On the conformal Calabi-Yau problem

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The Abel Symposium
NTNU, Trondheim, 5 July 2013

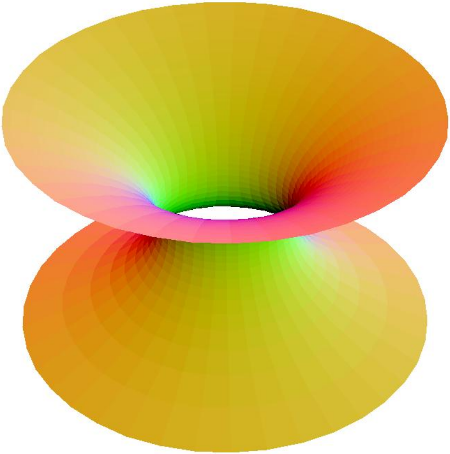
Plan of the talk

- Basics on minimal surfaces
- Connection with holomorphic null curves in $\mathbb{C}^3$
- Our contribution to the Calabi-Yau problem; brief history
- The main tools: Riemann-Hilbert problem for null curves, exposing points, gluing techniques
- Proper null curves in $\mathbb{C}^3$ with a bounded coordinate function
- Applications to null curves in $SL_2(\mathbb{C})$ and to Bryant surfaces in the hyperbolic 3-space

Based on joint work with Antonio Alarcón, University of Granada.

From Euler's surfaces of rotation...

1744 **Euler** The only area minimizing surfaces of rotation are planes and catenoids.



...via Lagrange's equation of minimal graphs...

1760 **Lagrange** Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain. Then a smooth graph $(x, y, f(x, y)) \subset \overline{\Omega} \times \mathbb{R}$ is a critical point of the *area functional* with prescribed boundary values iff

$$(1 + f_y^2) f_{xx} - 2 f_x f_y f_{xy} + (1 + f_x^2) f_{yy} = 0;$$

equivalently,

$$\mathrm{div}\left(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}\right) = 0.$$

This is known as the equation of minimal graphs.

1776 **Meusnier** A smooth surface $M \subset \mathbb{R}^3$ satisfies locally the above equation iff M is locally area minimizing (among surfaces with the same boundary), and

- this holds iff its *mean curvature* function $\Theta \colon M \to \mathbb{R}$ vanishes identically.

Definition (Minimal Surface)

A smoothly immersed surface $M \to \mathbb{R}^3$ is said to be **minimal** if its mean curvature is identically zero.

The helicoid (Archimedes' screw)

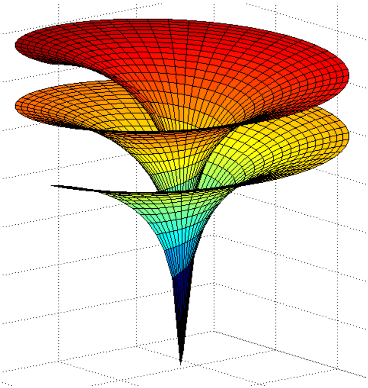
1776 **Meusnier** The helicoid is a minimal surface.

$$x = \rho\cos(\alpha\theta), \quad y = \rho\sin(\alpha\theta), \quad z = \theta$$

1842 **Catalan** The helicoid and the plane are the only ruled minimal surfaces in $\mathbb{R}^3$.

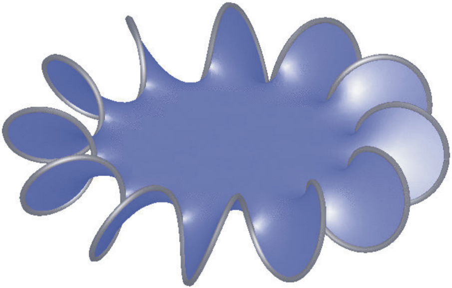
A relative of helicoid - Dini's surface

A surface with constant negative curvature, named after **Ulisse Dini** (1845 – 1918), an Italian mathematician and politician born in Pisa.



The Plateau Problem

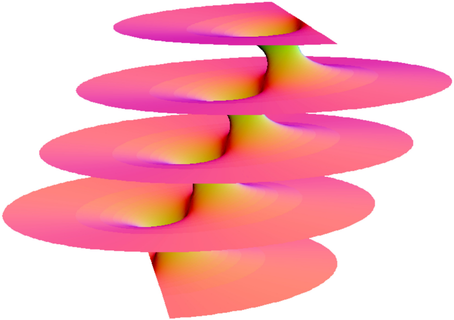
1873 **Plateau** Minimal surfaces can be obtained as soap films.



