Complex analytic methods in minimal surface theory

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Lecture 1: Survey

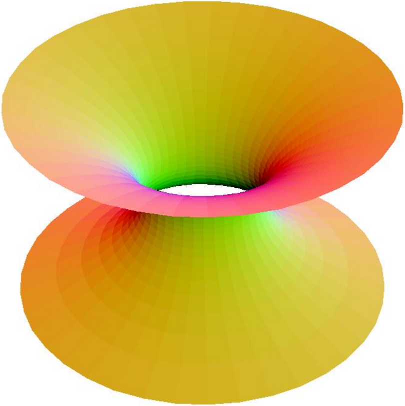
- A brief history of minimal surfaces
- Basics on minimal surfaces and holomorphic null curves
- The Oka principle for null curves
- Isotopies of conformal minimal immersions (CMI's)
- Desingularizing null holomorphic curves in $\mathbb{C}^n$ $(n \geq 3)$ and conformal minimal immersions in $\mathbb{R}^n$ for $n \geq 5$
- Proper conformal minimal immersions to $\mathbb{R}^3$ and embeddings to $\mathbb{R}^5$
- On the Calabi-Yau problem for minimal surfaces
- Minimal hull of a compact set in $\mathbb{R}^n$

Based on joint work with

- Antonio Alarcón and Francisco J. López, University of Granada
- Barbara Drinovec Drnovšek, University of Ljubljana.

From Euler's surfaces of rotation...

1744 **Euler** The only area minimizing surfaces of rotation in $\mathbb{R}^3$ are planes and catenoids.



...via Lagrange's equation of minimal graphs...

1760 **Lagrange** Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain and $f : \overline{\Omega} \to \mathbb{R}$ a smooth function. Then the graph

$$S = \{(x, y, f(x, y)) : (x, y) \in \overline{\Omega}\} \subset \mathbb{R}^3$$

is an **area minimizing surface** if and only if

$$(1 + f_y^2) f_{xx} - 2 f_x f_y f_{xy} + (1 + f_x^2) f_{yy} = 0;$$

equivalently,

$$\mathrm{div}\left(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}\right) = 0.$$

This is known as the **equation of minimal graphs**.

...to the concept of a minimal surface

1776 **Meusnier** A smooth surface $M \subset \mathbb{R}^3$ satisfies locally the above equation iff its mean curvature function vanishes identically.

Definition

A smoothly immersed surface $M \to \mathbb{R}^3$ is said to be a **minimal surface** if its **mean curvature function** $\mathrm{H} : M \to \mathbb{R}$ is identically zero: $\mathrm{H} = 0$.

We have

$$\mathrm{H} = \frac{\kappa_1 + \kappa_2}{2}$$

where κ_1, κ_2 are the **principal curvatures**. Their product

$$\mathrm{K} = \kappa_1 \kappa_2 : M \to \mathbb{R}$$

is the **Gauss curvature** function of M. Note that $\mathrm{H} = 0 \Rightarrow \mathrm{K} \leq 0$.

The helicoid (Archimedes' screw)

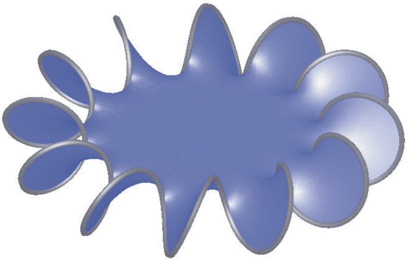
1776 **Meusnier** The helicoid is a minimal surface.

$$x = \rho\cos(\alpha\theta), \quad y = \rho\sin(\alpha\theta), \quad z = \theta$$

1842 **Catalan** The helicoid and the plane are the only **ruled** minimal surfaces in $\mathbb{R}^3$ (unions of straight lines).

The Plateau Problem

1873 **Plateau** Minimal surfaces can be obtained as soap films.



1932 **Douglas, Radó** Every continuous injective closed curve in $\mathbb{R}^3$ (a *Jordan curve*) spans a minimal surface.

Examples by Riemann

1865 **Riemann** On the way to this solution, Riemann and others discovered new examples of minimal surfaces using the **Weierstrass representation**.

Almost 150 years later:

2014 **Meeks, Pérez, Ros** **Riemann's minimal examples, catenoids, helicoids, and planes are the only properly embedded minimal planar domains in $\mathbb{R}^3$.** (Ann. of Math, in press)

This is the most recent great result in the theory.

