Complex analytic methods in minimal surface theory

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24–27 November 2014

Lecture 2: Complex analytic methods

The aim of Lecture 2 is to introduce the complex analytic methods which we use in minimal surface theory, and to outline the proof of some of the results announced in Lecture 1. The emphasis will be on

- the theory of **Oka manifolds**
- construction of **dominating and period dominating sprays**

- Antonio Alarcón and Franc Forstnerič, *Null curves and directed immersions of open Riemann surfaces.* Invent. Math. **196** (2014), no. 3, 733–771

- Franc Forstnerič: *Stein Manifolds and Holomorphic Mappings (The Homotopy Principle in Complex Analysis).* Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, 56. Springer-Verlag, Berlin-Heidelberg (2011)

- Franc Forstnerič: *Oka manifolds: from Oka to Stein and back.* With an appendix by F. Lárusson. Ann. Fac. Sci. Toulouse Math. (6) **22** (2013) no. 4., 747–809.

Oka theory...

...is about a tight relationship between homotopy theory and complex geometry involving maps from **Stein manifolds** to **Oka manifolds**.

The Oka Principle: There are only topological obstructions to solving certain complex-analytic problems on Stein manifolds and Stein spaces.

1939 **Oka** 2nd Cousin problem on domains of holomorphy has only topological obstructions.

1958 **Grauert** The **Oka-Grauert principle** for vector bundles and principal fiber bundles over Stein manifolds.

1989 **Gromov** The Oka principle for maps to elliptic complex manifolds.

2003-5 **Lárusson** Abstract homotopy theoretic framework for Oka theory.

2005-9 **Forstnerič** Convex approximation property implies the Oka property. Introduction of the class of **Oka manifolds** and **Oka maps**.

Stein manifolds (Karl Stein, 1951)

A complex manifold S is said to be a **Stein manifold** if

- **holomorphic functions separate points:**

  $x, x' \in S$, $x \neq x' \implies f(x) \neq f(x')$ for some $f \in \mathcal{O}(S)$, and

- S is **holomorphically convex**: For every compact set $K \subset S$, its $\mathcal{O}(S)$-convex hull $\widehat{K}$ is also compact:

  $$\widehat{K}_{\mathscr{O}(S)} = \{x \in S : |f(x)| \leq \sup_K |f|, \ \forall f \in \mathscr{O}(S)\}$$

  Equivalently, for every discrete sequence $a_j \in S$ there exists a holomorphic function f on S such that $|f(a_j)| \to +\infty$ as $j \to \infty$.

A **Stein space** is a complex space satisfying these axioms.

Embedding Stein manifolds in Euclidean spaces

1949   **Behnke**-**Stein** An open Riemann surface is a Stein manifold.

1956-61   **Remmert, Bishop, Narasimhan** A complex manifold S of dimension n is Stein if and only if it is embeddable as a closed complex submanifold of some $\mathbb{C}^N$; one can take $N = 2n + 1$.

Stein manifolds are relatives of affine algebraic manifolds.

1984   **Stout** Every relatively compact domain in a Stein manifold is biholomorphic to a domain in an affine algebraic manifold.

1992   **Eliashberg and Gromov; Schürmann (1997)** A Stein manifold of dimension $n > 1$ is embeddable in $\mathbb{C}^N$ with $N = \left[\frac{3n}{2}\right] + 1$.

1971   **Forster** This N is optimal for every $n > 1$.

Problem   Is every open Riemann surface biholomorphic to some closed nonsingular embedded complex curve in $\mathbb{C}^2$?

What should be the dual notion to a Stein manifold?

By definition, a Stein manifold S admits many holomorphic maps $f : S \to \mathbb{C}$. Replace $\mathbb{C}$ by a complex manifold X and ask the following

Main question: For which complex manifolds X do there exist many holomorphic maps $S \to X$ from any Stein manifold S?

What is a good way to interpret 'many maps'?

Start with two classical 19th century theorems:

Weierstrass Theorem. On a discrete subset of a domain $\Omega \subset \mathbb{C}$ we can prescribe the values of a holomorphic function $f : \Omega \to \mathbb{C}$.

Runge Theorem. If $K \subset \mathbb{C}$ is a compact set with no holes then every holomorphic function $K \to \mathbb{C}$ can be approximated uniformly on K by entire functions $\mathbb{C} \to \mathbb{C}$.

Cartan Extension Theorem. If T is a closed complex subvariety of a Stein manifold S, then every holomorphic function $T \to \mathbb{C}$ extends to a holomorphic function $S \to \mathbb{C}$.

Oka-Weil Approximation Theorem. Let $K = \widehat{K}_{\mathscr{O}(S)}$ be a compact *holomorphically convex* subset of a Stein manifold S. Here

$$\widehat{K}_{\mathscr{O}(S)} = \{p \in S \colon |f(p)| \leq \sup_K |f| \;\; \forall f \in \mathcal{O}(S)\}.$$

Then every holomorphic function $K \to \mathbb{C}$ can be approximated uniformly on K by holomorphic functions $S \to \mathbb{C}$.

These are fundamental properties of Stein manifolds.

We can also view them as properties of the target manifold, the complex number field $\mathbb{C}$.

