Nonlinear holomorphic approximation theory

Franc Forstnerič

Univerza v Ljubljani

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Abstract

I will discuss some developments in holomorphic approximation theory of **Runge, Mergelyan, Carleman and Arakelyan type**, with emphasis on manifold-valued maps.

A more comprehensive discussion of these topics is available in the survey

J.-E. Fornæss, F. Forstnerič, E.F. Wold:
Holomorphic approximation: the legacy of Weierstrass, Runge, Oka-Weil, and Mergelyan. Preprint, 2/018.

https://arxiv.org/abs/1802.03924

From Runge to Oka-Weil

- **Runge 1885** If K is a compact set with connected complement in $\mathbb{C}$ then every $f \in \mathcal{O}(K)$ is a uniform limit of holomorphic polynomials. More generally, for every compact $K \subset \mathbb{C}$ we can approximate functions in $\mathcal{O}(K)$ by rational functions with poles in $\mathbb{C} \setminus K$.

- **Behnke and Stein 1949, Koditz and Timmann 1975**
Let K be a compact set in a Riemann surface X. Every $f \in \mathcal{O}(K)$ can be approximated uniformly on K by meromorphic functions on X without poles in K, and by functions in $\mathcal{O}(X)$ if K has **no holes**.

- **K. Oka 1936, A. Weil 1935** If $X \subset \mathbb{C}^n$ is a domain of holomorphy (or a **Stein manifold**, **K. Stein 1951**) and $K \subset X$ is a compact $\mathcal{O}(X)$-convex set, then every $f \in \mathcal{O}(K)$ is a uniform limit of functions in $\mathcal{O}(X)$.

The Oka-Grauert Principle (nonlinear Runge theorem)

Theorem (Oka 1939, Grauert 1958)

Let X be a Stein space, $K \subset X$ be a compact $\mathcal{O}(X)$-convex subset, and $X_0 \subset X$ be a closed complex subvariety. Given a complex homogeneous manifold Y and a continuous map $f_0 \colon X \to Y$ which is holomorphic on $K \cup X_0$, there is a homotopy $f_t \colon X \to Y$ $(t \in [0,1])$ such that

- $f_t \in \mathcal{O}(K)$ *and* $f_t|_K$ *approximates* $f_0|_K$ *for all* $t \in [0,1]$,
- $f_t|_{X_0} = f_0|_{X_0}$ *for all* $t \in [0,1]$, *and*
- $f_1 : X \to Y$ *is holomorphic.*

For $Y = \mathbb{C}$, this is the **Oka-Weil approximation theorem** combined with the **Oka-Cartan extension theorem**.

The analogous result holds for families of maps depending on a parameter in a compact Hausdorff space. In particular, the natural inclusion

$$\mathcal{O}(X,Y) \longrightarrow \mathscr{C}(X,Y)$$

is a **weak homotopy equivalence**.

Elliptic manifolds and Oka manifolds

Gromov 1989 The same holds if the target manifold Y is **elliptic**, i.e., it admits a **dominating holomorphic spray** — a holomorphic map $s : E \to Y$ from the total space of a holomorphic vector bundle $E \to Y$ such that

$$s(0_y) = y, \quad ds_{0_y}(E_y) = T_y Y \quad \forall y \in Y.$$

This holds for example if TY is generated by $\mathbb{C}$-complete holomorphic vector fields (flexible manifolds; **Arzhantsev et al. 2013**).

F., 2005-2010 Let $h : Z \to X$ be any holomorphic fibre bundle with fibre Y over a Stein base space X. Then, the Oka-Grauert principle holds in all forms for sections $X \to Z$ iff Y satisfies the following condition:

Every holomorphic map $K \to Y$ from a compact convex set $K \subset \mathbb{C}^N$ (for any $N \in \mathbb{N}$) is a uniform limit of entire maps $\mathbb{C}^N \to Y$.

A complex manifold Y satisfying these equivalent conditions is called an

Oka manifold (F. 2009)

What do we know about Oka manifolds?

