

Corollary (4)

Assume that $n \geq 1$.

(a) *Every holomorphic vector bundle on $V(g, n)$ is holomorphically trivial.*

(b) *There exists a nowhere vanishing holomorphic vector field ξ on $V(g, n)$ which is tangent to the fibres of the projection $\pi : V(g, n) \to T(g, n)$.*

(c) *There exists a holomorphic 1-form θ on $V(g, n)$ satisfying $\langle \theta, \xi \rangle = 1$. In particular, θ is nowhere vanishing on the tangent bundle to any fibre of π.*

Problem

Let $n \geq 1$. Does there exist a holomorphic function $f : V(g, n) \to \mathbb{C}$ whose restriction to any fibre of $\pi : V(g, n) \to T(g, n)$ is an immersion?

Proof of the main theorem

We must prove the following.

Theorem

Assume that

- X *is a Stein manifold,*
- Z *is a connected complex manifold with* $\dim Z = \dim X + 1$,
- $\pi : Z \to X$ *is a surjective proper holomorphic submersion, and*
- $s_1, \ldots, s_n : X \to Z$ *are holomorphic sections with pairwise disjoint images for some* $n \geq 1$.

Then, the domain $\Omega = Z \setminus \bigcup_{i=1}^{n} s_i(X)$ *is Stein.*

Proof, 1

The conditions imply that every fibre $Z_x = \pi^{-1}(x)$, $x \in X$, is a compact Riemann surface, and the fibres are diffeomorphic but not necessarily biholomorphic to each other. In particular, $Z \to X$ is a smooth fibre bundle whose fibre M is a compact smooth surface. Hence, $\{Z_x\}_{x\in X}$ is a holomorphic family of compact Riemann surfaces, and

$$\Omega_x = Z_x \setminus \bigcup_{i=1}^{n} s_i(x)$$

is a holomorphic family of n-punctured Riemann surfaces.

Each $H_i = s_i(X)$ is a closed complex hypersurface in Z whose ideal sheaf is a principal ideal sheaf.

Grauert and Remmert: If Z is a Stein space and H is a closed complex hypersurface in Z whose ideal sheaf is a principal ideal sheaf, then $Z \setminus H$ is also Stein. If Z is nonsingular then the ideal sheaf of any closed complex hypersurface in Z is a principal ideal sheaf.

Hence, it suffices to prove the theorem for $n = 1$, that is, to show that the complement of the image of a single section $s : X \to Z$ is Stein.

Proof, 2

By Siu 1976, the Stein hypersurface $H = s(X)$ has a basis of open Stein neighbourhoods U in Z. Since $U \setminus H$ is Stein, it admits a strongly plurisubharmonic exhaustion function $\phi : U \setminus H \to \mathbb{R}_+$.

We will construct a strongly plurisubharmonic exhaustion function $Z \setminus H \to \mathbb{R}_+$; a theorem of Grauert 1958 will then imply that $Z \setminus H$ is Stein.

Fix a point $x_0 \in X$ and set $z_0 = s(x_0) \in H \subset Z$. Since $\pi : Z \to X$ is a holomorphic submersion with compact one-dimensional fibres, it is a smooth fibre bundle whose fibre M is a compact smooth surface. In particular, there is a neighbourhood $X_0 \subset X$ of x_0 such that the restricted bundle

$$Z|X_0 = \pi^{-1}(X_0) \to X_0$$

can be smoothly identified with the trivial bundle $X_0 \times M \to X_0$. In this identification, $z_0 = (x_0, p_0)$ with $p_0 \in M$.

Proof, 3

Since ϕ tends to $+\infty$ along H, there are small smoothly bounded open discs $p_0 \in D \Subset D' \Subset M$ such that

$$\inf_{p \in bD} \phi(x_0, p) > \max_{p \in bD'} \phi(x_0, p).$$

The set $O = M \setminus D$ is a compact surface with smooth boundary $bO = bD$, endowed with the complex structure inherited by the identification $M \cong Z_{x_0} = \pi^{-1}(x_0)$. Note that $bD' \subset \mathrm{Int}(O)$. There is a smooth strongly subharmonic function $u_0 : O \to \mathbb{R}_+$ such that

$$\phi(x_0, \cdot\,) > u_0 \text{ on } bO = bD \quad \text{and} \quad u_0 > \phi(x_0, \cdot\,) \text{ on } bD'.$$

Shrinking X_0 around x_0, the following conditions hold for every $x \in X_0$:

(a) $s(x) \in D$,

(b) the function $u(x, \cdot\,) := u_0$ is strongly subharmonic on O in the complex structure on $Z_x \cong M$,

(c) $\phi(x, \cdot\,) > u(x, \cdot\,)$ on $bO = bD$ and $u(x, \cdot\,) > \phi(x, \cdot\,)$ on bD'.

