Complex analytic methods in minimal surface theory

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Lecture 3: Riemann-Hilbert problem for null holomorphic curves, the Calabi-Yau problem, and minimal hulls

The aim of this lecture is to provide approximate solutions to certain Riemann-Hilbert boundary value problems for holomorphic null curves in the complex Euclidean space $\mathbb{C}^3$, and derive geometric applications.

References:

- Antonio Alarcón and Franc Forstnerič, *Null curves and directed immersions of open Riemann surfaces.* Invent. Math. **196** (2014), no. 3, 733–771

- Antonio Alarcón and Franc Forstnerič, *The Calabi-Yau problem, null curves, and Bryant surfaces.* http://arxiv.org/abs/1308.0903

- Barbara Drinovec Drnovšek and Franc Forstnerič, *Minimal hulls of compact sets in* $\mathbb{R}^3$. http://arxiv.org/abs/1409.6906

The classical Riemann-Hilbert problem

- Let $f \colon \overline{\mathbb{D}} \to \mathbb{C}^n$ be a $\mathscr{C}^0$ map that is holomorphic in $\mathbb{D}$. Let

$$g \colon b\mathbb{D} \times \overline{\mathbb{D}} \to \mathbb{C}^n$$

  be a $\mathscr{C}^0$ map such that $g(\xi, \cdot)$ is holomorphic in $\mathbb{D}$ and $g(\xi, 0) = f(\xi)$ for every $\xi \in b\mathbb{D}$.

  Let $K \subset \mathbb{D}$ be a compact set and let $\epsilon > 0$.

- Problem: Find a $\mathscr{C}^0$ map $\phi \colon \overline{\mathbb{D}} \to \mathbb{C}$, holomorphic in $\mathbb{D}$, such that
  - ϕ is ϵ-close to f over K, and
  - $\phi(\xi)$ is ϵ-close to the curve $g(\xi, b\overline{\mathbb{D}})$ for every $\xi \in b\mathbb{D}$.

- The model case: $n = 2$, $f(\xi) = (\xi, 0)$ and $g(\xi, \zeta) = (\xi, \zeta)$; take $\phi(\xi) = (\xi, \xi^N)$ for large $N \in \mathbb{N}$.

- Solutions exist even when $\overline{\mathbb{D}}$ is replaced by a bordered Riemann surface and $\mathbb{C}^n$ by any complex manifold (Forstnerič and Drinovec Drnovšek, Duke Math. J. 139 (2007)). This is useful in a variety of problems.

Null (holomorphic) curves in $\mathbb{C}^3$

Let M be either an open or a bordered Riemann surface.

- Recall that a holomorphic immersion

$$F = (F_1, F_2, F_3)\colon M \to \mathbb{C}^3$$

  is a null curve if it is directed by the conical quadric subvariety

$$\mathfrak{A} = \{z = (z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^2 + z_2^2 + z_3^2 = 0\},$$

  in the sense that the derivative $F' = (F_1', F_2', F_3')$ with respect to any local holomorphic coordinate on M takes values in $\mathfrak{A}_* = \mathfrak{A} \setminus \{0\}$.

- The real and the imaginary part of a null curve $F\colon M \to \mathbb{C}^3$ are conformal minimal immersions in $\mathbb{R}^3$.

- Every conformal minimal immersion $\mathbb{D} \to \mathbb{R}^3$ is the real part of a null curve $\mathbb{D} \to \mathbb{C}^3$. (This fails on non-simply connected Riemann surfaces due to the possibly nonzero periods of the harmonic conjugate.)

Geometry of the null quadric

$$\mathfrak{A} = \{z = (z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^2 + z_2^2 + z_3^2 = 0\}.$$

- $\mathfrak{A}$ is a complex cone with vertex at 0; $\mathfrak{A}_* = \mathfrak{A} \setminus \{0\}$ is smooth.

- $\mathfrak{A}_*$ is a holomorphic fiber bundle with fiber $\mathbb{C}_*$ over the curve $\Lambda = \{[z_1 : z_2 : z_3] : z_1^2 + z_2^2 + z_3^2 = 0\} \subset \mathbb{CP}^2$. We have $\Lambda \cong \mathbb{CP}^1$.