- A Riemann surface is Oka iff it is not Kobayashi hyperbolic.
- If $\pi\colon Z \to X$ is a holomorphic fibre bundle with an Oka fibre Y, then X is Oka iff Z is Oka. This holds in particular in a covering space.
- **Kobayashi & Ochiai 1974** A compact complex manifold of general Kodaira type is not dominable, and hence not Oka.
- **Lárusson & F. 2014**: About compact complex surfaces:
  1. $\kappa = -\infty$: Rational surfaces are Oka. A ruled surface is Oka if and only if its base is Oka. Minimal Hopf surfaces and Enoki surfaces are Oka. Inoue surfaces, Inoue-Hirzebruch surfaces, and intermediate surfaces, minimal or blown up, are not Oka. (This covers surfaces of class VII if the global spherical shell conjecture is true.)
  2. $\kappa = 0$: Bielliptic surfaces, Kodaira surfaces, and tori are Oka. It is unknown whether any K3 surfaces or Enriques surfaces are Oka.
  3. $\kappa = 1$: **Buzzard & Lu 2000** determined which properly elliptic surfaces are dominable. Nothing further is known about the Oka property. No example of an Oka surface with $\kappa = 1$ is known.
- Many classes of affine algebraic manifolds are known to be Oka (**Kaliman and Kutzschebauch, Andrist, Leuenberger,...**)

Generalizations and applications

The most general version of this theorem concerns sections of holomorphic submersions $h : Z \to X$ over stratified Stein spaces X with the property that every point in a stratum $X_0 \subset X$ has a neighborhood $U \subset X_0$ such that $Z|_U \to U$ admits a dominating fibre-spray.

Some applications:

- **Oka, Grauert:** Classification of principal bundles over Stein spaces.
- **Forster and Ramspott:** the number of equations needed to define a subvariety.
- **Eliashberg and Gromov; Schürmann:** Precise embedding dimension for Stein manifolds and Stein spaces.
- **Ivarsson and Kutzschebauch:** Holomorphic Vaserstein problem.
- **Heizner and Kutzschebauch; Kutzschebauch, Lárusson, Schwarz:** Equivariant h-principle for Stein spaces or bundles with holomorphic group actions.
- **Leiterer:** Similarity of holomorphic matrix-valued functions.
- **Alarcón, López, Drinovec-Drnovšek, F.:** Minimal surfaces in Euclidean spaces.

Mergelyan approximation

Mergelyan 1951 If K is a compact set in $\mathbb{C}$ without holes, then every function in $\mathscr{A}(K) = \mathscr{C}(K) \cap \mathcal{O}(\mathring{K})$ is a uniform limit of entire functions.

In view of Runge's theorem, Mergelyan's theorem is equivalent to

The Mergelyan property (MP): $\mathscr{A}(K) = \overline{\mathcal{O}}(K)$.

Vitushkin 1966 Characterization of MP in terms of continuous capacity.

Bishop 1958 (localization theorem) Let K be a compact set in a Riemann surface X. If every point $p \in K$ has a compact neighborhood $D_p \subset X$ such that $K \cap D_p$ has MP, then K has MP. In particular, a compact set without holes in an open Riemann surface has DP.

Boivin and Jiang 2004 (the converse to Bishop's theorem)
If a compact set K in a Riemann surface X has MP, then for every closed coordinate disc $D_p \subset X$ the set $K \cap D_p$ has MP.

Verdera 1986 Let K be compact set in $\mathbb{C}$. If $f \in \mathscr{C}^r_0(\mathbb{C})$ $(r \in \mathbb{N})$ is such that $\partial f/\partial \bar{z}$ vanishes on K to order $r-1$, then f can be approximated in $\mathscr{C}^r(\mathbb{C})$ by functions holomorphic in neighborhoods of K.

$\bar{\partial}$-proof of Bishop's localization theorem

Sakai 1972 Let $f \in \mathscr{A}(K)$, f continuous in a neighborhood of K. Cover K by finitely many compact sets D_j as in Bishop's theorem such that $\mathring{D}_j$ is an open cover of K. Let χ_j be a subordinate smooth partition of unity. By the assumption, for any $\epsilon > 0$ we have functions $f_j \in \mathscr{C}(D_j) \cap \mathcal{O}(K \cap D_j)$ such that $\|f_j - f\|_{\mathscr{C}(K \cap D_j)} < \epsilon$. Set

$$g = \sum_{j=1}^{m} \chi_j f_j.$$

Then on some open neighborhood U of K we have that

$$\|g - f\|_{\mathscr{C}(U)} = O(\epsilon), \quad \overline{\partial} g = \sum_{j=1}^{m} f_j \, \overline{\partial} \chi_j = \sum_{j=1}^{m} (f_j - f) \, \overline{\partial} \chi_j = O(\epsilon).$$

Let $\chi \in \mathscr{C}_0^\infty(U)$ be a cut-off function with $0 \le \phi \le 1$ and $\chi \equiv 1$ near K. Then $\|\chi \, \overline{\partial} g\|_{\mathscr{C}(\overline{U})} = O(\epsilon)$ and so $T(\chi \cdot \overline{\partial} g) = O(\epsilon)$, where T is a Cauchy-Green operator (**Behnke & Stein 1949**). Hence, the function $\tilde{f} = g - T(\chi \cdot \overline{\partial} g) \in \mathcal{O}(K)$ approximates f to a precision $O(\epsilon)$ on K.