1932 **Douglas, Radó** Every continuous injective closed curve in $\mathbb{R}^3$ spans a minimal surface.

Examples by Riemann

1865 **Riemann** On the way to this solution, Riemann and others discovered new examples of minimal surfaces using the Weierstrass representation.



Conformal minimal surfaces in $\mathbb{R}^3$

Assume that M is a **Riemann surface**, i.e., a smooth orientable surface with a choice of a conformal=complex structure.

Definition

A smooth immersion $M \to \mathbb{R}^3$ is **conformal** if it preserves angles, and is **minimal** if its mean curvature is identically zero.

- Every Riemann surface is conformally equivalent to a closed embedded surface in $\mathbb{R}^3$ (Rüedy 1971).
- Denote by $\Theta : M \to \mathbb{R}$ its mean curvature and by $\nu : M \to \mathbb{S}^2 \subset \mathbb{R}^3$ its Gauss map. Then
$$\Delta G = 2\Theta\,\nu.$$
- Hence a conformal immersion $M \to \mathbb{R}^3$ is minimal iff it is harmonic.
- A conformal immersion $G : M \to \mathbb{R}^3$ is minimal iff its Gauss map $\nu : M \to \mathbb{S}^2 \equiv \overline{\mathbb{C}}$ is conformal.

Complete bounded minimal surfaces in $\mathbb{R}^3$

- An immersion $G : M \to \mathbb{R}^3$ is said to be **complete** if the pullback $G^* ds^2$ of the Euclidean metric ds^2 on $\mathbb{R}^3$ is a complete metric on M. Equivalently, the G-image of any curve in M which ends on bM is infinitely long.

- We give a contribution to the **conformal Calabi-Yau problem**:

Theorem

Every bordered Riemann surface admits a complete conformal minimal (=harmonic) immersion into $\mathbb{R}^3$ with bounded image.

- What is new in comparison to all existing results is that we do not change the complex structure on the Riemann surface.

[A. Alarcón, F. Forstnerič: http://arxiv.org/abs/1308.0903]

This theorem is a corollary to a comparable result concerning *holomorphic null curves* in $\mathbb{C}^3$.

Definition (Null curves)

Let M be a Riemann surface. A holomorphic immersion

$$F = (F_1, F_2, F_3) \colon M \to \mathbb{C}^3$$

is a **null curve** if the derivative $F' = (F_1', F_2', F_3')$ with respect to any local holomorphic coordinate $\zeta = x + iy$ on M satisfies

$$(F_1')^2 + (F_2')^2 + (F_3')^2 = 0.$$

Connection between null curves and minimal surfaces

- If $F = G + iH : M \to \mathbb{C}^3$ is a holomorphic null curve, then

$$G = \Re F : M \to \mathbb{R}^3, \quad H = \Im F : M \to \mathbb{R}^3$$

are conformal harmonic (hence minimal) immersions into $\mathbb{R}^3$.

- Conversely, a conformal minimal immersion $G : \mathbb{D} \to \mathbb{R}^3$ of the disc $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ is the real part of a holomorphic null curve $F : \mathbb{D} \to \mathbb{C}^3$. (This fails on non-simply connected Riemann surfaces due to the period problem for the harmonic conjugate.)
- If $F = G + iH \colon M \to \mathbb{C}^3$ is a null curve then

$$F^* ds^2_{\mathbb{C}^3} = 2G^* ds^2_{\mathbb{R}^3} = 2H^* ds^2_{\mathbb{R}^3}.$$

- It follows that the real and the imaginary part of a complete null curve in $\mathbb{C}^3$ are complete conformal minimal surfaces in $\mathbb{R}^3$.