Basics on curvature of surfaces in $\mathbb{R}^n$

Assume that D is a domain in $\mathbb{R}^2_{(u_1,u_2)}$ and $\mathbf{x} = (x_1, \ldots, x_n) \colon D \to \mathbb{R}^n$ is a $\mathscr{C}^2$ embedding. Let $S = \mathbf{x}(D) \subset \mathbb{R}^n$, a parametrized surface in $\mathbb{R}^n$.

Every smooth embedded curve in S is of the form

$$\lambda(t) = \mathbf{x}(u_1(t), u_2(t)) \in S$$

where $t \mapsto (u_1(t), u_2(t))$ is a smooth embedded curve in D.

Let $s = s(t)$ denote the arc length on λ. The number

$$\kappa(\mathbf{T}, \mathbf{N}) := \frac{d^2\lambda}{ds^2} \cdot \mathbf{N} = \sum_{i,j=1}^{2} \left(\mathbf{x}_{u_i u_j} \cdot \mathbf{N} \right) \frac{du_i}{ds} \frac{du_j}{ds}$$

is the **normal curvature** of S at $p = \lambda(t) \in S$ in the tangent direction $\mathbf{T} = \lambda'(s) \in T_p S$ with respect to the normal vector $\mathbf{N} \in N_p S$.
(It only depends on $\mathbf{T}$ and $\mathbf{N}$.)

Curvature in terms of fundamental forms

In terms of t-derivatives we get

$$\kappa(\mathbf{T}, \mathbf{N}) = \frac{\sum_{i,j=1}^{2} \left(\mathbf{x}_{u_i u_j} \cdot \mathbf{N} \right) \dot{u}_i \dot{u}_j}{\sum_{i,j=1}^{2} g_{i,j} \dot{u}_i \dot{u}_j} = \frac{\text{second fundamental form}}{\text{first fundamental form}}$$

Fix a normal vector $\mathbf{N} \in N_p S$ and vary the unit tangent vector $\mathbf{T} \in T_p S$. The **principal curvatures** of S at p in direction $\mathbf{N}$ are the numbers

$$\kappa_1(\mathbf{N}) = \max_{\mathbf{T}} \kappa(\mathbf{T}, \mathbf{N}), \qquad \kappa_2(\mathbf{N}) = \min_{\mathbf{T}} \kappa(\mathbf{T}, \mathbf{N}).$$

Their average

$$\mathrm{H}(\mathbf{N}) = \frac{\kappa_1(\mathbf{N}) + \kappa_2(\mathbf{N})}{2} \in \mathbb{R}$$

is the **mean curvature** of S at p in the normal direction $\mathbf{N} \in N_p S$.

The mean curvature vector

Let $G = (g_{i,j})$ and $h(\mathbf{N}) = (h_{i,j}(\mathbf{N})) = (\mathbf{x}_{u_i u_j} \cdot \mathbf{N})$ denote the matrices of the 1st and the 2nd fundamental form, respectively.
The extremal values of $\kappa(\mathbf{T}, \mathbf{N})$ are roots of the equation

$$\det\big(h(\mathbf{N}) - \mu G\big) = 0$$

$$\det G \cdot \mu^2 - \big(g_{2,2} h_{1,1}(\mathbf{N}) + g_{1,1} h_{2,2}(\mathbf{N}) - 2 g_{1,2} h_{1,2}(\mathbf{N})\big)\mu + \det h(\mathbf{N}) = 0.$$

The Vieta formula gives

$$\mathrm{H}(\mathbf{N}) = \frac{\kappa_1 + \kappa_2}{2} = \frac{g_{2,2}\mathbf{x}_{u_1 u_1} + g_{1,1}\mathbf{x}_{u_2 u_2} - 2 g_{1,2}\mathbf{x}_{u_1 u_2}}{2 \det G} \cdot \mathbf{N}.$$

There is a unique normal vector $\mathbf{H} \in N_p S$ such that

$$\mathrm{H}(\mathbf{N}) = \mathbf{H} \cdot \mathbf{N} \quad \text{for all } \mathbf{N} \in N_p S.$$

This $\mathbf{H}$ is the **mean curvature vector** of the surface S at p.