Mergelyan's theorem in higher dimensions

Sakai's proof can be used for a Stein compact K in any complex manifold provided we have solution operators for the $\bar\partial$-equation satisfying the same bounds on a suitable basis of Stein neighborhoods of K. Here is a brief summary when K is the closure of a pseudoconvex domain D in $\mathbb{C}^n$:

Henkin 1969, Lieb 1969, Kerzman 1971: MP holds for strongly pseudoconvex domains D with smooth boundary: every function in $\mathscr{A}^r(D) = \mathscr{C}^r(\overline{D}) \cap \mathcal{O}(D)$ is a $\mathscr{C}^r(\overline{D})$-limit of functions in $\mathcal{O}(\overline{D})$.

Fornæss 1976: $\mathscr{C}^2$ boundary suffices for $\mathscr{C}^0$ approximation.

Beatrous & Range 1980 MP holds for $f \in \mathscr{A}(D)$ if D is weakly pseudoconvex and f can be approximated on a neighborhood of the set of weakly pseudoconvex boundary points (the degeneration set).

Diederich & Fornæss 1976: **MP fails on a worm domain**.

Laurent-Thiebaut & F. 2007: MP holds on a smooth pseudoconvex domain whose degeneration set is a Levi flat hypersurface with Levi foliation defined by a closed nowhere vanishing 1-form.

Approximation on totally real submanifolds

A submanifold M of a complex manifold X is **totally real** if $T_pM \cap \mathfrak{i} T_pM = \{0\}$ for all $p \in M$. Models are $\mathbb{R}^m \subset \mathbb{R}^n \subset \mathbb{C}^n$.

Weierstrass 1885 Every continuous function on a compact interval in $\mathbb{R}$ is a uniform limit of entire functions on $\mathbb{C}$.

Carleman 1927, Alexander 1979, Gauthier and Zeron 2002 Approximation in the fine topology on curves in $\mathbb{C}^n$.

Range & Siu, 1974 $\mathscr{C}^k$ approximation by holomorphic functions and $\bar\partial$-closed forms on $\mathscr{C}^k$ totally real submanifolds of a complex manifold.

Baouendi & Treves 1981 Local approximation of CR functions on CR submanifolds by entire functions (Gaussian kernel method).

F., Løw, Øvrelid 2001 Approximation of $\bar\partial$-flat functions in tubes around totally real manifolds (Henkin type integral kernel method).

Manne 1993, Manne, Øvrelid, Wold 2011 Carleman approximation on totally real submanifolds (Gaussian kernel method).

Mergelyan approximation on handlebodies

A compact set S in a complex manifold X is **admissible** if $S = K \cup M$, where S and K are Stein compacts and $M = S \setminus K$ is a totally real submanifold of X: $T_pM \cap \mathrm{i}T_pM = \{0\}$ for all $p \in M$.

Theorem (Hörmander & Wermer 1968; F. 2005; Manne, Øvrelid, Wold 2011; Fornæss, F., Wold 2018)

Let X be a complex manifold.

1. *Let $S = K \cup M$ be an admissible set in X with M of class $\mathscr{C}^k$. Then for any $f \in \mathscr{C}^k(S) \cap \mathcal{O}(K)$ there exists a sequence $f_j \in \mathcal{O}(S)$ such that $\lim_{j\to\infty} \|f_j - f\|_{\mathscr{C}^k(S)} = 0$.*
2. *Assume in addition that $K = \overline{D}$ is the closure of a strongly pseudoconvex domain $D \subset X$. Given $f \in \mathscr{C}(S) \cap \mathscr{A}(D)$ there is a sequence $f_j \in \mathcal{O}(S)$ such that $\lim_{j\to\infty} \|f_j - f\|_{\mathscr{C}(S)} = 0$.*
   If $f \in \mathscr{C}^k(S)$, the convergence takes place in $\mathscr{C}^k(S)$.