- $\mathfrak{A}_*$ is an Oka manifold; in particular, maps $M \to \mathfrak{A}_*$ satisfy the Runge and Mergelyan approximation properties.

- The spinor representation:

$$\pi \colon \mathbb{C}^2 \to \mathfrak{A}, \quad \pi(u, v) = \left(u^2 - v^2, \mathrm{i}(u^2 + v^2), 2uv\right).$$

The map $\pi \colon \mathbb{C}_*^2 \to \mathfrak{A}_*$ is a nonramified two-sheeted covering.

Let θ be a nowhere vanishing holomorphic 1-form on M.

There is a bijective correspondence (up to translations)

$$\left\{ F\colon M \to \mathbb{C}^3 \text{ null curve} \right\} \longleftrightarrow \left\{ f\colon M \to \mathfrak{A}_* \text{ holomorphic, } f\theta \text{ exact} \right\}$$

$$F(p) = F(p_0) + \int_{p_0}^{p} f\theta, \quad dF = f\theta.$$

Theorem (Lecture 1, Theorem 3)

The general position of null curves in $\mathbb{C}^3$ is embedded.

Riemann-Hilbert problem for null discs

Theorem

Let $F\colon \overline{\mathbb{D}} \to \mathbb{C}^3$ be a null curve, let $\vartheta \in \mathfrak{A}_*$ be a null vector, let

$$\mu\colon b\mathbb{D} = \{\zeta \in \mathbb{C}\colon |\zeta| = 1\} \to \mathbb{R}_+ := [0, +\infty)$$

be a continuous function (the size function), and set

$$\varkappa\colon b\mathbb{D} \times \overline{\mathbb{D}} \to \mathbb{C}^3, \quad \varkappa(\zeta, \xi) = F(\zeta) + \mu(\zeta)\, \xi\, \vartheta.$$

Given numbers $\epsilon > 0$ and $0 < r < 1$, there exist a number $r' \in [r, 1)$ and a null holomorphic immersion $G\colon \overline{\mathbb{D}} \to \mathbb{C}^3$ satisfying

- *G is ϵ-close to F in the $\mathscr{C}^1$ topology on $\{\zeta \in \mathbb{C}\colon |\zeta| \leq r'\}$,*
- *$\mathrm{dist}(G(\zeta), \varkappa(\zeta, b\mathbb{D})) < \epsilon$ for all $\zeta \in b\mathbb{D}$, and*
- *$\mathrm{dist}(G(\rho\zeta), \varkappa(\zeta, \overline{\mathbb{D}})) < \epsilon$ for all $\zeta \in b\mathbb{D}$ and all $\rho \in [r', 1)$.*

Furthermore, if I is a compact arc in $b\mathbb{D}$, the size function μ vanishes everywhere on $b\mathbb{D} \setminus I$, and U is an open neighborhood of I in $\overline{\mathbb{D}}$, then

- *one can choose G to be ϵ-close to F in the $\mathscr{C}^1$ topology on $\overline{\mathbb{D}} \setminus U$.*

Proof

$$
\begin{aligned}
F' &= \pi(u, v) = (u^2 - v^2, \mathfrak{i}(u^2 + v^2), 2uv) \colon \overline{\mathbb{D}} \to \mathfrak{A}_* \\
\vartheta &= \pi(p, q) = (p^2 - q^2, \mathfrak{i}(p^2 + q^2), 2pq) \in \mathfrak{A}_* \\
\eta &= \sqrt{\mu} \colon b\mathbb{D} \to \mathbb{R}_+ \\
\eta(\zeta) &\approx \tilde{\eta}(\zeta) = \sum_{j=1}^{N} A_j \zeta^{j-m} \quad \text{(rational approximation on } \mathbb{C}_*\text{)} \\
u_n(\xi) &= u(\xi) + \sqrt{2n+1}\,\tilde{\eta}(\xi)\,\xi^n p \quad (n > m,\ u_n(0) = u(0)) \\
v_n(\xi) &= v(\xi) + \sqrt{2n+1}\,\tilde{\eta}(\xi)\,\xi^n q \quad (v_n(0) = v(0)) \\
\Phi_n &= \pi(u_n, v_n) = \left(u_n^2 - v_n^2, \mathfrak{i}(u_n^2 + v_n^2), 2u_n v_n\right) \colon \overline{\mathbb{D}} \to \mathfrak{A}_* \\
G_n(\zeta) &= F(0) + \int_0^\zeta \Phi_n(\xi)\,d\xi, \qquad \zeta \in \overline{\mathbb{D}}.
\end{aligned}
$$

