Minimal surfaces from a complex analytic viewpoint

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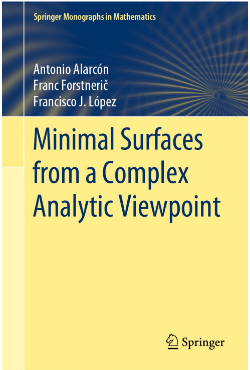
In this lecture...



I will describe some recent developments in the theory of minimal surfaces in Euclidean spaces $\mathbb{R}^n$ which have been obtained by complex analytic methods.
My collaborators on these projects:

- Antonio Alarcón and Francisco J. López, Granada
- Barbara Drinovec Drnovšek, Ljubljana
- David Kalaj, Podgorica
- Finnur Lárusson, Adelaide

1744 **Euler** A surface in $\mathbb{R}^3$ is called **minimal** if it locally minimizes the area among all nearby surfaces with the same boundary.

The only minimal surfaces of rotation are planes and catenoids.

$$x^2 + y^2 = \cosh^2 z$$

$$(t, z) \mapsto (\cos t \cdot \cosh z, \sin t \cdot \cosh z, z)$$

The catenoid is a paradigmatic example in the theory. Besides **Enneper's surface**, it is the only complete nonflat orientable minimal surface in $\mathbb{R}^3$ with the smallest absolute total Gaussian curvature 4π.

1762 **Lagrange:** Formula for the **first variation of the area** of a surface with fixed boundary. The **equation of minimal graphs**.

Minimal surfaces are stationary points of the area functional. Small pieces of such surfaces are area minimizers.

1776 **Meusnier A surface in $\mathbb{R}^3$ is a minimal surface if and only if its mean curvature vanishes at every point.**

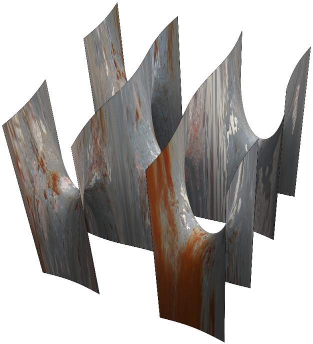
The helicoid is a minimal surface.

It is obtained by rotating a line and displacing it along the axis of rotation.

$(u, v) \mapsto$
$(\cos u \cdot \sinh v, \sin u \cdot \sinh v, u)$

1842 **Catalan** The helicoid and the plane are the only ruled minimal surfaces in $\mathbb{R}^3$.

1835 **Scherk** The first two new minimal surfaces since Meusnier (1776).



The first Scherk's surface is doubly periodic.

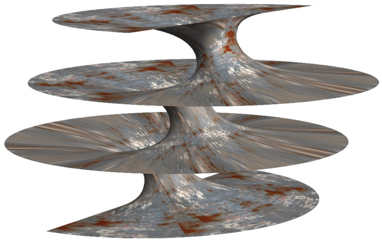
Its main branch is a graph over the square $P = (-\pi/2, \pi/2)^2$ given by

$$x_3 = \log \frac{\cos x_2}{\cos x_1}$$

Finn and Osserman (1964)
Sherk's surface S has the biggest absolute Gaussian curvature at $0 \in \mathbb{R}^3$ over all minimal graphs over P tangent to S at 0.

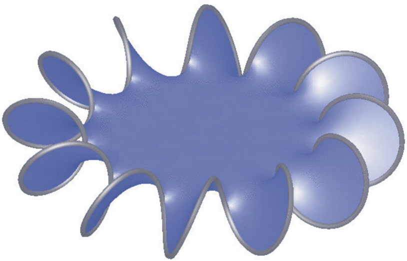
1917 **Bernstein** A minimal graph in $\mathbb{R}^3$ over the whole plane is a plane.

Bernhard Riemann discovered a family R_λ, $\lambda > 0$, of periodic planar domains, properly embedded as minimal surfaces in $\mathbb{R}^3$ such that every horizontal plane intersects each R_λ in a circle or a line. As $\lambda \to 0$ his surfaces converge to a vertical catenoid, and as $\lambda \to \infty$ they converge to a vertical helicoid.



2015 **Meeks, Pérez, Ros** Planes, catenoids, helicoids, and Riemann's examples are the only planar domains which can be properly embedded as minimal surfaces in $\mathbb{R}^3$.

1873 **Plateau** Soap films are minimal surfaces. A conformally parameterized minimal disc with a given Jordan boundary curve minimizes the internal tension within the surface.

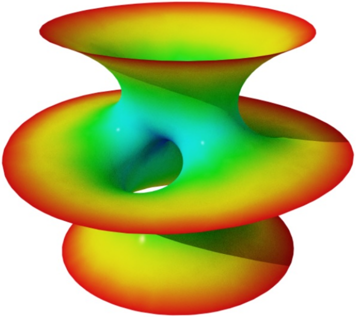


1932 **Douglas, Radó** Every Jordan curve Γ in $\mathbb{R}^3$ spans a minimal surface.

1976 **Meeks, S.-T. Yau** If Γ lies in the boundary of a convex domain, then the disc surface of smallest area with boundary Γ is embedded.

Costa's minimal surface

1982 **Celso J. Costa** discovered a complete embedded minimal surface in $\mathbb{R}^3$ of genus one, a middle planar end and two catenoidal ends, and total Gaussian curvature -12π. Its conformal type is that of a thrice-punctured torus. Hoffman and Meeks proved in 1985 that it is embedded.