The calculation

- Let $F = G + iH = (F^1, F^2, F^3) : M \to \mathbb{C}^3$ be a holomorphic null curve and $\zeta = x + iy$ a local holomorphic coordinate on M. Then

$$\begin{aligned} 0 &= \sum_{j=1}^{3} (F^j_\zeta)^2 = \sum_{j=1}^{3} (F^j_x)^2 = \sum_{j=1}^{3} \left(G^j_x + iH^j_x \right)^2 \\ &= \sum_{j=1}^{3} \left((G^j_x)^2 - (H^j_x)^2 \right) + 2i \sum_{j=1}^{3} G^j_x H^j_x. \end{aligned}$$

- Since $H_x = -G_y$ by the CR equations, this reads

$$0 = |G_x|^2 - |G_y|^2 - 2i\, G_x \cdot G_y \Longleftrightarrow |G_x| = |G_y|,\ G_x \cdot G_y = 0.$$

- It follows that G is conformal harmonic and

$$F^* ds^2_{\mathbb{C}^3} = |F_x|^2 (dx^2 + dy^2) = 2|G_x|^2 (dx^2 + dy^2) = 2G^* ds^2_{\mathbb{R}^3} = 2H^* ds^2_{\mathbb{R}^3}.$$

Example: catenoid and helicoid

Example: The **catenoid** and the **helicoid** are conjugate minimal surfaces – the real and the imaginary part of the same null curve

$$F(\zeta) = (\cos\zeta, \sin\zeta, -\imath\zeta), \qquad \zeta = x + \imath y \in \mathbb{C}.$$

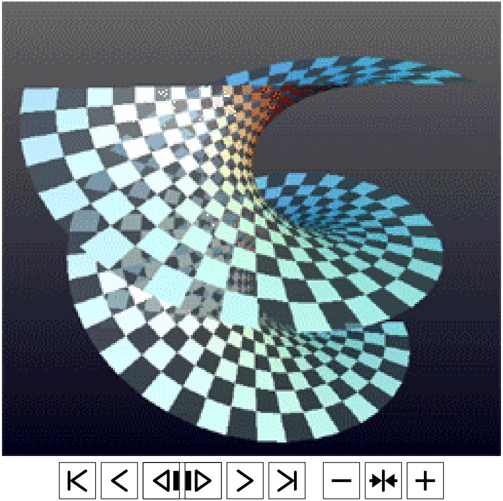
Consider the family of minimal surfaces ($t \in \mathbb{R}$):

$$\begin{aligned} G_t(\zeta) &= \Re\left(e^{\imath t} F(\zeta)\right) \\ &= \cos t \begin{pmatrix} \cos x \cdot \cosh y \\ \sin x \cdot \cosh y \\ y \end{pmatrix} + \sin t \begin{pmatrix} \sin x \cdot \sinh y \\ -\cos x \cdot \sinh y \\ x \end{pmatrix} \end{aligned}$$

At $t = 0$ we have a catenoid, and at $t = \pm\pi/2$ we have a (left or right handed) helicoid.

Helicatenoid (Source: Wikipedia)

The family of minimal surfaces $G_t(\zeta) = \Re\left(e^{\imath t} F(\zeta)\right)$, $t \in \mathbb{R}$:



The first main result

This shows that the existence of complete bounded conformal minimal immersions $M \to \mathbb{R}^3$ follows from part (B) of the following result.

Theorem

Let M be a bordered Riemann surface.

(A) *There exists a proper complete holomorphic immersion $M \to \mathbb{B}^2$ into the unit ball of $\mathbb{C}^2$.*

(B) *There exists a proper complete null holomorphic embedding $F\colon M \hookrightarrow \mathbb{B}^3$ into the unit ball of $\mathbb{C}^3$.*

(B) answers a question of Martín, Umehara and Yamada (2009).

Part (A) holds for immersions into any Stein manifold (X, ds^2) of dimension > 1 with a chosen Riemannian metric.

[A. Alarcón, F. Forstnerič: Every bordered Riemann surface is a complete proper curve in a ball. Math. Ann. 2013]

Higher dimensional examples

1985 **Løw** Every strongly pseudoconvex Stein domain M admits a proper holomorphic embedding $\phi\colon M \to \mathbb{D}^m$ into a polydisc.