The mean curvature in isothermal coordinates

The formulas simplify in **isothermal coordinates**:

$$G = (g_{i,j}) = \xi I,\ \det G = \xi^2; \quad \xi = ||\mathbf{x}_{u_1}||^2 = ||\mathbf{x}_{u_2}||^2,\ \mathbf{x}_{u_1} \cdot \mathbf{x}_{u_2} = 0$$

$$\mathrm{H}(\mathbf{N}) = \frac{\mathbf{x}_{u_1 u_1} + \mathbf{x}_{u_2 u_2}}{2\xi} \cdot \mathbf{N} = \frac{\triangle \mathbf{x}}{2\xi} \cdot \mathbf{N}.$$

Lemma

Assume that D is a domain in $\mathbb{R}^2_{(u_1,u_2)}$ and $\mathbf{x}\colon D \to \mathbb{R}^n$ is a conformal immersion *of class $\mathscr{C}^2$ (i.e., $u = (u_1, u_2)$ are isothermal for $\mathbf{x}$.) Then the Laplacian $\triangle \mathbf{x} = \mathbf{x}_{u_1 u_1} + \mathbf{x}_{u_2 u_2}$ is orthogonal to $S = \mathbf{x}(D)$ and satisfies*

$$\triangle \mathbf{x} = 2\xi \mathbf{H}$$

where $\mathbf{H}$ is the mean curvature vector and $\xi = ||\mathbf{x}_{u_1}||^2 = ||\mathbf{x}_{u_2}||^2$.

Proof of the lemma

Proof.

It suffices to show that the vector $\triangle \mathbf{x}(u)$ is orthogonal to the surface S at the point $\mathbf{x}(u)$ for every $u \in D$. If this holds, it follows from the preceding formula that the normal vector $(2\xi)^{-1}\triangle \mathbf{x}(u) \in N_{\mathbf{x}(u)}S$ fits the definition of the mean curvature vector $\mathbf{H}$, so it equals $\mathbf{H}$.

Conformality of the immersion $\mathbf{x}$ can be written as follows:

$$\mathbf{x}_{u_1} \cdot \mathbf{x}_{u_1} = \mathbf{x}_{u_2} \cdot \mathbf{x}_{u_2}, \qquad \mathbf{x}_{u_1} \cdot \mathbf{x}_{u_2} = 0.$$

Differentiating the first identity on u_1 and the second one on u_2 yields

$$\mathbf{x}_{u_1 u_1} \cdot \mathbf{x}_{u_1} = \mathbf{x}_{u_1 u_2} \cdot \mathbf{x}_{u_2} = -\mathbf{x}_{u_2 u_2} \cdot \mathbf{x}_{u_1},$$

whence $\triangle \mathbf{x} \cdot \mathbf{x}_{u_1} = 0$. Similarly we get $\triangle \mathbf{x} \cdot \mathbf{x}_{u_2} = 0$ by differentiating the first identity on u_2 and the second one on u_1. This proves the claim. $\square$

Lagrange's formula for the variation of area

The area of an immersed surface $\mathbf{x}\colon D \to \mathbb{R}^n$ with the 1st fundamental form $G = (g_{i,j})$ equals

$$\mathcal{A}(\mathbf{x}) = \int_D \sqrt{\det G} \cdot du_1 du_2.$$

Let $\mathbf{N}\colon D \to \mathbb{R}^n$ be a *normal vector field* along $\mathbf{x}$ which vanishes on bD. Consider the 1-parameter family of maps $\mathbf{x}^t\colon D \to \mathbb{R}^n$:

$$\mathbf{x}^t(u) = \mathbf{x}(u) + t\,\mathbf{N}(u), \quad u \in D,\ t \in \mathbb{R}.$$

A calculation gives the formula for the first variation of area:

$$\delta\mathcal{A}(\mathbf{x})\mathbf{N} = \frac{d}{dt}\bigg|_{t=0} \mathcal{A}(\mathbf{x}^t) = -2\int_D \mathbf{H}\cdot \mathbf{N}\,\sqrt{\det G} \cdot du_1 du_2.$$

It follows that $\qquad \delta\mathcal{A}(\mathbf{x}) = 0 \Longleftrightarrow \mathbf{H} = 0.$

Conformal minimal surfaces are harmonic

In view of the already established formula

$$\triangle \mathbf{x} = 2\xi\, \mathbf{H}$$

which holds for any conformal immersion $\mathbf{x}$ we get

Corollary

The following are equivalent for a smooth conformal immersion $\mathbf{x} : M \to \mathbb{R}^n$ from an open Riemann surface M to $\mathbb{R}^n$:

- $\mathbf{x}$ *is minimal (a stationary point of the area functional).*
- $\mathbf{x}$ *has vanishing mean curvature vector:* $\mathbf{H} = 0$.
- $\mathbf{x}$ *is harmonic:* $\triangle \mathbf{x} = 0$.