1991 **Costa** The only complete embedded minimal surfaces in $\mathbb{R}^3$ having genus one and three ends are the 1-parameter family of Costa–Hoffman–Meeks surfaces.

Assume that $D \subset \mathbb{R}^2_{(u,v)}$ is a bounded domain with smooth boundary and $X : \overline{D} \to \mathbb{R}^n$ is a smooth immersion. Precomposing X with a diffeomorphism from another such domain in $\mathbb{R}^2$, we may assume that X is **conformal**:

$$|X_u| = |X_v|, \qquad X_u \cdot X_v = 0.$$

Digression: If M is any smooth orientable surface and $X : M \to N$ is an immersion into a Riemannian manifold (N, ds^2) (for example, into $(\mathbb{R}^n, ds^2)$ with the Riemannian metric $ds^2 = dx_1^2 + \cdots + dx_n^2$), then X induces on M the structure of a Riemann surface such that X is a conformal immersion.

Indeed, let $g = X^* ds^2$ be the induced metric. Then, $X : (M, g) \to (N, ds^2)$ is an isometric immersion. At every point of M there exists a local oriented **isothermal coordinate** $z = x + \imath y$ in which

$$g = \lambda |dz|^2 = \lambda (dx^2 + dy^2), \quad \lambda > 0.$$

(Solve a **Beltrami equation**.) The transition map between any pair of such local charts is an orientation preserving conformal diffeomorphism between plane domains, hence a biholomorphic map.

If M is nonorientable, we can pass to its orientable double cover $\widetilde{M} \to M$.

Analytic description of conformal minimal surfaces

We consider the **area functional**

$$\mathrm{Area}(X) = \int_D |X_u \times X_v| \, dudv = \int_D \sqrt{|X_u|^2|X_v|^2 - |X_u \cdot X_v|^2} \, dudv$$

and the **Dirichlet energy functional**

$$\mathscr{D}(X) = \frac{1}{2} \int_D |\nabla X|^2 \, dudv = \frac{1}{2} \int_D \left(|X_u|^2 + |X_v|^2 \right) \, dudv.$$

From the elementary inequalities

$$|x|^2|y|^2 - |x \cdot y|^2 \le |x|^2|y|^2 \le \frac{1}{4} \left(|x|^2 + |y|^2 \right)^2, \quad x, y \in \mathbb{R}^n,$$

which are equalities iff x, y is a conformal frame, we infer that

$$\mathrm{Area}(X) \le \mathscr{D}(X), \quad \text{with equality iff } X \text{ is conformal.}$$

Hence, these two functionals have the same critical points.

A conformal immersion is minimal iff it is harmonic

If $G : \overline{D} \to \mathbb{R}^n$ is a smooth map vanishing on bD, then

$$\frac{d}{dt}\Big|_{t=0} \mathscr{D}(X+tG) = \int_D (X_u \cdot G_u + X_v \cdot G_v)\, dudv = -\int_D \Delta X \cdot G\, dudv.$$

We integrated by parts and used $G|_{bD} = 0$.

This vanishes for all such G iff $\Delta X = 0$, thereby proving the first part of the following theorem. The second one is an elementary calculation.

Theorem

A smooth conformal immersion $X : \overline{D} \to \mathbb{R}^n$ $(n \geq 3)$ parameterizes a minimal surface iff X is harmonic $(\Delta X = 0)$ iff the mean curvature vector field of X vanishes.

Connection to complex analysis

Let $z = u + \imath v$ be a complex coordinate on $\mathbb{C}$. Then,

$$dX = \partial X + \bar{\partial} X = X_z dz + X_{\bar{z}} d\bar{z} = \frac{1}{2}(X_u - \imath X_v)\, dz + \frac{1}{2}(X_u + \imath X_v)\, d\bar{z}.$$

From $\Delta X = 4(X_z)_{\bar{z}}$ we see that

X **is harmonic iff** X_z **is holomorphic**.

Furthermore:

$$X \text{ is conformal} \iff X_u \cdot X_v = 0,\ |X_u|^2 = |X_v|^2 \iff \sum_{k=1}^{n} (\partial X_k)^2 = 0.$$

The analogous conclusions hold if D is any **open Riemann surface**.

Every orientable immersed surface in $\mathbb{R}^n$ admits a conformal parameterization by a Riemann surface. For nonorientable surfaces, we pass to their orientable double cover.

Hence, a smooth immersion $X = (X_1, X_2, \cdots, X_n) : M \to \mathbb{R}^n$ from an open Riemann surface parameterizes a conformal minimal surface if and only if

$$\partial X = (\partial X_1, \ldots, \partial X_n) \text{ is a holomorphic 1-form and } \sum_{k=1}^{n} (\partial X_k)^2 = 0.$$

Conversely: a nowhere vanishing holomorphic 1-form $\Phi = (\phi_1, \ldots, \phi_n)$ on M satisfying the nullity condition $\sum_{k=1}^{n} \phi_k^2 = 0$ and the period vanishing conditions

$$\Re \int_C \Phi = 0 \in \mathbb{R}^n \quad \text{for every closed curve } C \text{ in } M$$

determines a conformal minimal immersion

$$X = \Re \int \Phi : M \to \mathbb{R}^n, \qquad 2\partial X = \Phi.$$

Since Φ is holomorphic, it suffices to test the period conditions on the homology basis of $H_1(M, \mathbb{Z})$. If M is simply connected, there are no period conditions.