Let $h\colon \mathbb{D} \to \mathbb{B}^2$ be a complete proper holomorphic immersion. Then

$$H\colon \mathbb{D}^m \to (\mathbb{B}^2)^m \subset \mathbb{C}^{2m}, \quad H(z_1, \ldots, z_m) = (h(z_1), \ldots, h(z_m))$$

is a complete proper holomorphic immersion. Similarly we get complete proper holomorphic embeddings $\mathbb{D}^m \to (\mathbb{B}^3)^m$.
Hence $F = H \circ \phi\colon M \to (\mathbb{B}^2)^m$ is a complete proper immersion.

Corollary

Every strongly pseudoconvex Stein domain admits a complete bounded holomorphic embedding into $\mathbb{C}^N$ for large N.

A brief history of the Calabi-Yau problem

1965 **E. Calabi** conjectured that there does not exist any complete minimal surface in $\mathbb{R}^3$ with a bounded coordinate function.

1977 **P. Yang** asked whether there exist any complete bounded complex submanifolds of $\mathbb{C}^n$ for $n > 1$. Note that complex submanifolds of complex Euclidean spaces are minimal.

1979 **P. Jones** constructed a complete bounded holomorphic immersion $\mathbb{D} \to \mathbb{C}^2$ of the disc, using BMO methods.

1980 **L.P. Jorge & F. Xavier** constructed complete minimal surfaces in $\mathbb{R}^3$ with a bounded coordinate function, thereby disproving Calabi's conjecture.

1996 **N. Nadirashvili** constructed a complete bounded conformal minimal immersion $\mathbb{D} \to \mathbb{R}^3$, hence a complete null curve in $\mathbb{C}^3$. His technique cannot be refined to control the imaginary part.

A brief history...continued

2000 **S.-T. Yau: Review of geometry and analysis** ("The Millenium Lecture"). Mathematics: frontiers and perspectives, AMS. The problem became known as the **Calabi-Yau problem**.

2008 **T.H. Colding and W.P. Minicozzi II:** An embedded complete minimal surface $M \hookrightarrow \mathbb{R}^3$ with finite genus and at most countably many ends is proper in $\mathbb{R}^3$, and M is algebraic.

2009 **F. Martín, M. Umehara and K. Yamada** constructed complete bounded holomorphic curves in $\mathbb{C}^2$ with arbitrary finite topology.

2012 **L. Ferrer, F. Martín and W.H. Meeks** found complete bounded minimal surfaces in $\mathbb{R}^3$ with arbitrary topology.

2013 **A. Alarcón and F.J. Lopez:** Examples of (i) complete bounded null curves in $\mathbb{C}^3$, (ii) complete bounded immersed holomorphic curves in $\mathbb{C}^2$ with arbitrary topology, and (iii) complete bounded *embedded* holomorphic curves in $\mathbb{C}^2$.

Geometry of the null quadric

- The directional variety of null curves:

$$A = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \colon z_1^2 + z_2^2 + z_3^2 = 0\}$$

- A is a complex cone with vertex at 0; $A^* = A \setminus \{0\}$ is smooth.
- $L = \{[z_1; z_2 : z_3] \in \mathbb{C}^3 \colon z_1^2 + z_2^2 + z_3^2 = 0\} \cong \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2$.
- $pr : A^* \to L$ is a holomorphic fiber bundle with fiber $\mathbb{C}^*$.
- It follows that A^* is an **Oka manifold**.
- The spinor representation:

$$\pi \colon \mathbb{C}^2 \to A, \quad \pi(u, v) = \left(u^2 - v^2, i(u^2 + v^2), 2uv\right).$$

The map $\pi \colon \mathbb{C}^2 \setminus \{0\} \to A^*$ is a nonramified two-sheeted covering.

Construction of holomorphic null curves

Let M be a bordered Riemann surface. Fix a nowhere vanishing holomorphic 1-form θ on M; such exists by the Oka-Grauert principle. There is a bijective correspondence (up to constants)

$$\{F\colon M \to \mathbb{C}^3 \text{ null curve}\} \longleftrightarrow \{f\colon M \to A^* \text{ holomorphic, } f\theta \text{ exact}\}$$

$$F(x) = F(p) + \int_p^x f\theta, \qquad dF = f\theta.$$

Theorem (The Oka principle for null curves)

Every continuous map $f_0\colon M \to A^*$ of an open Riemann surface M to A^* is homotopic to a holomorphic map $f\colon M \to A^*$ such that $f\theta$ has vanishing periods. Furthermore, a generic null curve is an embedding. The same holds whenever $A^* \subset \mathbb{C}^n$, $n \geq 3$, is an Oka manifold.