Proof

$$
\begin{aligned}
F' &= \pi(u, v) = \left(u^2 - v^2, \mathfrak{i}(u^2 + v^2), 2uv\right) \colon \overline{\mathbb{D}} \to \mathfrak{A}_* \\
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u_n(\xi) &= u(\xi) + \sqrt{2n+1}\, \tilde{\eta}(\xi)\, \xi^n p \quad (n > m,\ u_n(0) = u(0)) \\
v_n(\xi) &= v(\xi) + \sqrt{2n+1}\, \tilde{\eta}(\xi)\, \xi^n q \quad (v_n(0) = v(0)) \\
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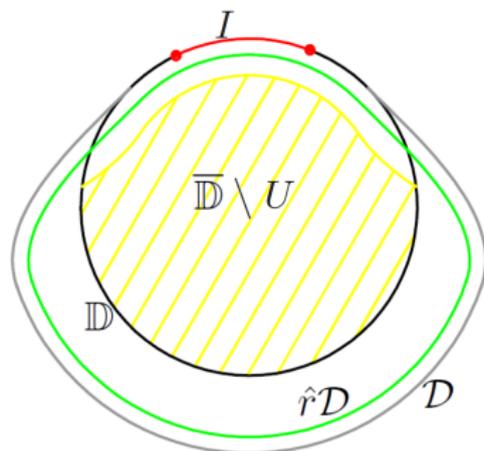G_n(\zeta) &= F(0) + \int_0^\zeta \Phi_n(\xi)\, d\xi, \qquad \zeta \in \overline{\mathbb{D}}.
\end{aligned}
$$

It follows that $G_n(\zeta) \approx F(\zeta) + \zeta^{2n+1} \mu(\zeta) \vartheta$. Take $G = G_n$ for large enough $n \in \mathbb{N}$.

Proof

Furthermore, if I is a compact arc in $b\mathbb{D}$, the size function μ vanishes everywhere on $b\mathbb{D} \setminus I$, and U is an open neighborhood of I in $\overline{\mathbb{D}}$, then

- one can choose G to be ϵ-close to F in the $\mathscr{C}^1$ topology on $\overline{\mathbb{D}} \setminus U$.

Riemann-Hilbert problem for null curves

Theorem

Let M be a bordered Riemann surface and let $F\colon M \to \mathbb{C}^3$ be a null holomorphic immersion (the *central curve*).

Let I be a compact arc in bM that is *not a connected component of* bM.

Choose a small annular neighborhood $A \subset M$ of the component C of bM containing I and a smooth retraction $\rho\colon A \to C$.

Let $\vartheta \in \mathfrak{A}_*$ be a null vector (the *direction vector*), let $\mu\colon C \to \mathbb{R}_+$ be a continuous function supported on I (the *size function*), and consider the continuous map

$$\varkappa\colon bM \times \overline{\mathbb{D}} \to \mathbb{C}^3$$

$$\varkappa(x, \xi) = \begin{cases} F(x); & x \in bM \setminus I \\ F(x) + \mu(x)\,\xi\,\vartheta; & x \in I. \end{cases}$$

Theorem (Continued)

Then for any number $\epsilon > 0$ there exist an arbitrarily small open neighborhood Ω of I in A and a null holomorphic immersion $G \colon M \to \mathbb{C}^3$ satisfying the following properties:

- *G is ϵ-close to F in the $\mathscr{C}^1$ topology on $M \setminus \Omega$.*
- $\mathrm{dist}(G(x), \varkappa(x, b\mathbb{D})) < \epsilon$ *for all $x \in bM$.*
- $\mathrm{dist}(G(x), \varkappa(\rho(x), \overline{\mathbb{D}})) < \epsilon$ *for all $x \in \Omega$.*

- We do not change the conformal structure on M.