In the sequel we shall always assume that $\mathbf{x}$ is conformal and hence

$$\delta \mathcal{A}(\mathbf{x}) \iff \mathbf{H} = 0 \iff \triangle \mathbf{x} = 0.$$

Minimal surfaces versus area minimizing surfaces

We wish to emphasize the difference between

- **minimal surfaces:** these are stationary (critical) points of the area functional, and are (only) locally area minimizing; and
- **area-minimizing surfaces:** these are surfaces which globally minimize the area among all nearby surfaces with the same boundary.
- Minimal surfaces which are **graphs** are globally area minimizing.
- Recent work on conditions ensuring that a minimal surface is globally are minimizing was done by C. Arezzo.

Holomorphic null curves in $\mathbb{C}^n$

We now explain the connection with complex analysis. In the sequel, M always denotes an open or a bordered Riemann surface.

Definition

A holomorphic immersion

$$F = (F_1, F_2, \ldots, F_n) \colon M \to \mathbb{C}^n$$

is a **null curve** if the derivative $F' = (F_1', F_2', \ldots, F_n')$ with respect to any local holomorphic coordinate $\zeta = x + \imath y$ on M satisfies

$$(F_1')^2 + (F_2')^2 + \ldots + (F_n')^2 = 0.$$

We denote by $\mathfrak{A} \subset \mathbb{C}^n$ the **null quadric**

$$\mathfrak{A} = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \colon \sum_{j=1}^n z_j^2 = 0\}.$$

The nullity condition is equivalent to $F'(\zeta) \in \mathfrak{A}_* = \mathfrak{A} \setminus \{0\}$.

Connection between null curves and minimal surfaces

- If $F = u + \imath v : M \to \mathbb{C}^n$ is a holomorphic null curve, then

$$u = \Re F : M \to \mathbb{R}^n, \quad v = \Im F : M \to \mathbb{R}^n$$

  are conformal harmonic (hence minimal) immersions into $\mathbb{R}^n$.

- Conversely, a conformal minimal immersion $u : \mathbb{D} \to \mathbb{R}^n$ of the disc $\mathbb{D} = \{\zeta \in \mathbb{C} \colon |\zeta| < 1\}$ is the real part of a holomorphic null curve $F : \mathbb{D} \to \mathbb{C}^n$. (This fails on multiply connected Riemann surfaces.)

- If $F = u + \imath v \colon M \to \mathbb{C}^n$ is a null curve then

$$F^* ds^2_{\mathbb{C}^n} = 2u^* ds^2_{\mathbb{R}^n} = 2v^* ds^2_{\mathbb{R}^n}.$$

- **Hence the real and the imaginary part of a complete null curve in $\mathbb{C}^n$ are complete conformal minimal surfaces in $\mathbb{R}^n$.**

  An immersion $M \to \mathbb{R}^n$ is **complete** if the pullback of the Euclidean metric on $\mathbb{R}^n$ is a complete metric on M. Equivalently, the image in $\mathbb{R}^n$ of any divergent curve in M has infinite length.

The calculation

- Let $F = u + \imath v = (F^1, \ldots, F^n) : M \to \mathbb{C}^n$ be a holomorphic null curve and $\zeta = x + \imath y$ a local holomorphic coordinate on M. Then

$$\begin{aligned} 0 &= \sum_{j=1}^{n} (F^j_\zeta)^2 = \sum_{j=1}^{n} (F^j_x)^2 = \sum_{j=1}^{n} \left(u^j_x + \imath v^j_x \right)^2 \\ &= \sum_{j=1}^{n} \left((u^j_x)^2 - (v^j_x)^2 \right) + 2\imath \sum_{j=1}^{n} u^j_x v^j_x. \end{aligned}$$

- Since $v_x = -u_y$ by the Cauchy-Riemann equations, this reads

$$0 = ||u_x||^2 - ||u_y||^2 - 2\imath\, u_x \cdot u_y \iff ||u_x|| = ||u_y||, \ u_x \cdot u_y = 0.$$

It follows that u is a conformal minimal immersion (CMI) and

$$F^* ds^2_{\mathbb{C}^n} = ||F_x||^2 (dx^2 + dy^2) = 2||u_x||^2 (dx^2 + dy^2) = 2u^* ds^2_{\mathbb{R}^n} = 2v^* ds^2_{\mathbb{R}^n}.$$

Conversely, ...