The null quadric

Let us introduce the **null quadric**:

$$\mathcal{A} = \left\{ z = (z_1, \dots, z_n) \in \mathbb{C}^n : \sum_{j=1}^{n} z_j^2 = 0 \right\}.$$

Fix a nowhere vanishing holomorphic 1-form θ on M. Then, with Φ as above, we have

$$\Phi = f\theta$$

where

$$f : M \to \mathcal{A}_* = \mathcal{A} \setminus \{0\}$$

is a holomorphic map having vanishing real periods:

$$\Re \int_C f\theta = 0 \in \mathbb{R}^n \quad \text{for every } [C] \in H_1(M, \mathbb{Z}).$$

If

$$\int_C f\theta = 0 \in \mathbb{C}^n \quad \text{for every } [C] \in H_1(M, \mathbb{Z}),$$

then $Z = \int f\theta : M \to \mathbb{C}^n$ is a well-defined holomorphic curve with $dZ = f\theta$; such are called **holomorphic null curves**.

Dimension $n = 3$

Let $X = (X_1, X_2, X_3) : M \to \mathbb{R}^3$ be a conformal minimal immersion, and let $(N_1, N_2, N_3) : M \to S^2 \subset \mathbb{R}^3$ denote its **classical Gauss map**. Then,

$$\mathfrak{g} = \frac{N_1 + \mathfrak{i}N_2}{1 - N_3} = \frac{\partial X_3}{\partial X_1 - \mathfrak{i}\partial X_2} : M \to \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$$

is a holomorphic map, called the **complex Gauss map** of X, and we have the **classical Enneper-Weierstrass formula**:

$$X = 2\Re \int \left(\frac{1}{2}\left(\frac{1}{\mathfrak{g}} - \mathfrak{g}\right), \frac{\mathfrak{i}}{2}\left(\frac{1}{\mathfrak{g}} + \mathfrak{g}\right), 1 \right) \partial X_3.$$

Many important quantities of a minimal surface can be computed from its Gauss map, in particular:

$$g = X^* ds^2 = 2(|\partial X_1|^2 + |\partial X_2|^2 + |\partial X_3|^2) = \frac{(1 + |\mathfrak{g}|^2)^2}{4|\mathfrak{g}|^2} |\partial X_3|^2$$

$$Kg = -\frac{4|d\mathfrak{g}|^2}{(1 + |\mathfrak{g}|^2)^2} = -\mathfrak{g}^*(\sigma^2_{\mathbb{CP}^1}) \qquad \text{the Gauss curvature function}$$

$$\mathrm{TC}(X) = \int_M K dA = -\mathrm{Area}_{\mathbb{CP}^1}(\mathfrak{g}(M)) \qquad \text{the total Gauss curvature}$$

Schwarz–Pick lemma for harmonic maps which are conformal at a point

Theorem (F. and Kalaj, 2021)

Let $\mathbb{D}$ denote the unit disc in $\mathbb{C}$ and $\mathbb{B}^n$ denote the unit ball of $\mathbb{R}^n$. Assume that $f : \mathbb{D} \to \mathbb{B}^n$ $(n \geq 2)$ is a harmonic map which is conformal at a point $z \in \mathbb{D}$. Denote by $R \in (0, 1]$ the radius of the affine disc $\Sigma = (f(z) + df_z(\mathbb{R}^2)) \cap \mathbb{B}^n$. Then

$$\|df_z\| \leq \frac{1}{R} \cdot \frac{1 - |f(z)|^2}{1 - |z|^2}.$$

Equality holds if and only if f is a conformal diffeomorphism of $\mathbb{D}$ onto Σ, and in this case equality holds at every point of $\mathbb{D}$.

The case $n = 2$, $R = 1$ generalizes the classical **Schwarz–Pick lemma** due to **H. A. Schwarz** 1869, **H. Poincaré** 1884, and **G. Pick** 1915.

The classical lemma pertains only to maps which are **conformal at every point** of $\mathbb{D}$, i.e., they are **holomorphic or antiholomorphic**.

Distance-decreasing property of conformal harmonic maps

The theorem implies that **conformal harmonic maps $\mathbb{D} \to \mathbb{B}^n$ are distance decreasing from the Poincaré metric $\frac{|dz|}{1-|z|^2}$ on the disc $\mathbb{D}$ to the Cayley–Klein metric on the ball $\mathbb{B}^n$:**

$$\mathcal{CK}^2 = \frac{(1-|x|^2)|dx|^2 + |x \cdot dx|^2}{(1-|x|^2)^2} = \frac{|dx|^2}{1-|x|^2} + \frac{|x \cdot dx|^2}{(1-|x|^2)^2}.$$

Extremal maps are the conformal embeddings of the disc $\mathbb{D}$ onto affine discs in the ball $\mathbb{B}^n$. Their differential at any point has norm one in the Poincaré metric on the disc $\mathbb{D}$ and the Cayley–Klein metric on the ball $\mathbb{B}^n$.

The same is true if we replace the disc by any hyperbolic Riemann surface (universally covered by the disc) endowed with the Poincaré metric.

The **Beltrami–Cayley–Klein model of hyperbolic geometry** was introduced by **Arthur Cayley** (1859) and **Eugenio Beltrami** (1968), and it was developed by **Felix Klein** (1871–1873). The underlying space is the ball $\mathbb{B}^n$, and geodesics are line segments with endpoints on the boundary sphere. This is a special case of a metric on convex domains in $\mathbb{R}^n$ introduced by **David Hilbert** in 1895.