The analogous results hold for manifold-valued maps.

The use of the Gaussian kernel

We define the bilinear form $\langle \cdot\,,\cdot\rangle$ on $\mathbb{C}^n$ by

$$\langle z, w\rangle = \sum_{i=1}^{n} z_i w_i, \qquad z^2 = \langle z, z\rangle = \sum_{i=1}^{n} z_i^2. \tag{1}$$

Consider first the standard real subspace $\mathbb{R}^n$ of $\mathbb{C}^n$. Recall that

$$\int_{\mathbb{R}^n} e^{-x^2} dx = \int_{\mathbb{R}^n} e^{-\sum_{i=1}^n x_i^2} dx_1 \cdots dx_n = \left(\int_{\mathbb{R}} e^{-t^2} dt\right)^n = \pi^{n/2}.$$

Hence $\int_{\mathbb{R}^n} e^{-x^2/\epsilon^2} dx = \epsilon^n \pi^{n/2}$, so the family $\pi^{-n/2}\epsilon^{-n} e^{-x^2/\epsilon^2}$ is an approximate identity on $\mathbb{R}^n$. Given $f \in \mathscr{C}_0^k(\mathbb{R}^n)$, the entire functions

$$f_\epsilon(z) = \frac{1}{\pi^{n/2}\epsilon^n} \int_{\mathbb{R}^n} f(x) e^{-(x-z)^2/\epsilon^2} dx, \qquad z \in \mathbb{C}^n,\ \epsilon > 0,$$

satisfy $f_\epsilon \to f$ in the $\mathscr{C}^k(\mathbb{R}^n)$ norm as $\epsilon \to 0$.

It is remarkable that the same procedure gives local approximation in the $\mathscr{C}^k$ norm on any totally real submanifold of class $\mathscr{C}^k$.

Approximation on graphs with small Lip norm

Let $\mathbb{B}^n_{\mathbb{R}} \subset \mathbb{R}^n$ denote the unit ball and $\mathbb{B}^n_{\mathbb{R}}(\epsilon) = \epsilon\mathbb{B}^n_{\mathbb{R}}$.

Lemma

Let $\psi : \mathbb{B}^n_{\mathbb{R}} \to \mathbb{R}^n$ be a $\mathscr{C}^k$ map $(k \in \mathbb{N})$ with $\psi(0) = 0$, $(d\psi)_0 = 0$, and set $\phi(x) = x + i\psi(x) \in \mathbb{C}^n$. Then there exists $0 < \delta < 1$ such that the following holds. Let $N \subset \mathbb{B}^n_{\mathbb{R}}$ be a closed set,

$$M = \phi(\mathbb{B}^n_{\mathbb{R}}(\delta) \cap N) \subset \mathbb{C}^n, \quad bM = \phi(b\mathbb{B}^n_{\mathbb{R}}(\delta) \cap N).$$

Given $f \in \mathscr{C}_0(M)$, there exist entire functions $f_\epsilon \in \mathcal{O}(\mathbb{C}^n)$ $(\epsilon > 0)$ satisfying the following conditions as $\epsilon \to 0$:

(a) *$f_\epsilon \to f$ uniformly on M, and*

(b) *$f_\epsilon \to 0$ on a neighborhood of bM.*

Moreover, if N is a $\mathscr{C}^k$-smooth submanifold of $\mathbb{B}^n_{\mathbb{R}}$ and $f \in \mathscr{C}^k_0(M)$, then the approximation in (a) may be achieved in the $\mathscr{C}^k$-norm on M.

Since functions on N extend to $\mathbb{B}^n_{\mathbb{R}}$ in the appropriate classes, it suffices to prove the lemma in the case $N = \mathbb{B}^n_{\mathbb{R}}$.

Proof of the lemma

We will need the following fact.

Hörmander 1976 If A is a symmetric $n \times n$ complex matrix with positive definite real part, then

$$(*) \qquad \int_{\mathbb{R}^n} e^{-\langle Au,u\rangle} du = \pi^{n/2} (\det A)^{-1/2}.$$

We shall use this with the matrix

$$A(x) = \phi'(x)^T \phi'(x), \quad \Re A(x) = I - \psi'(x)^T \psi'(x).$$

Since $\psi'(0) = 0$, there is a number $0 < \delta_0 < 1$ such that $\Re A(x) > 0$ is positive definite for all $x \in \mathbb{B}^n_{\mathbb{R}}(\delta_0)$, and ψ is Lip-α with $\alpha < 1$ on $\mathbb{B}^n_{\mathbb{R}}(\delta_0)$. By using a smooth cut-off function, we extend ψ to $\mathbb{R}^n$ such that $\mathrm{supp}\,(\psi) \subset \mathbb{B}^n_{\mathbb{R}}$, without changing its values on $\mathbb{B}^n_{\mathbb{R}}(\delta_0)$.