... an immersion $u = (u_1, \ldots, u_n) : M \to \mathbb{R}^n$ is conformal if and only if, in any local holomorphic coordinate $\zeta = x + \imath y$ on M, we have

$$||u_x|| = ||u_y|| > 0, \qquad u_x \cdot u_y = 0.$$

Equivalently, $u_x \pm \imath u_y \in \mathfrak{A}_*$ are null vectors. From

$$2\partial u = (u_x - \imath u_y) d\zeta$$

we infer that u is conformal if and only if

$$(\partial u_1)^2 + (\partial u_2)^2 + \cdots + (\partial u_n)^2 = 0,$$

and is conformal minimal (=conformal harmonic) iff

$\partial u = (\partial u_1, \ldots, \partial u_n)$ is a holomorphic 1-form with values in $\mathfrak{A}_*$.

If v is any local harmonic conjugate of u then CR equations imply

$$\partial(u + \imath v) = 2\partial u = 2\imath\, \partial v,$$

hence $F = u + \imath v$ is a null holomorphic immersion.

Construction of null curves and CMI's

Let M be an open or a bordered Riemann surface. Fix a nowhere vanishing holomorphic 1-form θ on M and a point $p \in M$. Recall:

$$\mathfrak{A} = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \colon \sum_{j=1}^{n} z_j^2 = 0\} \quad \textbf{the null quadric}.$$

The above discussion gives bijective correspondences (up to constants):

$$\{F \colon M \to \mathbb{C}^n \text{ null curve}\} \longleftrightarrow \{f \colon M \to \mathfrak{A}_* \text{ holomorphic, } f\theta \text{ exact}\}$$

$$F(x) = F(p) + \int_p^x f\theta; \qquad x \in M.$$

$$\{u \colon M \to \mathbb{R}^n \text{ conformal minimal}\} \longleftrightarrow \{f \colon M \to \mathfrak{A}_* \text{ holo., } \Re(f\theta) \text{ exact}\}$$

$$u(x) = u(p) + \int_p^x \Re(f\theta); \qquad x \in M.$$

Example: catenoid and helicoid

Example: The **catenoid** and the **helicoid** are **conjugate minimal surfaces** – the real and the imaginary part of the same null curve

$$F(\zeta) = (\cos\zeta, \sin\zeta, -\mathrm{i}\zeta) \in \mathbb{C}^3, \qquad \zeta = x + \mathrm{i}y \in \mathbb{C}.$$

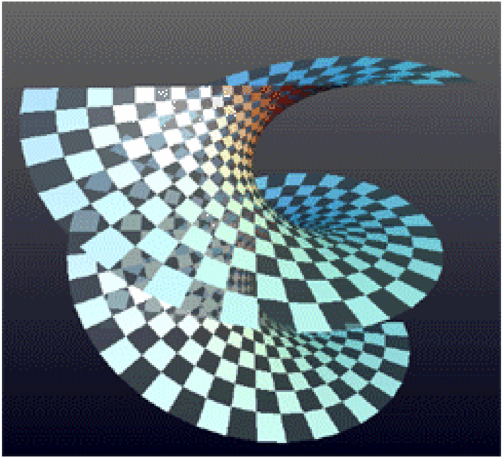
Consider the following family of minimal surfaces in $\mathbb{R}^3$ for $t \in \mathbb{R}$:

$$\begin{aligned} u_t(\zeta) &= \Re\left(e^{\mathrm{i}t} F(\zeta)\right) \\ &= \cos t \begin{pmatrix} \cos x \cdot \cosh y \\ \sin x \cdot \cosh y \\ y \end{pmatrix} + \sin t \begin{pmatrix} \sin x \cdot \sinh y \\ -\cos x \cdot \sinh y \\ x \end{pmatrix} \end{aligned}$$

At $t = 0$ we have a catenoid and at $t = \pm\pi/2$ a helicoid.

Helicatenoid (Source: Wikipedia)

The family of minimal surfaces $u_t(\zeta) = \Re\left(e^{it}F(\zeta)\right)$, $\zeta \in \mathbb{C}$, $t \in \mathbb{R}$:



The Oka principle (h-principle) for null curves

Theorem (1)

Every continuous map $f_0 \colon M \to \mathfrak{A}_*$ of an open Riemann surface M to the null quadric $\mathfrak{A}_* = \mathfrak{A} \setminus \{0\}$ is homotopic to a holomorphic map $f \colon M \to \mathfrak{A}_*$ such that $f\theta$ has vanishing periods, and hence

$$F(x) = F(p) + \int_p^x f\theta, \qquad x \in M$$

is a null holomorphic immersion $M \to \mathbb{C}^n$. F can be chosen **proper**.

The same holds if $\mathfrak{A} \subset \mathbb{C}^n$ is an irreducible cone with the only singularity at $0 \in \mathbb{C}^n$ such that $\mathfrak{A}_* = \mathfrak{A} \setminus \{0\}$ is an **Oka manifold**.