Proof, 4

We define a function $\rho_0 : (X_0 \times M) \setminus H \to \mathbb{R}_+$ by taking for every $x \in X_0$:

$$\rho_0(x,p) = \begin{cases} \phi(x,p), & p \in D \setminus \{s(x)\}; \\ \max\{\phi(x,p), u(x,p)\}, & p \in D' \setminus D; \\ u(x,p), & p \in M \setminus D'. \end{cases}$$

Note that ρ_0 is well defined, piecewise smooth, strongly subharmonic on each fibre $Z_x \setminus \{s(x)\}$ $(x \in X_0)$, and it agrees with ϕ on $(X_0 \times D) \setminus H$. By using the regularised maximum in the definition of ρ_0, we may assume that ρ_0 is smooth.

This gives an open locally finite cover $\{X_j\}_{j=1}^{\infty}$ of X with smooth fibre bundle trivialisations $Z|X_j \cong X_j \times M$, discs $D_j \subset M$ such that $s(x) \in D_j$ for all $x \in X_j$, and smooth functions $\rho_j : (Z|X_j) \setminus H = \pi^{-1}(X_j) \setminus H \to \mathbb{R}_+$ such that

- ρ_j is strongly subharmonic on each fibre $Z_x \setminus \{s(x)\}$ $(x \in X_j)$, and
- $\rho_j = \phi$ on $(X_j \times D_j) \setminus H$, so it is strongly plurisubharmonic there.

Proof, 5

Let $\{\chi_j\}_j$ be a smooth partition of unity on X with $\operatorname{supp}(\chi_j) \subset X_j$ for each j. Set

$$\rho = \sum_{j=1}^{\infty} \chi_j \rho_j : Z \setminus H \to \mathbb{R}_+.$$

The restriction of ρ to each fibre $Z_x \setminus \{s(x)\}$ $(x \in X)$ is strongly subharmonic, and there is an open neighbourhood $U_0 \subset U \subset Z$ of H such that $\rho = \phi$ holds on $U_0 \setminus H$. In particular, ρ is strongly plurisubharmonic on $U_0 \setminus H$.

Since ρ is strongly subharmonic on every fibre $Z_x \setminus \{s(x)\}$, $x \in X$, we have

$$(dd^c\rho)(z, \xi \wedge J\xi) > 0 \ \text{ if } z \in Z \setminus H \text{ and } 0 \neq \xi \in \ker d\pi_z.$$

Hence, the eigenvectors of the Levi form $(dd^c\rho)(z, \cdot\,)$ with non-positive eigenvalues lie in a closed cone $C_z \subset T_z Z$ with $C_z \cap \ker d\pi_z = \{0\}$.

Furthermore, for every compact set $K \subset X$, the set $\pi^{-1}(K) \setminus U_0 \subset Z \setminus H$ is compact. Hence, by choosing a strongly plurisubharmonic exhaustion function $\tau : X \to \mathbb{R}_+$ such that $dd^c\tau$ grows fast enough, we can ensure that

$$\rho + \tau \circ \pi : Z \setminus H \to \mathbb{R}_+ \ \text{ is a strongly plurisubharmonic exhaustion function.}$$

This proves the theorem.

A generalization

A minor modification of the proof gives the following more general result.

Theorem

Assume that

- X *is a connected Stein manifold,*
- $\pi : Z \to X$ *is a surjective proper holomorphic submersion with purely one dimensional fibres, and*
- H *is a closed complex subvariety of* Z *of pure codimension one which intersects every connected component of each fibre* $Z_x = \pi^{-1}(x)$, $x \in X$, *but it does not contain any such component.*

Then, $Z \setminus H$ *is a Stein manifold.*

Fibrewise algebraic functions on $V(g, n)$

Since the fibres of the Teichmüller projection $\pi : V(g, n) \to T(g, n)$ are algebraic curves, it is natural to expect that the universal family $V(g, n)$ carries many holomorphic functions which are algebraic on every fibre. Using Grauert's theorem on coherence of direct images of coherent analytic sheaves, I proved the following result which gives another proof that $V(g, n)$ is Stein.