We will show that the lemma holds for any number δ with $0 < \delta < \delta_0$.

Proof of condition (b)

Set

$$f_\epsilon(z) = \frac{1}{\pi^{n/2}\epsilon^n} \int_M f(w) e^{-(w-z)^2/\epsilon^2} dw, \qquad z \in \mathbb{C}^n.$$

Writing $z = x + iy \in \mathbb{C}^n$ and $w = u + iv = \mathbb{C}^n$, we have that

$$\left| e^{-(w-z)^2} \right| = e^{-\Re(w-z)^2} = e^{(y-v)^2-(x-u)^2}.$$

For a fixed $w = u + iv \in \mathbb{C}^n$ let

$$\Gamma_w = \{z = x + iy \in \mathbb{C}^n : (y - v)^2 < (x - u)^2\}.$$

On Γ_w, the function $e^{-(w-z)^2/\epsilon^2}$ converges to zero as $\epsilon \to 0$. Since ψ is Lip-α with $\alpha < 1$ on $\mathbb{B}^n_{\mathbb{R}}(\delta)$, we have

$$M \setminus \{w\} \subset \Gamma_w, \quad \forall w \in M = \phi(\mathbb{B}^n_{\mathbb{R}}(\delta)).$$

Given $f \in \mathscr{C}_0(M)$ there is and open neighborhood $U \subset \mathbb{C}^n$ of bM with $U \subset \bigcap_{w \in \mathrm{supp}\,(f)} \Gamma_w$. This establishes condition (b).

Proof of condition (a)

Fix a point $z_0 = \phi(x_0) \in M$ with $x_0 \in \mathbb{B}^n_{\mathbb{R}}(\delta)$. We have that

$$\begin{aligned} f_\epsilon(z_0) &= \frac{1}{\pi^{n/2}\epsilon^n} \int_M f(z) e^{-(z-z_0)^2/\epsilon^2}\, dz \\ &= \frac{1}{\pi^{n/2}\epsilon^n} \int_{\mathbb{R}^n} f(\phi(x))\, e^{-(\phi(x)-\phi(x_0))^2/\epsilon^2} \det \phi'(x)\, dx \\ &= \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} f(\phi(x_0+\epsilon u)) e^{-[u+\mathrm{i}(\psi(x_0+\epsilon u)-\psi(x_0))/\epsilon]^2} \det \phi'(x_0+\epsilon u) du. \end{aligned}$$

The Lipschitz condition on ψ gives

$$\left| e^{-[u+\mathrm{i}(\psi(x_0+\epsilon u)-\psi(x_0))/\epsilon]^2} \right| \leq e^{-(1-\alpha)|u|^2}$$

for all $x_0 \in \mathbb{B}^n_{\mathbb{R}}(\delta)$ and $0 < \epsilon < \delta_0 - \delta$. By dominated convergence,

$$\begin{aligned} \lim_{\epsilon\to 0} f_\epsilon(z_0) &= \pi^{-n/2} \int_{\mathbb{R}^n} f(\phi(x_0)) e^{-\langle \phi'(x_0)u, \phi'(x_0)u\rangle} \det \phi'(x_0)\, du \\ &= \pi^{-n/2} \int_{\mathbb{R}^n} f(z_0) e^{-\langle \phi'(x_0)^T \phi'(x_0)u, u\rangle} \det \phi'(x_0) du = f(z_0). \end{aligned}$$

The last equality follows from (*) applied with the matrix

$$A = \phi'(x_0)^T \phi'(x_0), \qquad \det A = \det \phi'(x_0)^2.$$

Proof of Mergelyan approximation on handlebodies

Recall that $S = K \cup M$ is an admissible set in X, with M a totally real submanifold of class $\mathscr{C}^k$, and $f \in \mathscr{C}^k(X) \cap \mathcal{O}(K)$.

Special case: $\mathrm{supp}\,(f) \subset M = S \setminus K$.