[A. Alarcón, F. Forstnerič: Null curves and directed immersions of open Riemann surfaces. Inventiones Math., in press]

Idea of the construction of complete bounded holomorphic immersions - Pythagora's theorem

- Let $F_0 \colon \overline{M} \to \mathbb{C}^n$ be a holomorphic immersion satisfying $|F_0| \geq r_0 > 0$ on bM. We try to increase the boundary distance on M with respect to the induced metric by a fixed number $\delta > 0$.
- To this end, we approximate F_0 uniformly on a compact set in M by an immersion $F_1 : \overline{M} \to \mathbb{C}^n$ which at a point $x \in bM$ adds a displacement for approximately δ in a direction $V \in \mathbb{C}^n$, $|V| = 1$, approximately orthogonal to the point $F_0(x) \in \mathbb{C}^n$. The boundary distance increases by $\approx \delta$, while the outer radius increases to

$$|F_1(x)| \approx \sqrt{|F_0(x)|^2 + \delta^2} \approx |F_0(x)| + \frac{\delta^2}{2|F_0(x)|} \leq |F_0(x)| + \frac{\delta^2}{2r_0}.$$

- By choosing a sequence $\delta_j > 0$ such that $\sum_j \delta_j = +\infty$ while $\sum_j \delta_j^2 < \infty$, we obtain by induction a limit immersion $F \colon M \to \mathbb{C}^n$ with bounded outer radius and with complete metric $F^* ds^2$.

The main tools

- This idea can be realized on short arcs $I \subset bM$, on which F_0 does not vary too much, by solving a **Riemann-Hilbert problem**.
- Globally this method alone could lead to 'sliding curtains', creating shortcuts in the new induced metric on M.
- To **localize the problem** and **eliminate any shortcuts**, we subdivide $bM = \cup_j I_j$ into a finite union of short arcs such that two adjacent arcs I_{j-1}, I_j meet at a common endpoint x_j. At the point $p_j = F(x_j) \in \mathbb{C}^n$ we attach to $F_0(\overline{M})$ a smooth real curve λ_j of length δ whose other endpoint q_j increases the outer radius by δ^2.
- By the method of **exposing boundary points** we modify the immersion so that $F_0(x_j) = q_j$. Hence any curve in M terminating on bM near x_j is elongated by approximately $\delta > 0$.
- In the next step we use a Riemann-Hilbert problem to increase the boundary distance on the arcs I_j by approximately δ. These local modifications are glued together by the method of sprays.

Riemann-Hilbert problem for null curves

Theorem (Riemann-Hilbert problem for null discs)

Let $F_0 : \overline{\mathbb{D}} \to \mathbb{C}^3$ *be a null holomorphic immersion, let* $V \in A^*$*, let* $\mu : b\mathbb{D} \to [0, +\infty)$ *be a continuous function, and consider the map*

$$Y : b\mathbb{D} \times \overline{\mathbb{D}} \to \mathbb{C}^3, \quad Y(\zeta, z) = F_0(\zeta) + \mu(\zeta) z V.$$

Given numbers $\varepsilon > 0$ *and* $0 < r < 1$*, there exist a number* $r' \in [r, 1)$ *and a null holomorphic immersion* $F : \overline{\mathbb{D}} \to \mathbb{C}^3$ *satisfying the following:*

- $\mathrm{dist}(F(\zeta), Y(\zeta, b\mathbb{D})) < \varepsilon$ *for* $\zeta \in b\mathbb{D}$.
- $\mathrm{dist}(F(\rho\zeta), Y(\zeta, \overline{\mathbb{D}})) < \varepsilon$ *for* $\zeta \in b\mathbb{D}$ *and* $\rho \in [r', 1)$.
- F *is* ε*-close to* F_0 *in the* $\mathscr{C}^1$ *topology on* $\{\zeta \in \mathbb{C} : |\zeta| \le r'\}$.

Furthermore, if J *is a compact arc in* $b\mathbb{D}$ *such that* μ *vanishes on* $b\mathbb{D} \setminus J$*, and* U *is an open neighborhood of* J *in* $\overline{\mathbb{D}}$*, then*

- *one can choose* F *to be* ε*-close to* F_0 *in the* $\mathscr{C}^1$ *topology on* $\overline{\mathbb{D}} \setminus U$.