Formulate them as properties of an arbitrary target X, with source any Stein manifold (or Stein space).

Oka properties of a complex manifold

The following are **Oka properties** (anti-hyperbolicity properties) which a given complex manifold X may or may not have.

- **Convex Approximation Property (CAP)**: every holomorphic map $K \to X$ from (a neighborhood of) a compact **convex** set $K \subset \mathbb{C}^n$ can be approximated uniformly on K by holomorphic maps $\mathbb{C}^n \to X$.

- **Basic Oka Property with Approximation (BOPA)**: Every continuous map $f_0 : S \to X$ from a Stein S which is holomorphic on a compact set $K = \widehat{K}_{\mathcal{O}(S)} \subset S$ can be deformed to a holomorphic map $f : S \to X$ by a deformation that is arbitrarily small on K.

- **Basic Oka Property with Interpolation (BOPI)**: Let $T \hookrightarrow S$ be a Stein inclusion. Every continuous map $S \to X$ which is holomorphic on T is homotopic (rel. T) to a holomorphic maps $S \to X$.

- **BOPAI** Combine BOPA and BOPI.

- **POPAI** The parametric version of BOPAI.

Main results of Oka theory

2009 **Forstnerič** All Oka properties introduced above are pairwise equivalent. The class of **Oka manifolds.**

Example A Riemann surface is Oka iff it is not hyperbolic.

1958-61 **Grauert** Complex Lie groups and homogeneous manifolds are Oka.

1989 **Gromov** Elliptic complex manifolds are Oka.

A complex manifold X is **elliptic** if it admits a **dominating spray**; i.e., a family of holomorphic maps $f_x \colon \mathbb{C}^n \to X$, depending holomorphically on $x \in X$, such that $f_x(0) = x$ and $df_x(0) \colon \mathbb{C}^n \to T_x X$ is surjective for every $x \in X$.

Example If X admits $\mathbb{C}$-complete holomorphic vector fields $V_1, \ldots, V_n$ which span $T_x X$ at every point $x \in X$, then the composition of their flows is a dominating spray on X:

$$f_x(t_1, \ldots, t_n) = \phi_{t_1}^1 \circ \phi_{t_2}^2 \circ \cdots \circ \phi_{t_n}^n(x), \quad t_1, \ldots, t_n \in \mathbb{C}.$$

The punctured null quadric is an Oka manifold

Recall that $\mathfrak{A} \subset \mathbb{C}^n$ is the **null quadric**

$$\mathfrak{A} = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \colon \sum_{j=1}^{n} z_j^2 = 0\}.$$

Theorem

$\mathfrak{A}_* = \mathfrak{A} \setminus \{0\}$ is an *elliptic manifold*, and hence an *Oka manifold*.

Proof. The holomorphic vector fields on $\mathbb{C}^n$,

$$V_{j,k}(z) = z_j \frac{\partial}{\partial z_k} - z_k \frac{\partial}{\partial z_j}, \quad 1 \leq j, k \leq n$$

are linear and hence $\mathbb{C}$-complete, their flows preserve $\mathfrak{A}_*$, and they span the tangent space of $\mathfrak{A}_*$ at every point. Thus $\mathfrak{A}_*$ is elliptic, and hence an Oka manifold by Gromov's theorem.

Classical approach: the Weierstrass Representation

- Let $\mathbf{x} = (\mathbf{x}_j)_{j=1,2,3} : M \to \mathbb{R}^3$ be a conformal minimal immersion.

- Set

$$\phi = \partial \mathbf{x} = (\phi_1, \phi_2, \phi_3), \qquad \sum_{j=1}^{3} \phi_j^2 = 0.$$

  Note that ϕ is a holomorphic 1-form on M with values in $\mathfrak{A}_*$ and $\mathbf{x} = \Re \int^z \phi$.

- The stereographic projection of the Gauss map of $\mathbf{x}$ is

$$g = \frac{\phi_3}{\phi_1 - i\phi_2} \quad \text{(meromorphic function on } M\text{)}.$$

- $\phi = (\phi_j)_{j=1,2,3} = \left(\frac{1}{2} \left(\frac{1}{g} - g \right), \frac{i}{2} \left(\frac{1}{g} + g \right), 1 \right) \phi_3.$

- $(g, \phi_3) \equiv$ Weierstrass Representation of $\mathbf{x}$.

Further instances of the Oka principle

Let $\pi : Z \to X$ be a holomorphic fiber bundle with Oka fiber. Then:

- X is Oka if and only if Z is Oka.

- If in addition X is Stein, then sections $f : X \to Z$ satisfy all forms of the Oka principle.

- Example: Any cone $\mathfrak{A} \subset \mathbb{C}^n$ is a $\mathbb{C}_*$-bundle over a projective manifold $X \subset \mathbb{CP}^{n-1}$. Hence $\mathfrak{A}_*$ is Oka iff X is Oka.
  For the null cone in $\mathbb{C}^3$ we have $X \cong \mathbb{CP}^1$ which is Oka.

- If $\pi : Z \to X$ is a holomorphic fiber bundle with an Oka fiber, then for any Stein manifold M we have the Oka principle for liftings of holomorphic maps $f : M \to X$ to maps $F : M \to Z$ (any continuous lifting can be deformed to a holomorphic lifting).