- By using a partition of unity, we may assume that $\mathrm{supp}\,(f) \subset M_0 \subset M$ where M_0 is small enough such that the lemma holds on M_0.
- Hence, we get a neighborhood $V \subset X$ of M_0 and $f_j \in \mathcal{O}(V)$ with $f_j \to f$ on M_0 and $f_j \to 0$ in a neighborhood U of bM_0 as $j \to \infty$.
- Since S is a Stein compact, we can **solve the Cousin-I problem** on a Cartan pair (A, B), where $S \setminus M_0 \subset A$ and $M_0 \subset B \subset V$ are open sets such that $A \cap B \subset U$ and $A \cup B$ is a Stein domain. That is, we glue $f_j \in \mathcal{O}(B)$ with the zero function on A.
- This gives $\mathscr{C}^k$ approximation of f on M_0 by functions $F_j \in \mathcal{O}(S)$ converging to 0 on $S \setminus \mathrm{supp}\,(f) \supset S \setminus M_0$.

Proof of Mergelyan approximation on handlebodies, 2

General case: $\operatorname{supp}(f)$ intersects K.
Let $U \supset K$ be an open neighborhood such that $f|_U \in \mathcal{O}(U)$.

- Since S is a Stein compact and we can (by the special case) approximate smooth functions supported on $M = S \setminus K$ by functions in $\mathcal{O}(S)$ which are small on K, there is a Stein neighborhood Ω of S such that $K_0 := \widehat{K}_{\mathcal{O}(\Omega)} \subset U$.
- Pick an $\mathcal{O}(\Omega)$-convex compact set $K_1 \subset U$ with $K_0 \subset \mathring{K}_1$ and a smooth cut-off function χ with $\operatorname{supp}(\chi) \subset K_1$ and $\chi = 1$ near K_0.
- By Oka-Weil, there exist functions $g_j \in \mathcal{O}(\Omega)$ such that

$$g_j \to f \text{ uniformly on } K_1 \text{ as } j \to \infty.$$

- It follows that

$$\tilde{f}_j := \chi g_j + (1-\chi) f = g_j + (1-\chi)(f - g_j) \stackrel{\mathscr{C}^k(S)}{\longrightarrow} f \quad \text{as } j \to \infty.$$

  As $g_j \in \mathcal{O}(S)$, it remains to approximate the functions $(1-\chi)(f-g_j) \in \mathscr{C}^k(S)$ (whose support does not intersect $K_0 \supset K$) by functions in $\mathcal{O}(S)$ that are small outside their support.

Mergelyan approximation of manifold-valued maps

The following simple lemma is useful in reducing the Mergelyan approximation problem for manifold-valued maps to the case of functions.

Lemma

Assume that $K \subset X$ is a compact set with MP, i.e., $\mathscr{A}(K) = \overline{\mathcal{O}}(K)$. Let Y be a complex manifold and $f \in \mathscr{A}(K, Y)$. If the graph $G_f = \big\{(x, f(x)) : x \in K\big\}$ has a Stein neighborhood, then $f \in \overline{\mathcal{O}}(K, Y)$.

Proof.

Let $W \subset X \times Y$ be a Stein neighborhood of the graph G_f.
Docquier-Grauert 1960 There are a proper holomorphic embedding $\iota : W \hookrightarrow \mathbb{C}^N$ and a holomorphic retraction $\rho : \Omega \to W$ from an open Stein neighborhood $\Omega \subset \mathbb{C}^N$.

Assuming that $\mathscr{A}(K) = \overline{\mathcal{O}}(K)$, we can approximate the map $K \ni x \mapsto \iota(x, f(x)) \in \mathbb{C}^N$ by holomorphic maps $F\colon U \to \Omega$ on open neighborhoods $U \subset X$ of K.

The map $pr_Y \circ \iota^{-1} \circ \rho \circ F : U \to Y$ then approximates f on K. □

A Stein neighborhood theorem of Poletsky

Theorem (E. Poletsky 2013)

Let K be a Stein compact in a complex manifold X and let $f \in \mathscr{A}(K, Y)$, where Y is a arbitrary complex manifold. Assume that every point $p \in K$ has a neighborhood $V_p \subset X$ such that

$$f|_{K\cap\overline{V}_p} \in \mathcal{O}(K \cap \overline{V}_p).$$

Then the graph G_f is a Stein compact in $X \times Y$.