- I can be replaced by a finite family of pairwise disjoint compact subarcs; it is allowed to use different direction vectors in each subarc.

Sketch of the proof

- Solve the problem in a small disc $D \subset \Omega \subset M \setminus A$ containing I. (Previous Theorem.) Let $G_0 \colon \overline{D} \to \mathbb{C}^3$ be a solution.

- Let θ be a nowhere vanishing holomorphic 1-form on M and write

$$dG_0 = g_0\theta, \quad g_0 \colon \overline{D} \to \mathfrak{A}_*,$$

$$dF = f_0\theta, \quad f_0 \colon M \to \mathfrak{A}_*.$$

- Embed f_0 as the core map of a dominating and period dominating holomorphic spray of maps (see Lecture 2):

$$\left\{ f_t \colon M \to \mathfrak{A}_* \right\}_{t \in B}, \quad 0 \in B \subset \mathbb{C}^N \quad (N \text{ large}).$$

- Find a dominating spray of holomorphic maps

$$\left\{ g_t \colon \overline{D} \to \mathfrak{A}_* \right\}_{t \in B}.$$

which are approximate solutions to the Riemann-Hilbert problem for the maps f_t in the spray $\{f_t\}_{t \in B}$ in $\overline{D}$.

Sketch of the proof

- Glue the two (dominating) sprays $\{f_t\}_{t\in B}$ and $\{g_t\}_{t\in B}$ into a single spray of holomorphic maps

$$\{h_t \colon M \to \mathfrak{A}_*\}_{t\in B}$$

such that h_t is close to g_t on $\overline{D}$ and to f_t on $\overline{M \setminus D}$.

- Since $f_0\theta$ is exact and the spray f_t is period dominating, so is h_t provided the approximation is close enough. Hence the period map

$$B \ni t \mapsto \mathcal{P}(h_t) = \left(\int_\gamma h_t\theta \right)_{\gamma \in \mathcal{H}_1(M,\mathbb{Z})}$$

has maximal rank at $t = 0$. The Implicit Function Theorem gives $t_0 \in B$ close to 0 such that $h_{t_0}\theta$ has vanishing periods.

- Fix a point $p_0 \in M \setminus D$. The map $G \colon M \to \mathbb{C}^3$,

$$G(p) = F(p_0) + \int_{p_0}^p h_{t_0}\theta$$

proves the theorem.

The conformal Calabi-Yau problem

Theorem

Every bordered Riemann surface carries a **complete proper null holomorphic embedding** *into a* **ball** *of* $\mathbb{C}^3$.

Corollary

Every bordered Riemann surface M carries a **complete conformal minimal immersion** $M \to \mathbb{R}^3$ *with* **bounded image**.

- This answers a question by Martín, Umehara, and Yamada 2009.
- What is new in comparison to all existing results in the literature is that

<div align="center">

**we do not change the complex structure
on the source Riemann surface.**

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(Compare with the work of Nadirashvili, Martín, Morales, López, Alarcón, Ferrer, Umehara, Yamada, Meeks, etc.)

Idea of the construction of complete bounded holomorphic immersions - Pythagoras' theorem

- Let $F_0 \colon M \to \mathbb{C}^n$ be a holomorphic immersion satisfying $|F_0| \geq r_0 > 0$ on bM. We try to increase the boundary distance on M with respect to the induced metric by a fixed number $\delta > 0$.

- To this end, we approximate F_0 uniformly on a compact set in M by an immersion $F_1 \colon M \to \mathbb{C}^n$ which at a point $x \in bM$ adds a displacement for approximately δ in a direction $V \in \mathbb{C}^n$, $|V| = 1$, approximately orthogonal to the point $F_0(x) \in \mathbb{C}^n$. The boundary distance increases by $\approx \delta$, while the outer radius increases to

$$|F_1(x)| \approx \sqrt{|F_0(x)|^2 + \delta^2} \approx |F_0(x)| + \frac{\delta^2}{2|F_0(x)|} \leq |F_0(x)| + \frac{\delta^2}{2r_0}.$$

- By choosing a sequence $\delta_j > 0$ such that $\sum_j \delta_j = +\infty$ while $\sum_j \delta_j^2 < \infty$, we obtain by induction a limit immersion $F \colon M \to \mathbb{C}^n$ with bounded outer radius and with complete metric $F^* ds^2$.