- Example: If $\mathfrak{A} \subset \mathbb{C}^3$ is the null quadric then the coordinate projection $\pi : \mathfrak{A}_* \to \mathbb{C}_*$ has Oka fiber ($\mathbb{C}_*$), and hence liftings enjoy the Oka principle.

Dominating holomorphic sprays of maps

Let M be a compact bordered Riemann surface (or a compact complex manifold with boundary), and let X be a complex manifold.

$\mathscr{A}^r(M, X)$: the space of $\mathscr{C}^r$ maps $M \to X$ which are holomorphic in $\mathring{M}$.

Definition

- A holomorphic spray of maps of class $\mathscr{A}^r(M, X)$ is a family of maps $f_t \in \mathscr{A}^r(M, X)$ depending holomorphically on a parameter t in a ball $0 \in B \subset \mathbb{C}^N$. The map f_0 is the core of the spray.
- A spray f_t is dominating if $\partial_t|_{t=0} f_t(x) : \mathbb{C}^N \to T_{f_0(x)} X$ is surjective for every $x \in M$.

Lemma (Existence of dominating sprays)

If M is a bordered Riemann surface (or a strongly pseudoconvex domain in a Stein manifold) and X is any complex manifold, then every map $f_0 \in \mathscr{A}^r(M, X)$ ($r \in \mathbb{Z}_+$) is the core map of a dominating holomorphic spray $f_t \in \mathscr{A}^r(M, X)$ ($t \in B \subset \mathbb{C}^N$, N large).

The period map

Let M be a compact bordered Riemann surface with $bM \neq \varnothing$. Fix a nowhere vanishing holomorphic 1-form θ on M.

Pick a basis $\{\gamma_j\}_{j=1}^{l}$ of the 1st homology group $H_1(M; \mathbb{Z})$. Let

$$\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_l) \colon \mathscr{A}^0(M, \mathbb{C}^n) \to (\mathbb{C}^n)^l$$

be the period map. The j-th component, applied to $f \in \mathscr{A}^0(M, \mathbb{C}^n)$, is

$$\mathcal{P}_j(f) = \int_{\gamma_j} f\theta = \int_0^1 f(\gamma_j(s))\, \theta(\gamma_j(s), \dot{\gamma}_j(s))\, ds \in \mathbb{C}^n.$$

By Stokes' theorem, the period map does not change under homotopic deformations of the loops $\gamma_j \colon [0, 1] \to M$.

The 1-form $f\theta$ is exact if and only if its periods vanish:

$$\mathcal{P}_j(f) = \int_{\gamma_j} f\theta = 0, \quad j = 1, \ldots, l.$$

Period dominating holomorphic sprays

Let $\mathfrak{A}$ denote the null quadric in $\mathbb{C}^n$ ($n \geq 3$) and $\mathfrak{A}_* = \mathfrak{A} \setminus \{0\}$.

Lemma (Existence of period dominating sprays)

Let M, θ and $\gamma_1, \ldots, \gamma_l$ be as above, and let $r \in \mathbb{Z}_+$.

Assume that $f_0 \colon M \to \mathfrak{A}_*$ is a map of class $\mathscr{A}^r(M, \mathfrak{A}_*)$ which is nonflat, in the sense that its range does not belong to a ray of the null quadric $\mathfrak{A}_*$.

Then there exists a dominating and period dominating spray $f_t \colon M \to \mathfrak{A}_*$ of class $\mathscr{A}^r(M)$ ($t \in B \subset \mathbb{C}^N$, N large):

$$\frac{\partial}{\partial t}\bigg|_{t=0} \mathcal{P}(f_t) : \mathbb{C}^N \to (\mathbb{C}^n)^l \text{ is surjective.}$$

Furthermore, every flat map $f_0 \in \mathscr{A}^r(M, \mathfrak{A}_*)$ can be approximated arbitrarily closely by nonflat maps in $\mathscr{A}^r(M, \mathfrak{A}_*)$.

Construction of period dominating sprays

We may assume that $M \Subset \widetilde{M}$ is a smoothly bounded domain in an open Riemann surface $\widetilde{M}$. Let $C_j = \gamma_j([0,1]) \subset M$ $(j = 1, \ldots, l)$ be a basis of $H_1(M, \mathbb{Z})$ such that $C = \cup_j C_j$ is $\mathscr{O}(M)$-convex.

- The graph $G = \{(x, f_0(x)) \colon x \in M\}$ has an open Stein neighborhood $\Omega \subset \widetilde{M} \times \mathfrak{A}_*$.

- Choose holomorphic vector fields $V_1, \ldots, V_m$ on Ω which are tangential to $\mathfrak{A}_*$ and span the tangent space at each point.

- Consider the spray f_t given by

$$f_t(x) = \phi^1_{h_1(x)t_1} \circ \cdots \circ \phi^N_{h_N(x)t_N}(x), \quad t = (t_1, \ldots, t_N),$$

where each ϕ^j is the flow of one of the vector fields V_k (possibly with repetitions) and h_j are holomorphic functions on M.