Theorem

Assume that X is a connected Stein manifold, $\pi : Z \to X$ is a surjective proper holomorphic submersion with connected one dimensional fibres, and H is a closed complex subvariety of Z of pure codimension one which does not contain any fibre of π. Then there exist an integer $N \in \mathbb{N}$ and a holomorphic map $f : \Omega = Z \setminus H \to \mathbb{C}^N$ which restricts to a proper algebraic embedding $\Omega_x \hookrightarrow \mathbb{C}^N$ on every fibre $\Omega_x = \Omega \cap \pi^{-1}(x)$, $x \in X$.

In particular, the algebra $\mathcal{A}(V(g, n))$ of holomorpic fibrewise algebraic functions is dense in the algebra $\mathcal{O}(V(g, n))$ of holomorphic functions.

Proof, 1

Let g denote the genus of the Riemann surface $Z_x = \pi^{-1}(x)$ and $n \geq 1$ the degree of the divisor $[H]_{Z_x}$; these numbers are independent of $x \in X$.

Choose $k \in \mathbb{N}$ such that $kn > 2g$ and consider the divisor $D = k[H]$ on Z. By Riemann–Roch we have for any $x \in X$ that

$$h^0(Z_x, \mathcal{O}_Z(D)|_{Z_x}) = kn + 1 - g > g + 1, \qquad h^1(Z_x, \mathcal{O}_Z(D)|_{Z_x}) = 0.$$

Here, h^0 and h^1 denote the dimensions of the cohomology groups H^0 and H^1.

By a theorem of Grauert and Remmert (see Theorem, p. 211 in the book *Coherent analytic sheaves*), it follows that the direct image

$$\mathcal{E} = \pi_* \mathcal{O}_Z(D)$$

is a locally free sheaf on X, that is, the sheaf of germs of holomorphic sections of a holomorphic vector bundle $E \to X$.

Proof, 2

Since X is Stein, $\mathcal{E}$ is generated by finitely many global sections $\xi_1, \ldots, \xi_N$.

The restrictions of the sections $\xi_1, \ldots, \xi_N$ to any fibre Z_x span the vector space $H^0(Z_x, \mathcal{O}_Z(D)|_{Z_x})$.

Since $kn > 2g$, the divisor $\mathcal{O}_Z(D)|_{Z_x}$ is very ample for every $x \in X$. (Hartshorne 1977, IV. Corollary 3.2). Hence:

Any basis of $H^0(Z_x, \mathcal{O}_Z(D)|_{Z_x})$ consisting of N elements defines an embedding $\Omega_x = Z_x \setminus H_x \hookrightarrow \mathbb{C}^N$.

Thus, the map

$$f = (f_1, \ldots, f_N) : Z \setminus H = \Omega \to \mathbb{C}^N$$

of class $\mathcal{A}(\Omega)$, defined by an N-tuple of generating sections of $\mathcal{E}$, restricts to a proper algebraic embedding on each fibre Ω_x, $x \in X$.

A relative Oka principle for fibrewise algebraic maps

An algebraic manifold Y is special algebraically elliptic if there are a trivial vector bundle $E \cong Y \times \mathbb{C}^N \to Y$ and an algebraic map $s : E \to Y$ such that

$$s(0_y) = y \quad \text{and} \quad ds_{0_y}(E_y) = T_y Y \quad \text{for every } y \in Y.$$

Theorem

Assume that Y is an algebraically special elliptic manifold. Given

- *a map $f : V(g,n) \to Y$ of class $\mathcal{A}(V(g,n), Y)$,*
- *a compact holomorphically convex set $K \subset V(g,n)$, and*
- *a homotopy of holomorphic maps $f_t : U \to Y$ $(t \in [0,1)$ on a neighbourhood $U \supset K$, with $f_0 = f|_U$,*

there is a holomorphic map $F : V(g,n) \times \mathbb{C} \to Y$ such that $F(\cdot\,, 0) = f$ and $F(\cdot, t) \in \mathcal{A}(V(g,n), Y)$ approximates f_t uniformly on K and in $t \in [0,1]$.

Problem

Does the conclusion hold for every algebraically elliptic manifold Y?

THANKS FOR YOUR ATTENTION