[A. Alarcón, F. Forstnerič: Null curves and directed immersions of open Riemann surfaces. Inventiones Math. 196 (2014), 733–771]

The flux of a conformal minimal immersion

The **flux map** of a conformal minimal immersion $u \colon M \to \mathbb{R}^n$ is the group homomorphism

$$\mathrm{Flux}_u \colon H_1(M; \mathbb{Z}) \to \mathbb{R}^n$$

given on any closed curve Γ in M by

$$\mathrm{Flux}_u(\Gamma) = \int_\Gamma d^c u.$$

Here $d^c u$ is the **conjugate differential**

$$d^c u = -du \circ J = \mathfrak{i}(\bar\partial u - \partial u)$$

where $J : TM \to TM$ denotes the almost complex structure operator.

Note that $dd^c u = 0$ if (and only if) u is harmonic.

A conformal harmonic immersion $u \colon M \to \mathbb{R}^n$ is the real part of a holomorphic null curve $M \to \mathbb{C}^n$ if and only if $\mathrm{Flux}_u = 0$.

Conformal minimal immersions with prescribed flux

Theorem (2)

Let M be an open Riemann surface and $n \geq 3$.

- *For every conformal minimal immersion $u_0 \colon M \to \mathbb{R}^n$ there exists a smooth isotopy $u_t \colon M \to \mathbb{R}^n$ $(t \in [0,1])$ of conformal minimal immersions such that u_1 is the real part of a null curve $M \to \mathbb{C}^n$.*
- *If u_0 is nonflat and $\mathfrak{p} \colon H_1(M;\mathbb{Z}) \to \mathbb{R}^n$ is a group homomorphism then there exists a smooth isotopy $u_t \colon M \to \mathbb{R}^n$ $(t \in [0,1])$ of conformal minimal immersions such that*

$$u_1 \text{ is complete and } \mathrm{Flux}_{u_1} = \mathfrak{p}.$$

- *If u_0 is complete then we can choose u_t (as above) to be complete for every $t \in [0,1]$.*

[A. Alarcón, F. Forstnerič: Every conformal minimal surface in $\mathbb{R}^3$ is isotopic to the real part of a holomorphic null curve. arxiv.org/abs/1408.5315]

General position theorems

Theorem (3)

Let M be an open Riemann surface or a bordered Riemann surface.

(a) *Every holomorphic null curve $M \to \mathbb{C}^n$ $(n \geq 3)$ can be approximated uniformly on compacts in M by* **embedded holomorphic null curves** $M \hookrightarrow \mathbb{C}^n$.

(b) *Every conformal minimal immersion $M \to \mathbb{R}^n$ $(n \geq 5)$ can be approximated uniformly on compacts in M by* **conformal minimal embeddings** $M \hookrightarrow \mathbb{R}^n$.

[(a) A. Alarcón, F. Forstnerič: Null curves and directed immersions of open Riemann surfaces. Inventiones Math. (2014)]
[(b) A. Alarcón, F. Forstnerič & F.J. López: Embedded minimal surfaces in $\mathbb{R}^n$. http://arxiv.org/abs/1409.6901]

Proper embeddings of manifolds

General problem: When is an abstract manifold of a certain kind embeddable as a submanifold of a Euclidean space?

- Whitney: every smooth n-manifold embeds smoothly in $\mathbb{R}^{2n+1}$.
- Nash, Gromov: isometric immersions and embeddings of Riemannian manifolds in $\mathbb{R}^N$.
- Greene and Wu: every Riemannian manifold of dimension n admits a harmonic embedding in $\mathbb{R}^{2n+1}$.
- Remmert, Bishop, Narasimhan: every n-dimensional Stein manifold embeds properly holomorphically in $\mathbb{C}^{2n+1}$.
- In particular, every open Riemann surface embeds properly in $\mathbb{C}^3$.
- Eliashberg and Gromov, Schurmann: every Stein manifold of dimension $n > 1$ embeds in $\mathbb{C}^{[3n/2]+1}$ and this is sharp.
- **Open problem:** does every open Riemann surface embed (properly or nonproperly) in $\mathbb{C}^2$?

Properly immersed or embedded minimal surfaces

Theorem (4)

Every open Riemann surface admits

(a) *a proper conformal minimal embedding into* $\mathbb{R}^5$,

(b) *a proper conformal minimal immersion into* $\mathbb{R}^3$, *and*

(c) *a proper null holomorphic embedding into* $\mathbb{C}^3$.