- We have $\left.\frac{\partial}{\partial t_j}\right|_{t=0} f_t(x) = h_j(x) V_j(f_0(x))$. Choose the values of h_j in a suitable way on $C = \cup_j C_j$ and apply Mergelyan's approximation theorem to ensure that f_t is dominating and period dominating.

Banach manifold structure

Theorem

Let M be a compact bordered Riemann surface and $r \in \mathbb{Z}_+$. The set

$$\{f \in \mathscr{A}^r(M, \mathfrak{A}_*): f \text{ nonflat}, \Re\mathcal{P}(f) = 0\}$$

is a real Banach manifold containing the complex Banach manifold

$$\{f \in \mathscr{A}^r(M, \mathfrak{A}_*): f \text{ nonflat}, \mathcal{P}(f) = 0\}$$

as a closed Banach submanifold.

Proof.

The set $\mathscr{A}^r(M, X)$ is a complex Banach manifold for any $r \in \mathbb{Z}_+$ and any complex manifold X.

The existence of period dominating sprays shows that the equations

$$\Re\mathcal{P}(f) = 0, \qquad \mathcal{P}(f) = 0 \qquad (f \in \mathscr{A}^r(M, \mathfrak{A}_*))$$

are of maximal rank at each nonflat point. The claims follow. $\qquad \square$

Banach manifold structure

By integrating these classes of maps we get the corresponding statements for the space of nonflat conformal minimal immersions in $\mathbb{R}^n$:

$$u \colon M \to \mathbb{R}^n, \quad u(x) = \int_{p_0}^x \Re(f\theta) \quad (x \in M),$$

and also for the space of nonflat holomorphic null curves in $\mathbb{C}^n$:

$$F \colon M \to \mathbb{C}^n, \quad F(x) = \int_{p_0}^x f\theta \quad (x \in M).$$

Theorem

Let M be a compact bordered Riemann surface and $n \geq 3, r \geq 1$.

(a) *The space $\mathrm{CMI}_*^r(M, \mathbb{R}^n)$ of nonflat conformal minimal immersions $M \to \mathbb{R}^n$ of class $\mathscr{C}^r$ is a real analytic Banach manifold.*

(b) *The space $\mathrm{NC}_*^r(M, \mathbb{C}^n)$ of nonflat holomorphic null curves $M \to \mathbb{C}^n$ of class $\mathscr{C}^r$ is a complex Banach manifold.*

(c) *The space $\{\Re F \colon F \in \mathrm{NC}_*^r(M, \mathbb{C}^n)\}$ is a Banach submanifold of $\mathrm{CMI}_*^r(M, \mathbb{R}^n)$.*

Oka principle

Theorem (Oka principle; Lecture 1, Theorem 1)

Every continuous map $f_0 \colon M \to \mathfrak{A}_*$ of an open Riemann surface M to the null quadric $\mathfrak{A}_* = \mathfrak{A} \setminus \{0\}$ is homotopic to a holomorphic map $f \colon M \to \mathfrak{A}_*$ such that $f\theta$ has vanishing periods, and hence

$$F(x) = F(p) + \int_p^x f\theta, \qquad x \in M$$

is a null holomorphic immersion $M \to \mathbb{C}^n$. F can be chosen **proper**.

Proof. Choose an exhaustion $M_1 \subset M_2 \subset \cdots \subset \cup_{j=1}^{\infty} M_j = M$ such that every M_j is a smooth Runge domain in M and M_{j+1} is obtained from M_j in one of the following ways:

(a) Noncritical case: M_{j+1} can be deformed onto M_j.

(b) Critical case, index 0: A new connected component.

(b) Critical case, index 1: $M_{j+1} = M_j \cup (1\text{-handle})$.

The noncritical case

We inductively construct a sequence of holomorphic maps $f_j \colon M_j \to \mathfrak{A}_*$ such that $f_j\theta$ has vanishing periods on M_j (finitely many integral conditions) and $f_{j+1}|_{M_j}$ approximates f_j as close as desired. The limit map $f = \lim_{j\to\infty} f_j \colon M \to \mathfrak{A}_*$ will solve the problem.

The noncritical case: $M_j \subset M_{j+1}$ are smoothly bounded compact sets in M such that M_{j+1} retracts onto M_j (no change of topology).

- We embed the map $f_j \colon M_j \to \mathfrak{A}_*$ into a period dominating holomorphic spray $f_{j,t} \colon M_j \to \mathfrak{A}_*$ ($t \in \mathbb{B} \subset \mathbb{C}^N$).
- Since $\mathfrak{A}_*$ is an Oka manifold and $M \times \mathbb{B}$ is a Stein manifold, we can approximate the spray $f_{j,t}$ by a holomorphic spray $f_{j+1,t} \colon M_{j+1} \to \mathfrak{A}_*$ for t in a slightly smaller ball $0 \in B' \subset \mathbb{B}$.
- If the approximation is close enough then, because of period domination condition, there exists $t_0 \in \mathbb{B}'$ close to 0 such that the map $f_{j+1,t_0} \colon M_{j+1} \to \mathfrak{A}_*$ has zero periods.

This completes the noncritical case, and also the critical case of index 0 (when a new connected component appears).

The critical case

The critical case: Now M_{j+1} retracts onto the union of M_j and a smooth arc $E_j \cong [0,1] \subset M_{j+1}$ whose endpoints a_j, b_j belong to M_j and which is otherwise disjoint from M_j, and $M_j \cup E_j$ is Runge in M_{j+1}.