The main tools

- This idea can be realized on short arcs $I \subset bM$, on which F_0 does not vary too much, by solving a **Riemann-Hilbert problem**.

- Globally this method alone could lead to 'sliding curtains', creating shortcuts in the new induced metric on M.

- To **localize the problem** and **eliminate any shortcuts**, we subdivide $bM = \cup_j I_j$ into a finite union of short arcs such that two adjacent arcs I_{j-1}, I_j meet at a common endpoint x_j. At the point $p_j = F(x_j) \in \mathbb{C}^n$ we attach to $F_0(M)$ a smooth real curve λ_j of length δ whose other endpoint q_j increases the outer radius by δ^2.

- By the method of **exposing boundary points** we modify the immersion so that $F_0(x_j) = q_j$. Hence any curve in M terminating on bM near x_j is elongated by approximately $\delta > 0$.

- In the next step we use a Riemann-Hilbert problem to increase the boundary distance on the arcs I_j by approximately δ. These local modifications are glued together by the method of sprays.

Null curves with a bounded coordinate

Theorem

Every bordered Riemann surface M carries a proper holomorphic null embedding $F = (F_1, F_2, F_3) \colon M \to \mathbb{C}^3$ such that F_3 is bounded on M. (Thus $(F_1, F_2) \colon M \to \mathbb{C}^2$ is a proper map.)

- These are the first examples of proper null curves in $\mathbb{C}^3$ with a bounded coordinate function.

- Every open Riemann surface properly immerses in $\mathbb{C}^2$ as a complex curve (Remmert 1956, Bishop 1960, and Narasimhan 1961).

- The only properly immersed minimal surfaces in $\mathbb{R}^3$ contained in a half-space are planes (Hoffman and Meeks 1990). In particular, with **F** as in the theorem, $\Re(e^{it}F) \colon M \to \mathbb{R}^3$ is not proper for any $t \in \mathbb{R}$.

- Say a word about the proof.

Null curves in $SL_2(\mathbb{C})$ and Bryant surfaces

- A **null curve** in the special linear group

$$SL_2(\mathbb{C}) = \left\{ z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \in \mathbb{C}^4 \colon \det z = z_{11}z_{22} - z_{12}z_{21} = 1 \right\}$$

is a holomorphic immersion $F \colon M \to SL_2(\mathbb{C})$ of an open Riemann surface M into $SL_2(\mathbb{C})$ which is directed by the quadric variety

$$\left\{ z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \colon \det z = z_{11}z_{22} - z_{12}z_{21} = 0 \right\} \subset \mathbb{C}^4.$$

- The projection of a null curve in $SL_2(\mathbb{C})$ to the hyperbolic 3-space $\mathbb{H}^3 = SL_2(\mathbb{C})/SU(2)$ is a conformal **Bryant surface** (a conformal immersion with constant mean curvature one in $\mathbb{H}^3$).
Bryant surfaces are closely related to minimal surfaces in $\mathbb{R}^3$.

Null curves in $SL_2(\mathbb{C})$ and CMC-1 surfaces in $\mathbb{H}^3$

- The biholomorphic map $\mathbf{T}\colon \mathbb{C}^3 \setminus \{z_3 = 0\} \to SL_2(\mathbb{C}) \setminus \{z_{11} = 0\}$,

$$\mathbf{T}(z_1, z_2, z_3) = \frac{1}{z_3} \begin{pmatrix} 1 & z_1 + \mathrm{i}z_2 \\ z_1 - \mathrm{i}z_2 & z_1^2 + z_2^2 + z_3^2 \end{pmatrix},$$

  carries null curves to null curves (Martín, Umehara & Yamada 2009).
- If $F = (F_1, F_2, F_3)\colon M \to \mathbb{C}^3$ is a **proper** null curve such that $c_1 < |F_3| < c_2$ on M for a pair of constants $0 < c_1 < c_2$, then $\mathbf{T} \circ F\colon M \to SL_2(\mathbb{C})$ is a **proper** null curve in $SL_2(\mathbb{C})$.
- Since $SU(2)$ is compact, **proper** null curves in $SL_2(\mathbb{C})$ project into **proper** Bryant surfaces in $\mathbb{H}^3$.