Poletsky's proof is similar in spirit to the proof of **Siu's theorem (1976)** that every Stein subvariety in a complex space admits a basis of open Stein neighborhoods.

It relies on the technique of **fusing plurisubharmonic functions**.

Mergelyan's theorem for manifold-valued maps

Corollary

- *If a compact set K in a Riemann surface X has the MP for functions, it has the MP for maps to an arbitrary complex manifold.*
- *The same holds if K is a Stein compact with $\mathscr{C}^1$ boundary in an arbitrary complex manifold X.*
- *In particular, a compact strongly pseudoconvex Stein domain satisfies MP for maps to any complex manifold.*

Proof.

Let $f \in \mathscr{A}(K, Y)$. We cover K by the interiors of closed coordinate discs $D_1, \ldots, D_k$ such that each $f(D_j)$ is contained in a coordinate chart of Y. Since K has MP for functions, the theorem of **Boivin and Jiang (2004)** shows that $f|_{K \cap D_j} \in \mathcal{O}(K \cap D_j, Y)$ for every $j = 1, \ldots, k$.
By Poletsky's theorem the graph G_f is a Stein compact. The assumption that K has MP then implies $f \in \overline{\mathcal{O}}(K, Y)$ in view of the lemma.

If K has $\mathscr{C}^1$ boundary then it obviously satisfies the local Mergelyan property for functions, so the above proof applies. □

Carleman approximation on totally real manifolds

Let X be a Stein manifold and $M \subset X$ be a closed subset. Set

$$\widehat{M}_{\mathcal{O}(X)} = \widehat{M} = \bigcup_{j=1}^{\infty} \widehat{M}_j$$

where M_j is a normal exhaustion of M by compacts.

The set M has **bounded exhaustion hulls** if for any compact $K \subset X$ there is a bigger compact $K' \subset X$ such that

$$\widehat{K \cup M} \subset (K \cup M) \cup K'.$$

For closed sets $M \subset \mathbb{C}$ this is the classical **Arakelyan condition**.

Manne 1993 If X is Stein and $M \subset X$ is a closed $\mathscr{C}^k$ totally real submanifold that is $\mathcal{O}(X)$-convex and has bounded exhaustion hulls, then M admits $\mathscr{C}^k$-Carleman approximation by entire functions.

Magnusson & Wold 2016 If closed holomorphically convex set M in a Stein manifold X admits $\mathscr{C}^0$ Carleman approximation, then M has bounded exhaustion hulls.

Carleman approximation of maps to Oka manifolds

Theorem (Chenoweth 2018)

Let X be a Stein manifold and Y be an Oka manifold. Assume that $K \subset X$ is a compact $\mathscr{O}(X)$-convex subset and $M \subset X$ is a closed totally real submanifold of class $\mathscr{C}^k$ ($k \in \mathbb{N}$) which is $\mathscr{O}(X)$-convex, has bounded exhaustion hulls, and such that $S = K \cup M$ is $\mathscr{O}(X)$-convex.

Then, every map $f \in \mathscr{C}^k(X, Y)$ which is $\bar{\partial}$-flat to order k on S and holomorphic on a neighbourhood of K can be approximated in the fine $\mathscr{C}^k$ topology on S by holomorphic maps $F \colon X \to Y$.

The proof combines most of the methods presented above:

- Mergelyan approximation of functions on handlebodies,
- existence of Stein neighborhoods of graphs over handlebodies (this gives Mergelyan approximation of manifold-valued maps),
- the techniques of Oka theory.

Arakelyan theorem for maps from plane domains to compact homogeneous manifolds

Arakelyan 1964–1971 The following conditions are equivalent for a closed set E in a domain $X \subset \mathbb{C}$:

(a) Every function in $\mathscr{A}(E)$ is a uniform limit of functions in $\mathcal{O}(X)$.

(b) The complement $X^* \setminus E$ of E in the one point compactification $X^* = X \cup \{*\}$ of X is connected and locally connected. (Equivalently, E has bounded exhaustion hulls.)

Scheinberg 1978 Generalization to X an open Riemann surface.

Theorem (F. 2018)

Assume that Y is a compact complex homogeneous manifold. If E is an Arakelyan set in a domain $X \subset \mathbb{C}$, then every continuous map $X \to Y$ which is holomorphic in $\mathring{E}$ can be approximated uniformly on E by holomorphic maps $X \to Y$.

THANK YOU

FOR YOUR ATTENTION