There are two possibilities:

(a) The points a_j, b_j belong to different connected components of M_j.

(b) The points a_j, b_j belong to the same connected component of M_j.

In case (a) no new nontrivial loop appears in the 1st homology group and we proceed as follows:

- Extend f_j to a smooth map $\tilde{f}_j \colon M_j \cup E_j \to \mathfrak{A}_*$, holomorphic on M_j.
- Embed $\tilde{f}_j$ into a period dominating spray of maps $\tilde{f}_{j,t}$ ($t \in \mathbb{B} \subset \mathbb{C}^N$) of the same class. (See the noncritical case.)
- By Mergelyan's theorem and the Oka property of $\mathfrak{A}_*$ we can approximate $\tilde{f}_{j,t}$ by a holomorphic spray $f_{j+1,t}$ over M_{j+1}.
- If the approximation is close enough then a member f_{j+1,t_0} of this spray has vanishing periods over all closed curves in M_{j+1}.

The critical case–continued

In case (b) the arc E_j closes to a loop $\Gamma \subset M_j \cup E_j \subset M_{j+1}$.

We extend f_j to a smooth map $\tilde{f}_j \colon M_j \cup E_j \to \mathfrak{A}_*$ such that

$$\int_\Gamma \tilde{f}_j \theta = 0.$$

This is possible since the convex hull of the algebraic variety $\mathfrak{A}$ equals $\mathbb{C}^n$.

The basic idea: Assume that E_j is parametrized by $\gamma \colon [0,1] \to E_j$ with $\theta(\dot{\gamma}(t)) = b > 0$ independent of t. Fix a point $\mathbf{v} \in \mathbb{C}^n$. Choose points $\mathbf{v}_1, \dots, \mathbf{v}_m \in \mathfrak{A}_*$ and numbers $\lambda_1, \dots, \lambda_m > 0$ such that

$$\sum_j \lambda_j = 1 \quad \text{and} \quad \sum_j \lambda_j \mathbf{v}_j = b^{-1}\mathbf{v}.$$

Choose $\tilde{f}_j \colon E_j \to \mathfrak{A}_*$ so that it spends approximately the time λ_j at the point $\mathbf{v}_j$ and goes very quickly from one point to another. Then

$$\int_{E_j} \tilde{f}_j \theta = \int_0^1 \tilde{f}_j(\gamma(t))\theta(\dot{\gamma}(t))\, dt \approx b\sum_j \lambda_j \mathbf{v}_j = \mathbf{v}.$$

A suitable choice of $\mathbf{v}$ gives $\int_\Gamma \tilde{f}_j \theta = 0$. Now proceed as in case (a).

General position theorems

Theorem (Lecture 1, Theorem 3)

Let M be an open Riemann surface.

(a) *Every immersed holomorphic null curve $F_0 \colon M \to \mathbb{C}^n$ ($n \geq 3$) can be approximated uniformly on compacts in M by embedded holomorphic null curves $F \colon M \to \mathbb{C}^n$.*

(b) *Every conformal minimal immersion $u_0 \colon M \to \mathbb{R}^n$ ($n \geq 5$) can be approximated uniformly on compacts in M by conformal minimal embeddings $u \colon M \to \mathbb{R}^n$.*

Sketch of proof: It suffices to consider the case when M is a compact bordered Riemann surface. Consider the difference map

$$\delta F_0 \colon M \times M \to \mathbb{C}^n, \quad \delta F_0(x, y) = F_0(x) - F_0(y).$$

Clearly F is an embedding if and only if

$$(\delta F)^{-1}(0) = D_M = \{(x, x) \colon x \in M\} \subset M \times M.$$

Since F_0 is an immersion, it is locally injective, and there is an open neighborhood $U \subset M \times M$ of the diagonal D_M such that δF_0 does not assume the value $0 \in \mathbb{C}^n$ in $\overline{U} \setminus D_M$.

To prove the theorem, it suffices to find arbitrarily close to F_0 another holomorphic null curve $F \colon M \to \mathbb{C}^n$ whose difference map δF, restricted to $M \times M \setminus U$, is transverse to the origin $0 \in \mathbb{C}^n$.

Since $\dim_{\mathbb{C}} M \times M = 2 < n$, this will imply that δF does not assume the value zero in $(M \times M) \setminus U$, so $F(x) \neq F(y)$ if $(x, y) \in M \times M \setminus U$. It also does not assume the value 0 on $U \setminus D_M$ if F is close enough to F_0.

Hence F is an embedding.

Such a map F is obtained by a transversality argument.

General position theorems–proof

We begin by finding a holomorphic spray of maps $H: \mathbb{B} \times M \to \mathbb{C}^n$, where $0 \in \mathbb{B} \subset \mathbb{C}^N$, such that $H(0, \cdot) = F_0$ and the difference map $\delta H: \mathbb{B} \times M \times M \to \mathbb{C}^n$, defined by

$$\delta H(\zeta, x, y) = H(\zeta, x) - H(\zeta, y), \qquad \zeta \in \mathbb{B}, \quad x, y \in M,$$

is a submersive family, meaning that its partial differential

$$d_\zeta|_{\zeta=0}\, \delta H(\zeta, x, y): \mathbb{C}^N \to \mathbb{C}^n$$

is surjective for every $(x, y) \in M \times M \setminus U$.