Corollary

Every bordered Riemann surface carries a proper holomorphic null embedding into $SL_2(\mathbb{C})$ and a proper conformal Bryant surface in $\mathbb{H}^3$.

- These are the first examples of **proper** null curves in $SL_2(\mathbb{C})$, and Bryant surfaces in $\mathbb{H}^3$, with finite topology and **hyperbolic** conformal structure.
  Morales 2003 Analogous problem for minimal surfaces in $\mathbb{R}^3$.

Final remarks and open problems

- Our method enables one to construct examples with control on the conformal structure. If we do not take care of the conformal structure, our method gives null curves in $\mathbb{C}^3$ with arbitrary topology.

- Is every open Riemann surface conformally equivalent to a properly immersed Bryant surface in $\mathbb{H}^3$?
  Does every open Riemann surface admit a proper holomorphic null embedding (or immersion) into $SL_2(\mathbb{C})$?

- Work in progress: Every bordered Riemann surface admits a conformal complete proper minimal immersion into a ball of $\mathbb{R}^3$, and also a bounded complete immersion in $\mathbb{R}^3$ which extends continuously to the boundary and maps the boundary onto a union of Jordan curves. There exits embeddings with these properties into $\mathbb{R}^5$.

Minimal hulls of compact sets in $\mathbb{R}^3$

(With Barbara Drinovec Drnovšek.)

When discussing hulls in various geometries, one typically deals with dual sets of objects.

Given a set $\mathcal{P}$ of real functions on a manifold X, the $\mathcal{P}$-hull of a compact subset $K \subset X$ is

$$\widehat{K}_{\mathcal{P}} = \{x \in X : f(x) \leq \sup_K f \quad \forall f \in \mathcal{P}\}.$$

Suppose that $\mathcal{G}$ is a class of geometric objects in X such that the restriction $f|_C$ satisfies the maximum principle for every $f \in \mathcal{P}$ and $C \in \mathcal{G}$. Then $C \subset \widehat{K}_{\mathcal{P}}$ for every $C \in \mathcal{G}$ with boundary $bC \subset K$.

The main question is how closely is the hull described by such objects.

A basic example is the convex hull $\mathrm{Co}(K)$ of a compact set in an affine space $X \cong \mathbb{R}^n$; here $\mathcal{P}$ is the class of all affine linear functions on X and $\mathcal{G}$ is the collection of straight line segments in X.

Polynomial hulls

Another classical case is the polynomial hull, $\widehat{K}$, of a compact set $K \subset \mathbb{C}^n$. This is the hull $\widehat{K}_{\mathcal{P}}$ for $\mathcal{P} = \{|f| : f \in \mathcal{O}(\mathbb{C}^n)\}$ or $\mathcal{P} = \mathrm{Psh}(\mathbb{C}^n)$ (plurisubharmonic functions). A natural dual class $\mathcal{G}$ consists of complex curves.

Wermer (1958), Stolzenberg (1966), Alexander (1971): If K is a compact sets of finite linear measure then $A = \widehat{K} \setminus K$ is a (possibly empty) one dimensional closed complex subvariety of $\mathbb{C}^n \setminus K$.

Stolzenberg' (1963): in general one cannot describe $\widehat{K}$ by bounded complex curves with boundaries exactly in K.

Theorem (Poletsky 1991, 1993)

Let K be a compact set in $\mathbb{C}^n$, and let $B \subset \mathbb{C}^n$ be a ball containing K. A point $p \in B$ belongs to $\widehat{K}$ iff there exists a sequence of holomorphic discs $f_j : \overline{\mathbb{D}} \to B$ satisfying the following for every $j = 1, 2, \ldots$:

$$f_j(0) = p \quad \text{and} \quad \left|\{t \in [0, 2\pi] : \mathrm{dist}(f(e^{it}), K) < 1/j\}\right| \geq 2\pi - 1/j.$$

Minimal plurisubharmonic functions

Is there a similar notion of the minimal hull, $\widehat{K}_{\mathfrak{M}}$, of a compact set $K \subset \mathbb{R}^n$? Which class of function should one use?