[(a) A. Alarcón, F. Forstnerič, F.J. López, 2014, arxiv.org/abs/1409.6901]
[(b) A. Alarcón, F.J. López, J. Differential Geom. **90** (2012) 351–382; Trans. Amer. Math. Soc. **366** (2014) 5139–5154]
[(c) A. Alarcón, F. Forstnerič: Null curves and directed immersions of open Riemann surfaces. Inventiones Math. (2014)]

The existence of a proper conformal minimal **embedding** $M \hookrightarrow \mathbb{R}^3$ is a very restrictive condition on M; see e.g. the surveys

[Meeks III, W.H.; Pérez, J: The classical theory of minimal surfaces. Bull. Amer. Math. Soc. (N.S.) **48** (2011) 325–407;
A survey on classical minimal surface theory. University Lecture Series, 60. American Mathematical Society, Providence, RI, 2012]

Embeddings into $\mathbb{C}^2 \cong \mathbb{R}^4$

Let M be an open Riemann surface. Note that every holomorphic embedding $M \hookrightarrow \mathbb{C}^2$ is also a conformal harmonic embedding $M \hookrightarrow \mathbb{R}^4$.

Theorem

Let M be a compact bordered Riemann surface with $bM \neq \emptyset$.

If M admits a (nonproper) holomorphic embedding $M \hookrightarrow \mathbb{C}^2$, then the interior $\mathring{M}$ of M admits a **proper** *holomorphic embedding $\mathring{M} \hookrightarrow \mathbb{C}^2$.*

F. Forstnerič, E.F. Wold: Bordered Riemann surfaces in $\mathbb{C}^2$. J. Math. Pures Appl. 91 (2009) 100–114

Theorem

Every domain in the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ with at most countably many boundary components, none of which are points, admits a proper holomorphic embedding in $\mathbb{C}^2$.

F. Forstnerič, E.F. Wold: Embeddings of infinitely connected planar domains into $\mathbb{C}^2$. Analysis & PDE 6 (2013) 499–514

The conformal Calabi-Yau problem

Recall that an immersion $M \to \mathbb{R}^n$ is **complete** if the image of any divergent curve in M has infinite length in $\mathbb{R}^n$.

Theorem (5)

Every bordered Riemann surface M admits a complete conformal minimal (=harmonic) immersion $\mathbf{x}\colon M \to \mathbb{R}^3$ with bounded image.

[A. Alarcón, F. Forstnerič, http://arxiv.org/abs/1308.0903]

What is new in comparison to all existing results is that we do not change the complex structure on the Riemann surface M.

We expect to prove the following extensions of Theorem 5:

- For every M as in the theorem there is a complete conformal minimal immersion $\mathbf{x}\colon M \to \mathbb{R}^3$ which is **proper into the ball** of $\mathbb{R}^3$.
- There exist **conformal minimal embeddings** $M \hookrightarrow \mathbb{R}^n$ for $n \geq 5$ with these properties.

Complete bounded holomorphic null curves in $\mathbb{C}^3$

Theorem (6)

Let M be a bordered Riemann surface.

(A) *There exists a proper complete holomorphic immersion $M \to \mathbb{B}^2$ into the unit ball of $\mathbb{C}^2$.*

(B) *There exists a proper complete null holomorphic embedding $M \hookrightarrow \mathbb{B}^3$ into the unit ball of $\mathbb{C}^3$.*

[A. Alarcón, F. Forstnerič: Every bordered Riemann surface is a complete proper curve in a ball. Math. Ann. 2013]

(B) answers a question of Martín, Umehara and Yamada (2009).

(A) holds for immersions into any Stein manifold (X, ds^2) of complex dimension > 1 with a chosen Riemannian metric.

Higher dimensional examples

1985 **Løw** Every strongly pseudoconvex Stein domain M admits a proper holomorphic embedding $\phi\colon M \to \mathbb{D}^m$ into a polydisc.

Let $h\colon \mathbb{D} \to \mathbb{B}^2$ be a complete proper holomorphic immersion. Then

$$H\colon \mathbb{D}^m \to (\mathbb{B}^2)^m \subset \mathbb{C}^{2m}, \quad H(z_1,\ldots,z_m) = (h(z_1),\ldots,h(z_m))$$

is a complete proper holomorphic immersion.

Hence $F = H \circ \phi\colon M \to (\mathbb{B}^2)^m$ is a complete proper immersion.

Similarly we get complete proper holomorphic embeddings $\mathbb{D}^m \to (\mathbb{B}^3)^m \subset \mathbb{C}^{3m}$.