Up to constants, the Cayley–Klein metric is the restriction to $\mathbb{B}^n$ of the Kobayashi metric and the Bergman metric on the complex ball $\mathbb{B}^n_{\mathbb{C}} \subset \mathbb{C}^n$.

Runge-Mergelyan approximation theorem for minimal surfaces

Theorem (Alarcón, López, F., 2012–2017)

Let K be a compact smoothly bounded domain without holes in an open Riemann surface M (such K is called a **Runge compact** *in M). Then:*

- *Every conformal minimal immersion $X : K \to \mathbb{R}^n$ $(n \geq 3)$ can be approximated uniformly on K by proper conformal minimal immersions $\widetilde{X} : M \to \mathbb{R}^n$.*
- **General position theorem:** *$\widetilde{X}$ can be chosen to have only simple double points if $n = 4$ and to be an embedding if $n \geq 5$.*
- *Analogous results hold for* **nonorientable minimal surfaces** *in $\mathbb{R}^n$ and for* **holomorphic null curves** *in $\mathbb{C}^n$, $n \geq 3$.*

2019 **Alarcón, Castro-Infantes**: In addition, one can prescribe the values of $\widetilde{X}$ on any closed discrete subset of M (Weierstrass-type interpolation).

2019 **Alarcón, López**: The analogous approximation result holds for complete minimal surfaces of finite total Gaussian curvature. In this case, M is a finitely punctured compact Riemann surface and $\partial\widetilde{X}$ is algebraic with an effective pole at every puncture.

Techniques used in the proof

Fix a nonvanishing holomorphic 1-form θ on M. (For example, $\theta = dz$ on $\mathbb{C}$.)

By Enneper–Weierstrass, it suffices to prove the Runge–Mergelyan approximation theorem for holomorphic maps $f : M \to \mathcal{A}_*$ satisfying the period vanishing conditions

$$\Re \int_C f\theta = 0 \quad \text{for all } [C] \in H_1(M, \mathbb{Z}).$$

To this end, we use two main properties of punctured null quadric. The first one is:

The punctured null quadric $\mathcal{A}_*$ is an Oka manifold.

In fact, $\mathcal{A}_*$ is a complex homogeneous space of the complex orthogonal group $O_n(\mathbb{C})$.

A complex manifold Y is an **Oka manifold** if, in the absence of topological obstructions, the Runge approximation theorem holds for holomorphic maps $M \to Y$ from any Stein manifold, in particular, from any open Riemann surface.

1958 **Grauert** Every complex homogeneous manifold is Oka.

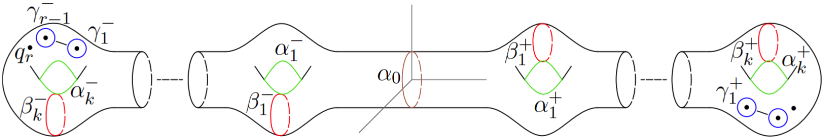
To control the period vanishing conditions, we also need that

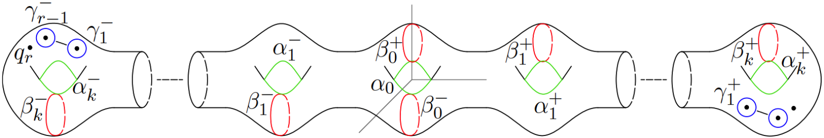
the convex hull of $\mathcal{A}_*$ equals $\mathbb{C}^n$.

It follows that any pair of points in $\mathbb{C}^n$ can be connected by a path whose derivative belongs to $\mathcal{A}_*$. (M. Gromov: Convex integration lemma.)

Assume that $K \subset L$ are connected, smoothly bounded Runge compacts in M, $X : K \to \mathbb{R}^n$ is a conformal minimal surface, and $f = 2\partial X/\theta : K \to \mathcal{A}_*$. We may assume that $f(K)$ is not contained in a ray of $\mathcal{A}_*$.

The noncritical case: there is no change of topology from K to L. Let $C_1, \ldots, C_\ell \subset K$ be closed curves forming a homology basis of K such that $C = \bigcup_{j=1}^{\ell} C_j$ is Runge.





The noncritical case

Let $\mathbb{B}^n$ denote the unit ball of $\mathbb{C}^n$. By using flows of vector fields tangent to $\mathcal{A}_*$ and the fact that $\mathrm{Co}(\mathcal{A}_*) = \mathbb{C}^n$, we construct a holomorphic map

$$F : K \times \mathbb{B}^{n\ell} \to \mathcal{A}_*, \quad F(\cdot\,, 0) = f = 2\partial X/\theta$$

such that the **period map**

$$\mathbb{B}^{n\ell} \ni t \longmapsto \left(\int_{C_j} F(\cdot\,, t)\theta\right)_{j=1}^{\ell} \in \mathbb{C}^{n\ell}$$

is **biholomorphic onto its image**. Such a map can be found of the form

$$F(p,t) = \phi^1_{g_1(p)t_1} \circ \phi^2_{g_2(p)t_2} \circ \cdots \circ \phi^{n\ell}_{g_{n\ell}(p)t_{n\ell}}(f(p)) \in \mathcal{A}_*, \quad p \in K,$$

where each ϕ^j is the flow of a holomorphic vector field tangent to $\mathcal{A}$ and $g_j \in \mathcal{O}(K)$.
Since $\mathcal{A}_*$ is Oka, we can approximate F by a holomorphic map $\widetilde{F} : M \times \mathbb{B}^{n\ell} \to \mathcal{A}_*$.