Definition (Harvey & Lawson; Drinovec Drnovšek & F.)

An upper semicontinuous function u on a domain $\omega \subset \mathbb{R}^n$ ($n \geq 3$) is minimal plurisubharmonic ($\mathfrak{M}$-psh) if the restriction of u to any affine 2-dimensional plane $L \subset \mathbb{R}^n$ is subharmonic on $L \cap \omega$ (in isothermal coordinates). The set of all such functions is denoted by $\mathfrak{M}\mathrm{Psh}(\omega)$.

Observations:

- The restriction of an $\mathfrak{M}$-psh function to an immersed minimal surface is subharmonic (in isothermal coordinates on the surface).
- A $\mathscr{C}^2$ function u is $\mathfrak{M}$-psh if and only if the sum of the two smallest eigenvalues of $\mathrm{Hess}\, u$ is nonnegative at every point.
- A function $u \colon \omega \to \mathbb{R}$ on $\omega \subset \mathbb{R}^n$ is $\mathfrak{M}$-psh iff the function $U \colon \omega \times \mathbb{R}^n \to \mathbb{R}$, $U(x + \mathrm{i}y) = u(x)$, is null plurisubharmonic, i.e., subharmonic on any null holomorphic curve.

Minimal hulls

Definition

The minimal hull of a compact set $K \subset \mathbb{R}^n$ $(n \geq 3)$ is the set

$$\widehat{K}_{\mathfrak{M}} = \{x \in \mathbb{R}^n : u(x) \leq \sup_K u \quad \forall u \in \mathfrak{M}\mathrm{Psh}(\mathbb{R}^n)\}.$$

By the max principle, every minimal surface M with $bM \subset K$ lies in $\widehat{K}_{\mathfrak{M}}$.

Theorem

Let K be a compact set in $\mathbb{R}^3$, and let $\omega \Subset \mathbb{R}^3$ be a bounded open convex set containing K. A point $p \in \omega$ belongs to the minimal hull $\widehat{K}_{\mathfrak{M}}$ if and only if there exists a sequence of conformal minimal discs $f_j : \overline{\mathbb{D}} \to \omega$ such that for all $j = 1, 2, \ldots$ we have $f_j(0) = p$ and

$$\left|\{t \in [0, 2\pi] : \mathrm{dist}(f_j(e^{it}), K) < 1/j\}\right| \geq 2\pi - 1/j.$$

[Barbara Drinovec Drnovšek & Franc Forstnerič, Minimal hulls of compact sets in $\mathbb{R}^3$. http://arxiv.org/abs/1409.6906

Largest $\mathfrak{M}$-psh minorant

This is a consequence of the following result, analogous to the Poletsky and Bu-Schachermayer theorem (1991, 1993) for psh minorants.

Theorem (B. Drinovec Drnovšek & Forstnerič, 2014)

Assume that ω is a domain in $\mathbb{R}^3$ and $\phi: \omega \to \mathbb{R} \cup \{-\infty\}$ is an upper semicontinuous function. Then the function $u: \omega \to \mathbb{R} \cup \{-\infty\}$, given for any $x \in \omega$ by

$$u(x) = \inf \left\{ \int_0^{2\pi} \phi(f(e^{\mathrm{i}t})) \frac{dt}{2\pi} : \ f \in \mathrm{CMI}(\overline{\mathbb{D}}, \omega), \ f(0) = x \right\}$$

is minimal plurisubharmonic or identically $-\infty$; moreover, u is the supremum of functions in $\mathfrak{M}\mathrm{Psh}(\omega)$ which are bounded above by ϕ.

The proof uses the approximate solution to the Riemann-Hilbert problem for null holomorphic discs in $\mathbb{C}^3$, together with the relationship between $\mathfrak{M}$-psh functions on domains $\omega \subset \mathbb{R}^3$ and null-psh functions on tubes $\omega \times \mathrm{i}\mathbb{R}^3 \subset \mathbb{C}^3$.