It follows that $d_\zeta(\delta H)$ is surjective for all ζ in a neighborhood $\mathbb{B}' \subset \mathbb{B}$ of the origin in $\mathbb{C}^N$. Hence the map $\delta H: \mathbb{B}' \times (M \times M \setminus U) \to \mathbb{C}^n$ is transverse to any submanifold of $\mathbb{C}^n$, in particular, to the origin $0 \in \mathbb{C}^n$.

For a generic member $H(\zeta, \cdot): M \to \mathbb{C}^n$ of this family, the difference map $\delta H(\zeta, \cdot)$ is also transverse to $0 \in \mathbb{C}^n$ on $M \times M \setminus U$. By choosing ζ close to 0 we obtain an embedded null curve $F = H(\zeta, \cdot)$ close to F_0.

Proper null curves and conformal minimal immersions

Theorem (Lecture 1, Theorem 4)

Every open Riemann surface admits

(a) *a proper conformal minimal embedding into $\mathbb{R}^5$,*

(b) *a proper conformal minimal immersion into $\mathbb{R}^3$, and*

(c) *a proper null holomorphic embedding into $\mathbb{C}^3$.*

We shall present the proof of part (c); the other two are similar.

The main part of the proof is contained in the following lemma which explaines how to approximate a given holomorphic null curve on a compact smooth domain $U \subset M$ by a holomorphic null curve on a bigger domain $V \subset M$ provided that U is a strong deformation retract of V (i.e., there is no change of topology).

Later on we shall also explain how to handle the critical case where the topology of the domain changes.

The main lemma

Lemma

Let $U \subset \mathring{V} \subset V$ be smoothly bounded compact Runge domains in M such that U is a deformation retract of V.

Let $F = (F_1, F_2, F_3) \colon U \to \mathbb{C}^3$ be a null immersion such that

$$\max\{|F_1(x)|, |F_2(x)|\} > \rho \quad \text{for some } \rho > 0 \text{ and all } x \in bU.$$

Then there is a null immersion $\widetilde{F} = (\widetilde{F}_1, \widetilde{F}_2, \widetilde{F}_3) \colon V \to \mathbb{C}^3$ such that

(i) *$\widetilde{F}$ is as close as desired to F in the $\mathcal{C}^1(U)$ topology,*

(ii) *$\max\{|\widetilde{F}_1(x)|, |\widetilde{F}_2(x)|\} > \rho$ for all $x \in V \setminus \mathring{U}$, and*

(iii) *$\max\{|\widetilde{F}_1(x)|, |\widetilde{F}_2(x)|\} > \rho + 1$ for all $x \in bV$.*

Construction of proper null curves

Proof of the Lemma. Note that $V \setminus \mathring{U} = \cup_{i=1}^{\mathrm{i}} \mathcal{A}_i$ where the $\mathcal{A}_i$'s are pairwise disjoint annuli,

$$b\mathcal{A}_i = \alpha_i \cup \beta_i \text{ where } \alpha_i \subset bU, \ \beta_i \subset bV.$$

There exist $\mathrm{j} \in \mathbb{N}$, subsets I_1 and I_2 of $I := \{1, \ldots, \mathrm{i}\} \times \mathbb{Z}_\mathrm{j}$ and compact connected subarcs $\{\alpha_{i,j} : (i,j) \in I\}$ satisfying the following conditions:

(a1) $\cup_{j=1}^{\mathrm{j}} \alpha_{i,j} = \alpha_i$ for $i = 1, \ldots, \mathrm{i}$.

(a2) $\alpha_{i,j}$ and $\alpha_{i,j+1}$ only meet in a common endpoint $p_{i,j}$.

(a3) $I_1 \cup I_2 = I$ and $I_1 \cap I_2 = \varnothing$.

(a4) $|F_k(x)| > \rho$ for all $x \in \alpha_{i,j}$ and all $(i,j) \in I_k$, $k = 1, 2$.

From (a4) one also has that

(a5) if $(i,j) \in I_h$ and $(i,j+1) \in I_l$, $h \neq l$, then $|F_k(p_{i,j})| > \rho$ for $k \in \{1, 2\}$.

Construction of proper null curves

For every $(i, j) \in I$ we choose a smooth embedded arc $\gamma_{i,j} \subset \mathcal{A}_i$ with the following properties:

- $\gamma_{i,j}$ is attached to U at the endpoint $p_{i,j}$,
- it intersects the arc α_i transversely at that point,
- $\gamma_{i,j} \cap \alpha_i = \{p_{i,j}\}$,
- the other endpoint $q_{i,j}$ of the arc $\gamma_{i,j}$ lies in β_i, $\gamma_{i,j}$ intersects β_i transversely at that point, and $\gamma_{i,j} \cap \beta_i = \{q_{i,j}\}$, and
- The arcs $\gamma_{i,j}$, $j \in \mathbb{Z}_j$, are pairwise disjoint.