By the implicit function theorem, there exists $\tilde{t} \in \mathbb{B}^{n\ell}$ close to 0 such that the map $\tilde{f} = F(\cdot\,, \tilde{t}) : M \to \mathcal{A}_*$ has vanishing real periods on the curves $C_1, \ldots, C_\ell$.

Hence, fixing a point $p_0 \in K$, the map $\widetilde{X} : L \to \mathbb{R}^n$ given by

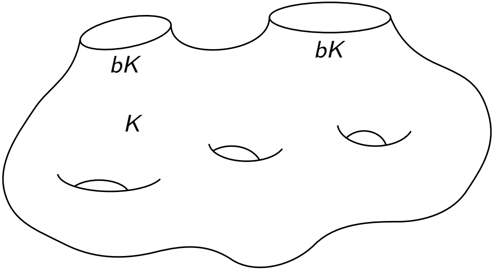
$$\widetilde{X}(p) = X(p_0) + \Re \int_{p_0}^{p} \tilde{f}\theta, \qquad p \in L$$

is a conformal minimal immersion which approximates $X : K \to \mathbb{R}^n$ on K.

The critical case

Assume now that $E \subset L \setminus \mathring{K}$ is an arc attached with its endpoints to K such that $K \cup E$ is a deformation retract of L. This situation arises when passing a critical point of index 1 of an exhaustion function on M. We illustrate **attaching a pair of pants**, which increases the genus by one.

1° The domain K



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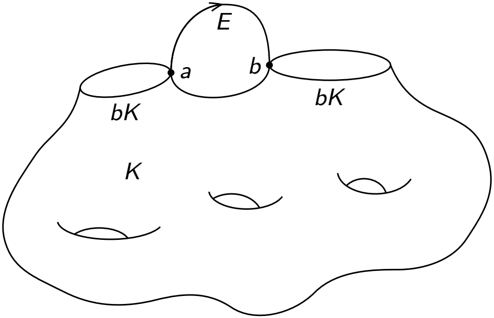

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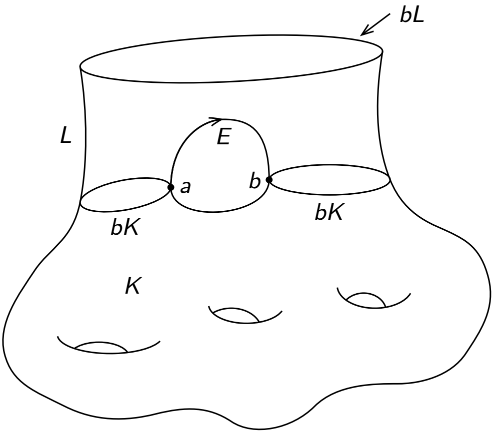
1° The domain K

2° Add the arc E – the seam of the pants

3° Add the pair of pants



The critical case

Let $a, b \in bK$ be the endpoints of the arc E. Fix $p_0 \in K$ and write

$$X(p) = X(p_0) + \Re \int_{p_0}^{p} f\theta, \qquad p \in K.$$

Extend f smoothly across E to a map $f : K \cup E \to \mathcal{A}_*$ such that

$$\Re \int_E f\theta = X(b) - X(a) \in \mathbb{R}^n.$$

This is possible since the convex hull of $\mathcal{A}_*$ equals $\mathbb{C}^n$.

Now, proceed as in the noncritical case, constructing a period dominating spray and applying Mergelyan approximation on $K \cup E$.

The proof of the approximation theorem follows by induction on an exhaustion of M such that every step is of one of the two types described above. Interpolation on a discrete set of points is obtained by the same scheme.

However, general position theorems and the existence of proper conformal minimal immersions or embeddings require substantial additional work.

Topological structure of the space of conformal minimal immersions

Let $\mathrm{CMI}(M, \mathbb{R}^n)$ denote the space of all conformal minimal immersions $M \to \mathbb{R}^n$, $n \geq 3$. Consider the map

$$\phi : \mathrm{CMI}(M, \mathbb{R}^n) \to \mathscr{O}(M, \mathcal{A}_*) \hookrightarrow \mathscr{C}(M, \mathcal{A}_*), \quad \phi(X) = 2\partial X/\theta.$$

Let $\mathrm{CMI}_{\mathrm{nf}}(M, \mathbb{R}^n)$ denote the subspace of $\mathrm{CMI}(M, \mathbb{R}^n)$ consisting of **nonflat** conformal minimal immersions $M \to \mathbb{R}^n$.

Theorem

1. **F., Lárusson 2019** *The map $\phi : \mathrm{CMI}_{\mathrm{nf}}(M, \mathbb{R}^n) \to \mathscr{C}(M, \mathcal{A}_*)$ is a weak homotopy equivalence, and a homotopy equivalence if M has finite topological type.*
2. **Alarcón, F., López 2019** *The map $\phi : \mathrm{CMI}(M, \mathbb{R}^n) \to \mathscr{C}(M, \mathcal{A}_*)$ induces a bijection of path components of the two spaces. Hence,*

$$\pi_0(\mathrm{CMI}(M, \mathbb{R}^n)) = \begin{cases} \mathbb{Z}_2^\ell, & n = 3,\ H_1(M, \mathbb{Z}) = \mathbb{Z}^\ell; \\ 0, & n > 3. \end{cases}$$

3. **Alarcón, F., Lárusson, 2018–2019** *The inclusion of the space of real parts of holomorphic null curves $M \to \mathbb{C}^n$ into the space of conformal minimal immersions $M \to \mathbb{R}^n$ is a weak homotopy equivalence.*

Minimal surfaces with the given Gauss map

Let $X = (X_1, \ldots, X_n) : M \to \mathbb{R}^n$ be a conformal minimal immersion.