Proof

Consider the unbranched two-sheeted holomorphic covering

$$\pi\colon \mathbb{C}^2 \setminus \{(0,0)\} \to A^*, \quad \pi(u,v) = \big(u^2 - v^2, i(u^2+v^2), 2uv\big).$$

Since $\overline{\mathbb{D}}$ is simply connected, the map $F_0'\colon \overline{\mathbb{D}} \to A^*$ lifts to a map $(u,v)\colon \overline{\mathbb{D}} \to \mathbb{C}^2 \setminus \{(0,0)\}$. Hence we have

$$\begin{aligned}
F_0' &= \pi(u,v) = \Big(u^2 - v^2, i(u^2+v^2), 2uv\Big) \in A^* \\
V &= \pi(a,b) = \Big(a^2 - b^2, i(a^2+b^2), 2ab\Big) \in A^* \\
\eta &= \sqrt{\mu}\colon b\mathbb{D} \to [0,\infty) \\
\eta(\zeta) &\approx \tilde{\eta}(\zeta) = \sum_{j=1}^{N} A_j \zeta^{j-m} \ \text{(rational approximation)} \\
\mu(\zeta) &\approx \tilde{\eta}^2(\zeta) = \sum_{j=1}^{2N} B_j \zeta^{j-2m}.
\end{aligned}$$

Proof–continued

For any integer $n \in \mathbb{N}$ we consider the following functions and maps

$$\begin{aligned}
u_n(\xi) &= u(\xi) + \sqrt{2n+1}\,\tilde{\eta}(\xi)\xi^n a, \\
v_n(\xi) &= v(\xi) + \sqrt{2n+1}\,\tilde{\eta}(\xi)\xi^n b, \\
\Phi_n(\xi) &= \pi(u_n(\xi), v_n(\xi)) = \big(u_n^2 - v_n^2, i(u_n^2 - v_n^2), 2u_n v_n\big) : \overline{\mathbb{D}} \to A^*, \\
F_n(\zeta) &= F_0(0) + \int_0^\zeta \Phi_n(\xi)\, d\xi, \qquad \zeta \in \overline{\mathbb{D}}.
\end{aligned}$$

Then $F_n \colon \overline{\mathbb{D}} \to \mathbb{C}^3$ is a null disc of the form

$$F_n(\zeta) = F_0(\zeta) + \mathbf{B}_n(\zeta)\, V + \mathbf{A}_n(\zeta).$$

Proof–continued

The $\mathbb{C}$-valued term $\mathbf{B}_n$ equals

$$\begin{aligned}\mathbf{B}_n(\zeta) &= (2n+1)\sum_{j=1}^{2N}\int_0^{\zeta} B_j \xi^{2n+j-2m}\, d\xi \\ &= \sum_{j=1}^{2N}\frac{2n+1}{2n+1+j-2m}B_j\,\zeta^{2n+1+j-2m}.\end{aligned}$$

Since the coefficients $(2n+1)/(2n+1+j-2m)$ in the sum for $\mathbf{B}_n$ converge to 1 as $n\to+\infty$, we have

$$\sup_{|\zeta|\le 1}\left|\mathbf{B}_n(\zeta)-\zeta^{2n+1}\tilde{\eta}^2(\zeta)\right|\to 0 \quad \text{as } n\to\infty.$$

Proof–continued

The remainder $\mathbb{C}^3$-valued term $\mathbf{A}_n(\zeta)$ equals

$$\begin{aligned}
\mathbf{A}_n(\zeta) &= 2\sqrt{2n+1}\int_0^{\zeta}\sum_{j=1}^{N} A_j\,\xi^{n+j-m}\big(u(\xi)(a,\imath a,b)+v(\xi)(-b,\imath b,a)\big)\,d\xi \\
|\mathbf{A}_n(\zeta)| &\leq 2\sqrt{2n+1}\,C_0\sum_{j=1}^{N}|A_j|\int_0^{|\zeta|}|\xi|^{n+j-m}\,d|\xi| \\
&\leq 2C_0\sum_{j=1}^{N}\frac{\sqrt{2n+1}}{n+1+j-m}|A_j|.
\end{aligned}$$

It follows that $|\mathbf{A}_n|\to 0$ uniformly on $\overline{\mathbb{D}}$ as $n\to+\infty$. Hence

$$F_n(\zeta)\approx F_0(\zeta)+\zeta^{2n+1}\tilde{\mu}(\zeta)V, \qquad \zeta\in\overline{\mathbb{D}}.$$

The theorem follows from this estimate.