Corollary

Every strongly pseudoconvex Stein domain admits a complete bounded holomorphic embedding into $\mathbb{C}^N$ for large N.

A brief history of the Calabi-Yau problem

1965 **E. Calabi** conjectured that there does not exist any complete minimal surface in $\mathbb{R}^3$ with a bounded coordinate function. (All known examples of complete minimal surfaces at the time were proper in $\mathbb{R}^3$.)

1977 **P. Yang** asked whether there exist complete bounded complex submanifolds of $\mathbb{C}^n$ for $n > 1$.

1979 **P. Jones** constructed a complete bounded holomorphic immersion $\mathbb{D} \to \mathbb{C}^2$ of the disc, using BMO methods.

1980 **L.P. Jorge & F. Xavier** constructed complete minimal surfaces in $\mathbb{R}^3$ with a bounded coordinate, disproving Calabi's conjecture.

1996 **N. Nadirashvili** constructed a complete bounded conformal minimal immersion $\mathbb{D} \to \mathbb{R}^3$, hence a complete null curve in $\mathbb{C}^3$. His technique cannot be refined to control the imaginary part.

A brief history...continued

2000 **S.-T. Yau: Review of geometry and analysis** ("The Millenium Lecture"). Mathematics: frontiers and perspectives, AMS. The problem became known as the **Calabi-Yau problem**.

2008 **T.H. Colding and W.P. Minicozzi II:** An embedded complete minimal surface $M \hookrightarrow \mathbb{R}^3$ with finite genus and at most countably many ends is proper in $\mathbb{R}^3$, and M is algebraic. An extension to surfaces with finite genus and countably many ends is announced by Meeks, Pérez and Ros.

2009 **F. Martín, M. Umehara and K. Yamada** constructed complete bounded holomorphic curves in $\mathbb{C}^2$ with arbitrary finite topology.

2012 **L. Ferrer, F. Martín and W.H. Meeks** found complete bounded minimal surfaces in $\mathbb{R}^3$ with arbitrary topology.

2013 **A. Alarcón and F.J. Lopez:** Examples of (i) complete bounded null curves in $\mathbb{C}^3$, (ii) complete bounded immersed holomorphic curves in $\mathbb{C}^2$ with arbitrary topology, and (iii) complete bounded *embedded* holomorphic curves in $\mathbb{C}^2$.

Globevnik's recent solution of a problem of Paul Yang

1977 **Paul Yang:** Do there exist bounded complete complex submanifolds of $\mathbb{C}^n$ for $n > 1$?

2014 **Josip Globevnik:** For any $n > 1$ there exists a holomorphic function f on the unit ball $\mathbb{B}$ of $\mathbb{C}^n$ such that every level set $\{f = c\}$ is complete (i.e., divergent curves in $\{f = c\}$ have infinite length).

Since most such hypersurfaces are smooth, **this gives an optimal answer to Yang's question.**

- **Idea of proof:** exhaust the ball $\mathbb{B}$ by a sequence of polyhedra $P_1 \subset P_2 \subset \ldots \subset \bigcup_{j=1}^{\infty} P_j = \mathbb{B}$. Construct $f \in \mathscr{O}(\mathbb{B})$ which is big on all faces of every P_j, except near the edges. Ensure by suitable choice of the P_j's that every divergent curve in $\mathbb{B}$ on which f is bounded has infinite length.

- **What is the possible topology and complex structure on such hypersurfaces?**

A few open problems

- Does there exist a complete bounded holomorphic **embedding** $\mathbb{D} \hookrightarrow \mathbb{C}^2$ of the disc? Of an arbitrary bordered Riemann surface?
- Is it possible to immerse or embed the ball $\mathbb{B}^2 \subset \mathbb{C}^2$ as a complete bounded complex submanifold of $\mathbb{C}^3$ or $\mathbb{C}^4$? Can it be complete proper in the ball of $\mathbb{C}^N$ for some big N?
- **Conjecture** (well known, likely very difficult): An orientable surface of finite topology with genus g and m ends properly embeds in $\mathbb{R}^3$ as a minimal surface if and only if $m \leq g + 2$.
- **The embedded Calabi-Yau problem in $\mathbb{R}^3$:** We know that a complete bounded embedded minimal surface in $\mathbb{R}^3$ must have infinite genus or uncountably many ends. It is still open whether such a surface exists or not.
- **Calabi's conjecture** is still open in dimensions $n > 3$: Do there exist complete bounded minimal hypersurfaces of $\mathbb{R}^n$ when $n > 3$?