Since the 1-form $\partial X = (\partial X_1, \ldots, \partial X_n)$ is holomorphic and nowhere vanishing, it determines the Kodaira type holomorphic map

$$G_X : M \to \mathbb{CP}^{n-1}, \quad G_X(p) = [\partial X_1(p) : \cdots : \partial X_n(p)], \quad p \in M,$$

called the **Gauss map of X**, with values in the projective hyperquadric

$$Q^{n-2} = \big\{[z_1 : \ldots : z_n] \in \mathbb{CP}^{n-1} : z_1^2 + z_2^2 + \cdots + z_n^2 = 0\big\}.$$

Theorem (Alarcón, F., López, 2019)

Every holomorphic map $\mathscr{G} : M \to Q^{n-2}$ is the Gauss map of a conformal minimal immersion $X : M \to \mathbb{R}^n$.

In particular, $Q^1 \cong \mathbb{CP}^1$, and every meromorphic function on M is the complex Gauss map of a conformal minimal surface $X : M \to \mathbb{R}^3$.

The Calabi-Yau problem for minimal surfaces

By results of **Chern–Osserman** (1967) and **Jorge–Meeks** (1983),
a complete minimal surface in $\mathbb{R}^n$ of finite total curvature is proper in $\mathbb{R}^n$.

1965 **Calabi's Conjecture** There is no complete nonflat minimal hypersurface in $\mathbb{R}^n$, $n \geq 3$, with a bounded coordinate function.

1980 **Jorge and Xavier** Calabi's conjecture is false in $\mathbb{R}^3$: there is a complete immersed minimal disc $\mathbb{D} \to \mathbb{R}^2 \times (-1, +1)$ with the third coordinate $\Re z$.

1996 **Nadirashvili** There is a bounded complete immersed minimal disc in $\mathbb{R}^3$.

2000 **S.-T. Yau** *Review of geometry and analysis*. Two main questions:

1. **What are the possible conformal types of complete bounded minimal surfaces?**
2. **What can be said about their boundary behaviour?**

2007 **Martín & Nadirashvili** There is a continuous map $X : \overline{\mathbb{D}} \to \mathbb{R}^3$ such that $X : \mathbb{D} \to \mathbb{R}^3$ is a complete conformal minimal immersion.

2008 **Colding & Minicozzi** Every complete **embedded** minimal surface in $\mathbb{R}^3$ of finite topological type is proper in $\mathbb{R}^3$ (so Calabi's conjecture holds).

2018 **Meeks, Pérez, Ros:** The same is true for embedded minimal surfaces in $\mathbb{R}^3$ of finite genus and countably many ends.

Theorem (Alarcón, Drinovec, F., López, 2015)

Let M be any bordered Riemann surface. Every conformal minimal immersion $X_0 : \overline{M} \to \mathbb{R}^n$ $(n \geq 3)$ can be uniformly approximated by continuous maps $X : \overline{M} \to \mathbb{R}^n$ (embeddings if $n \geq 5$) such that

$X : M \to \mathbb{R}^n$ *is a* **complete conformal minimal immersion, and the boundary $X(bM) \subset \mathbb{R}^n$ is a union of Jordan curves.**

Theorem (Alarcón and F., 2021)

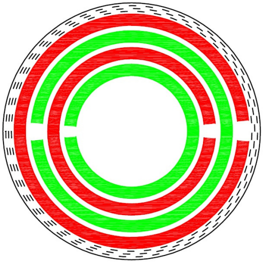
The same holds if M is a bordered Riemann surface of the form

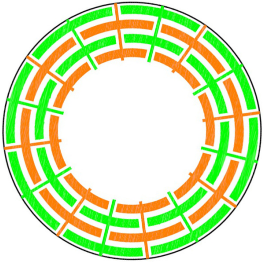
$$M = R \setminus \bigcup_i D_i$$

where R is a compact Riemann surface and D_i is a finite or countable family of pairwise disjoint compact geometric discs in R.

Labyrinths used by Jorge-Xavier and Nadirashvili

All constructions of complete bounded minimal surfaces up to 2015 were based on those by Jorge-Xavier **(1980)** and Nadirashvili **(1996)**, using Runge's theorem to modify X so that every path crossing a piece of a suitable labyrinth becomes long, while any divergent curve avoiding the labyrinth is also long.





In both constructions one must cut aways small pieces of the surface in order to keep the image bounded in $\mathbb{R}^n$. Hence, **this method does not provide control of the conformal type on topologically nontrivial surfaces.**

Novelties and techniques behind our theorems

The main novelties:

1. There is no change of the conformal structure on M (any conformal structure of the indicated type can arise).
2. The image surface is bounded by pairwise disjoint Jordan curves.
3. Countably many nonpoint boundary components are allowed.
4. The analogous results have been proved in several other geometries, in particular for holomorphic curves, holomorphic null curves, holomorphic Legendrian curves in complex contact manifolds, and superminimal surfaces in self-dual or anti-self-dual Einstein 4-manifolds (via the Pensor twistor space and Bryant correspondence).