Null curves with a bounded coordinate

The Riemann-Hilbert problem for null curves also gives the following.

Theorem

Every bordered Riemann surface M carries a proper holomorphic null embedding $F = (F_1, F_2, F_3)\colon M \to \mathbb{C}^3$ such that the function F_3 is bounded on M. (Thus $(F_1, F_2)\colon M \to \mathbb{C}^2$ is a proper map.)

- This contrasts the theorem of **Hoffman and Meeks** (1990) that the only properly immersed minimal surfaces in $\mathbb{R}^3$ contained in a half-space are planes.
- This result has a nontrivial line of corollaries. A null curve in $SL_2(\mathbb{C})$ is a holomorphic immersion $F\colon M \to SL_2(\mathbb{C})$ of an open Riemann surface M which is directed by the variety

$$\mathscr{B} = \left\{ z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} : \det z = z_{11}z_{22} - z_{12}z_{21} = 0 \right\} \subset \mathbb{C}^4.$$

Null curves in $SL_2(\mathbb{C})$

- The biholomorphic map $\mathscr{T}: \mathbb{C}^3 \setminus \{z_3 = 0\} \to SL_2(\mathbb{C}) \setminus \{z_{11} = 0\}$,

$$\mathscr{T}(z_1, z_2, z_3) = \frac{1}{z_3} \begin{pmatrix} 1 & z_1 + iz_2 \\ z_1 - iz_2 & z_1^2 + z_2^2 + z_3^2 \end{pmatrix},$$

  carries null curves into null curves.

- Furthermore, if $F = (F_1, F_2, F_3) \colon M \to \mathbb{C}^3$ is a proper null curve such that $1/2 < |F_3| < 1$ on M, then $G = \mathscr{T} \circ F \colon M \to SL_2(\mathbb{C})$ is a proper null curve in $SL_2(\mathbb{C})$. This proves the following.

Corollary

Every bordered Riemann surface carries a proper holomorphic null embedding into $SL_2(\mathbb{C})$.

Bryant surfaces in hyperbolic 3-space

- The projection of a null curve in $SL_2(\mathbb{C})$ to the hyperbolic 3-space $\mathscr{H}^3 = SL_2(\mathbb{C})/SU(2)$ is a **Bryant surface**, i.e., a conformally immersed surface with constant mean curvature one in $\mathscr{H}^3$.

Corollary

Every bordered Riemann surface is conformally equivalent to a properly immersed Bryant surface in the hyperbolic 3-space $\mathscr{H}^3$.

2002 **Collin-Hauswirth-Rosenberg** Properly *embedded* Bryant surfaces in $\mathscr{H}^3$ of finite topology have finite total curvature and regular ends. Hence our examples cannot embedded.

- To the best of our knowledge, these are the first examples of proper null curves in $SL_2(\mathbb{C})$, and Bryant surfaces in $\mathscr{H}^3$, with finite topology and *hyperbolic conformal structure*.

A few open problems

- Does there exist a complete bounded holomorphic **embedding** $\mathbb{D} \hookrightarrow \mathbb{C}^2$ of the disc? Of an arbitrary bordered Riemann surface?
- Does there exist a **proper** minimal conformal immersion $M \hookrightarrow \mathbb{B}^3$ of an arbitrary bordered Riemann surface M?
- Is it possible to immerse or embed the ball $\mathbb{B}^2 \subset \mathbb{C}^2$ as a complete bounded complex submanifold of $\mathbb{C}^3$, $\mathbb{C}^4$,...
- **Conjecture** (well known, likely very difficult): An orientable surface of finite topology with genus g and m ends properly embeds in $\mathbb{R}^3$ as a minimal surface if and only if $m \leq g+2$.
- **Calabi's conjecture** is still open in dimensions $n > 3$: Do there exist complete bounded minimal hypersurfaces of $\mathbb{R}^n$ when $n > 3$?