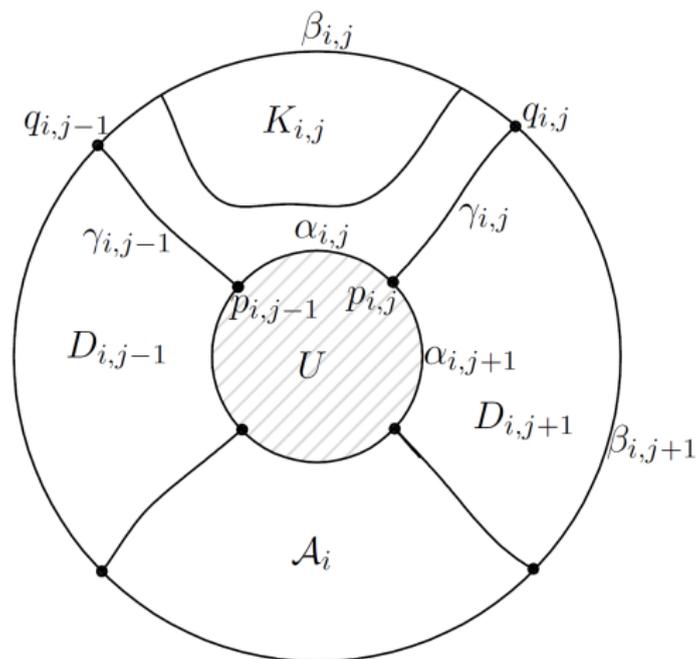The situation is shown on the following figure.

Figure : The annulus $\mathcal{A}_i$.

Construction of proper null curves

Let $z = (z_1, z_2, z_3)$ be the coordinates on $\mathbb{C}^3$ and $\pi_k \colon \mathbb{C}^3 \to \mathbb{C}$ the coordinate projections $\pi(z_1, z_2, z_3) = z_k$ for $k = 1, 2, 3$.

Choose compact smooth embedded arcs $\lambda_{i,j} \subset \mathbb{C}^3$, $(i, j) \in I$, satisfying:

(b1) $\lambda_{i,j}$ agrees with $F(\gamma_{i,j})$ near the endpoint $F(p_{i,j})$.

(b2) $|\pi_k(z)| > \rho$ for all $z \in \lambda_{i,j-1} \cup \lambda_{i,j}$, $(i, j) \in I_k$, $k = 1, 2$.

(b3) $|\pi_k(z)| > \rho + 1$ for all $z \in \{v_{i,j-1}, v_{i,j}\}$, $(i, j) \in I_k$, $k = 1, 2$, where $v_{i,l} \in \mathbb{C}^3$ denotes the other endpoint of the arc $\lambda_{i,l}$.

(b4) The unit tangent vector field to the arc $\lambda_{i,j}$ assumes values in $\mathfrak{A}_*$.

The existence of such arcs $\lambda_{i,j}$ is clear. Since $p_{i,j-1}$ and $p_{i,j}$ are the endpoints of $\alpha_{i,j}$, properties (b1) and (b2) are compatible thanks to (a4). On the other hand, (b3) is always possible.

Taking into account (b1), we can find a smooth map $\widehat{G} \colon V \to \mathbb{C}^n$ that agrees with F in an open neighborhood of U and maps the arc $\gamma_{i,j}$ diffeomorphically onto $\lambda_{i,j}$ for every $(i, j) \in I$.

Construction of proper null curves

The set

$$S := U \cup \big(\cup_{(i,j) \in I} \gamma_{i,j} \big) \subset M$$

is admissible. Condition (b4) shows that the map $\widehat{G}|_S \colon S \to \mathbb{C}^3$ is a generalized null curve. Hence the Mergelyan theorem furnishes a null immersion $G = (G_1, G_2, G_3) \colon V \to \mathbb{C}^3$ such that

(c1) G is as close as desired to F in the $\mathcal{C}^1(U)$ topology,

(c2) $|G_k(x)| > \rho$ for all $x \in \gamma_{i,j-1} \cup \alpha_{i,j} \cup \gamma_{i,j}$, $(i,j) \in I_k$, $k = 1, 2$, and

(c3) $|G_k(x)| > \rho + 1$ for all $x \in \{q_{i,j-1}, q_{i,j}\}$, $(i,j) \in I_k$, and $k = 1, 2$.

To obtain (c2) we take into account (a4) and (b2), whereas (c3) follows from (b3). Furthermore, (c2) and (c3) give

(c4) if $(i,j) \in I_h$ and $(i, j+1) \in I_l$, $h \neq l$, then for $k \in \{1, 2\}$ we have

$$|G_k(x)| > \rho \text{ for all } x \in \gamma_{i,j}, \text{ and } |G_k(q_{i,j})| > \rho + 1.$$

Therefore G satisfies the Lemma on the admissible set S.

Construction of proper null curves

Let $\beta_{i,j}$ be the Jordan arc in β_i which connects the points $q_{i,j-1}$ and $q_{i,j}$ and does not intersect the set $\{q_{i,h} \colon h \in \mathbb{Z}_j \setminus \{j-1, j\}\}$.