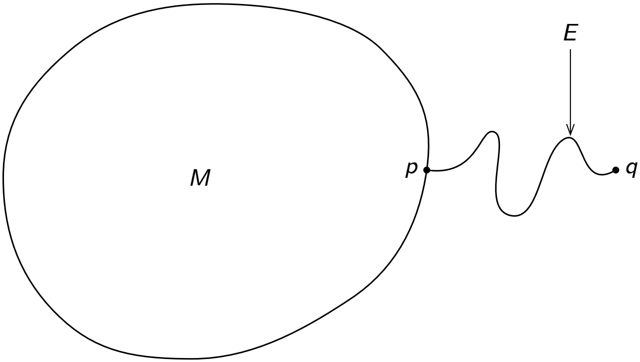
The method of proof: We construct a uniformly convergent sequence of spiralling modifications which inductively increase the intrinsic metric on M and make the limit minimal surface complete. The main tools:

(a) **Exposing boundary points of Riemann surfaces**

(b) **The Riemann-Hilbert boundary value problem for minimal surfaces**

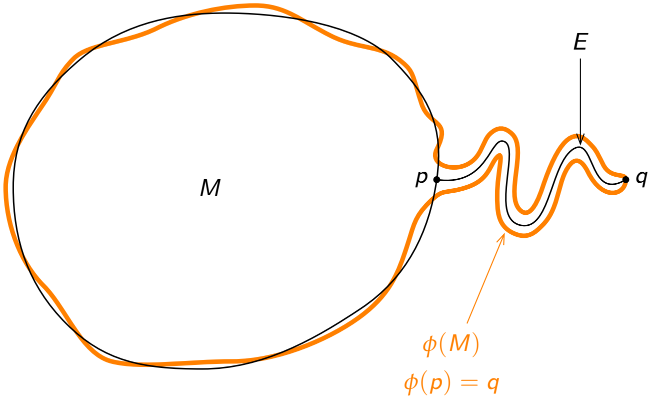
The idea of proof is shown in pictures.

Exposing a boundary point



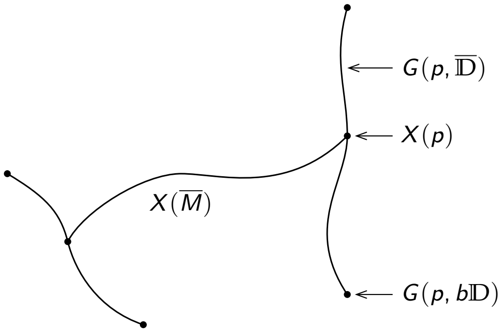
M is a bordered domain in a Riemann surface R. We attach to M a smooth arc E in the complement of $\overline{M}$ with one endpoint $p \in bM$. We extend $X : M \to \mathbb{R}^n$ smoothly to E such that the arc $X(E) \subset \mathbb{R}^n$ is long but stays close to $X(M)$. By Mergelyan approximation, X can be made a conformal minimal immersion in a neighbourhood of $M \cup E$. We do this at (finitely) many points of bM.

Exposing a boundary point



We then construct a conformal diffeomorphism $\phi : \overline{M} \to \phi(\overline{M}) \subset R$ which maps p to the other endpoint q of the arc E, it adds a thin tube around E, and it is close to the identity on $\overline{M}$ outside a small neighbourhood of p. The conformal minimal immersion $X \circ \phi : \overline{M} \to \mathbb{R}^n$ then has a much larger intrinsic distance to $p \in bM$.

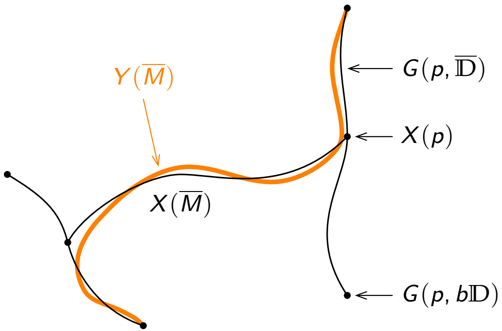
The Riemann–Hilbert deformation, I



On each of the short arcs $I \subset bM$ between two consecutive exposed points, we push the image $X(\overline{M})$ in the direction roughly orthogonal to the position vector in order to enlarge the intrinsic boundary distance by a given amount.

This is done by attaching to $X(\overline{M})$ a family of conformal minimal discs $G(p, \cdot\,) : \overline{\mathbb{D}} \to \mathbb{R}^n$, $p \in I$, and taking an approximate solution of the Riemann–Hilbert problem as shown on the next slide.

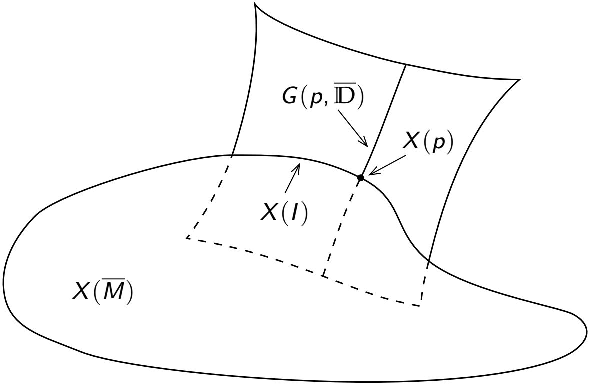
The Riemann–Hilbert deformation, I



The new disc Y is close to X outside a small neighbourhood of the arc $I \subset bM$, and for $p \in I$ the point $Y(p)$ is close to the circle $G(p, b\mathbb{D})$. The modification is tempered at the endpoints of I to keep what was done in the first step.

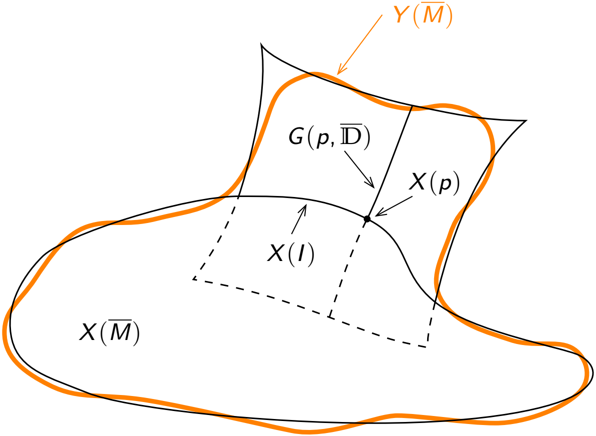
The model case is the disc $\overline{\mathbb{D}} \ni z \to (z, 0) \in \mathbb{C}^2$ deformed to the discs $z \mapsto (z, cz^N)$ for large $N \in \mathbb{N}$.

The Riemann–Hilbert deformation, II



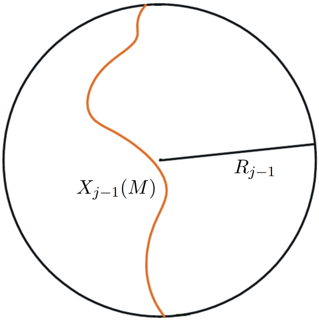
On this 3-dimensional picture of the deformation, we see a family of conformal minimal discs $G(p, \cdot\,) : \overline{\mathbb{D}} \to \mathbb{R}^n$ centred at points $G(p, 0) = X(p)$, $p \in I$.

The Riemann–Hilbert deformation, II



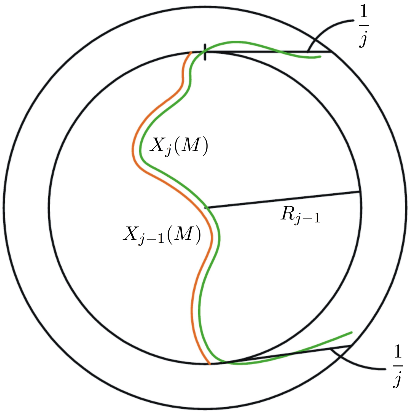
The modification Y has now been made. Its image follows closely that of X until we come near the arc I, where it turns in the direction of the discs $G(p, \cdot\,)$ and maps the point $p \in I$ close to the circle $G(p, b\mathbb{D})$.

An inductive application of this technique



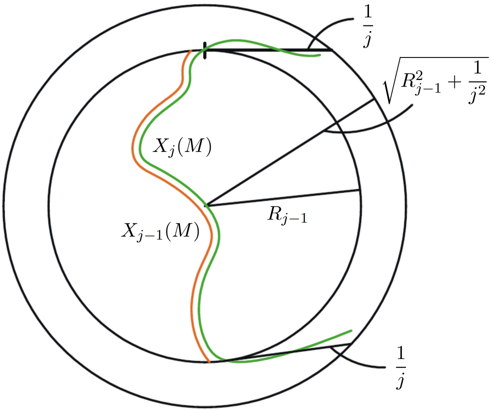
The illustration shows the inductive step in the construction of a bounded complete conformal minimal surface $X = \lim_{j\to\infty} X_j : M \to \mathbb{R}^n$. At the j-th step, the inner radius increases for roughly $\delta_j = 1/j$, while the outer radius increases for at most $\delta_j^2 = 1/j^2$. Since $\sum_{j=1}^{\infty} \delta_j = +\infty$, the limit surface is complete, and since $\sum_{j=1}^{\infty} \delta_j^2 < \infty$, it is bounded.

An inductive application of this technique



A Riemann–Hilbert deformation for approximately $\delta_j = 1/j$ in the direction tangential to the position vector.

An inductive application of this technique



The intrinsic radius of the surface increased for approximately $1/j$, while the new extrinsic radius is roughly $R_j = \sqrt{R_{j-1}^2 + \frac{1}{j^2}} \approx R_{j-1} + \frac{1}{2j^2 R_{j-1}} \leq R_{j-1} + \frac{c}{j^2}$.

To create surfaces with Jordan boundaries, we apply the same method locally around the initial curves in $X(bM)$.

A few open problems

(a) Let R be a compact Riemann surface and $M \subsetneq R$ be an open domain which admits a nonconstant bounded harmonic function that does not extend to a harmonic function on any bigger domain.

Is M the conformal structure of a complete bounded minimal surface?

(b) Is there an example of a complete bounded minimal surface in $\mathbb{R}^3$ whose underlying complex structure is $\mathbb{C} \setminus K$, where K is a Cantor set in $\mathbb{C}$?

(c) Does the Calabi–Yau property hold for minimal surfaces in every Riemannian manifold? (It holds for superminimal surfaces in self-dual Einstein 4-manifolds by using the Penrose twistor construction, the Bryant correspondence, and the corresponding theory for holomorphic Legendrian curves in complex contact manifolds.)

∼ Thank you for your attention ∼