For every $(i,j) \in I$ we denote by $D_{i,j}$ the closed disc in $\mathcal{A}_i$ bounded by the arcs $\alpha_{i,j}$, $\gamma_{i,j-1}$, $\gamma_{i,j}$, and $\beta_{i,j}$. Then $\mathcal{A}_i = \cup_{j=1}^{j} D_{i,j}$ for every i.

Since G is continuous, properties (c2) and (c3) extend to small open neighbourhoods of the compact sets $\gamma_{i,j-1} \cup \alpha_{i,j} \cup \gamma_{i,j}$ and $\{q_{i,j-1}, q_{i,j}\}$, respectively. Therefore, we can choose for every $k \in \{1, 2\}$ and every $(i,j) \in I_k$ a closed disc $K_{i,j} \subset D_{i,j} \setminus (\gamma_{i,j-1} \cup \alpha_{i,j} \cup \gamma_{i,j})$ such that

(d1) $K_{i,j} \cap \beta_{i,j}$ is a compact connected Jordan arc,

(d2) $|G_k(x)| > \rho$ for all $x \in \overline{D_{i,j} \setminus K_{i,j}}$, and

(d3) $|G_k(x)| > \rho + 1$ for all $x \in \overline{\beta_{i,j} \setminus K_{i,j}}$.

Obviously we have

$$V \setminus \mathring{U} = \cup_{(i,j) \in I} \big(K_{i,j} \cup \overline{D_{i,j} \setminus K_{i,j}} \big), \tag{1}$$

$$bV = \cup_{(i,j) \in I} \big((\beta_{i,j} \cap K_{i,j}) \cup \overline{\beta_{i,j} \setminus K_{i,j}} \big). \tag{2}$$

Construction of proper null curves

Assume that $I_1 \neq \varnothing$; otherwise $I_2 = I \neq \varnothing$ and reason symmetrically.

We now deform G into another null immersion $H\colon V \to \mathbb{C}^3$ satisfying the Lemma on the set $U \cup \left(\cup_{(i,j) \in I_1} D_{i,j} \right)$. We will apply to G a perturbation that is large in $K_{i,j}$ for all $(i,j) \in I_1$, small on U, and controlled elsewhere. Such control will be insured by demanding that $H_1 = G_1$.

The compact set

$$S_1 := \left(U \cup \left(\cup_{(i,j) \in I_2} D_{i,j} \right) \right) \cup \left(\cup_{(i,j) \in I_1} K_{i,j} \right) \subset M$$

is admissible. Note that the compact sets $U \cup \left(\cup_{(i,j) \in I_2} D_{i,j} \right)$ and $\cup_{(i,j) \in I_1} K_{i,j}$ are disjoint.

Choose a point $\xi = (0, \xi_2, \xi_3) \in \mathbb{C}^3 \cap \{z_1 = 0\}$ such that

$$|\xi_2 + G_2(x)| > \rho + 1 \quad \text{for all } x \in \cup_{(i,j) \in I_1} K_{i,j}. \tag{3}$$

In fact, any ξ with ξ_2 large enough satisfies this condition.

Construction of proper null curves

The map $\widehat{H} = (\widehat{H}_1, \widehat{H}_2, \widehat{H}_3) \colon S_1 \to \mathbb{C}^3$, given by

(e1) $\widehat{H}(x) = G(x)$ for all $x \in U \cup \left(\cup_{(i,j) \in I_2} D_{i,j} \right)$, and

(e2) $\widehat{H}(x) = (G_1(x), G_2(x) + \xi_2, G_3(x) + \xi_3)$ for all $x \in \cup_{(i,j) \in I_1} K_{i,j}$,

is a null immersion.

Mergelyan's Theorem provides a null holomorphic immersion
$H = (H_1, H_2, H_3) \colon V \to \mathbb{C}^3$ such that

$$H_1 = G_1 \tag{4}$$

and the following properties hold:

(f1) H is as close to $\widehat{H}$ as desired in the $\mathscr{C}^1(S_1)$ topology.

(f2) $|H_k(x)| > \rho$ for all $x \in \overline{D_{i,j} \setminus K_{i,j}}$, $(i,j) \in I_k$, $k = 1, 2$.

(f3) $|H_k(x)| > \rho + 1$ for all $x \in \overline{\beta_{i,j} \setminus K_{i,j}}$, $(i,j) \in I_k$, $k = 1, 2$.

(f4) $|H_2(x)| > \rho + 1$ for all $x \in K_{i,j}$ and all $(i,j) \in I_1 = I \setminus I_2$.

If $I_2 = \varnothing$ then the Lemma already holds with $\widetilde{F} = H$.

Indeed, in such case $I = I_1$; hence, taking into account (1), properties (f2) and (f4) imply item (ii) in the lemma. Likewise, (iii) follows from (f3), (f4), and (2). Finally, item (i) is insured by (f1) and (c1).

Assume now that $I_2 \neq \varnothing$. In the next step we deform H to obtain a null immersion $\widetilde{F} \colon V \to \mathbb{C}^3$ satisfying the Lemma. The deformation is big on $\cup_{(i,j) \in I_2} D_{i,j}$ and small elsewhere. The deformation procedure is symmetric with respect to the one in the previous step of the proof.

The properness theorem for null curves $M \to \mathbb{C}^3$ is obtained by an obvious inductive application of the Lemma.